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Adaptive Reduced Basis Domain Decomposition Methods

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MM Mathematics Münster Cluster of Excellence

Outline

1. Reduced Basis Methods for Elliptic Problems
2. Localized Reduced Basis Methods for Elliptic Problems
3. Localized Reduced Basis Domain Decomposition Methods for Elliptic Problems

Reduced Basis Methods for Elliptic Problems

Reduced Basis Methods

Parametric linear elliptic problem (full order model)

For given parameter $\mu \in \mathcal{P}$, find $u_h(\mu) \in V_h$ s.t.

$$\begin{aligned} a(u_h(\mu), v_h; \mu) &= f(v_h) \qquad \forall v_h \in V_h \\ y_h(\mu) &= g(u_h(\mu)) \end{aligned}$$

Reduced Basis Methods

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Parametric linear elliptic problem (reduced order model)

For given $V_N \subset V_h$, let $u_N(\mu) \in V_N$ be given by Galerkin proj. onto V_N, i.e.

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a(u_N(\mu), v_N; \mu) &= f(v_N) \qquad && \forall v_N \in V_N \\
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RB Methods – Computing V_N

Weak greedy basis generation

function $\textsc{Weak-Greedy}(\mathcal{S}_{train} \subset \mathcal{P}, \varepsilon)$
    $V_N \leftarrow \{0\}$
    while $\max_{\mu \in \mathcal{S}_{train}} \textsc{Err-Est}(\textsc{ROM-Solve}(\mu), \mu) > \varepsilon$ **do**
        $\mu^* \leftarrow \text{arg-max}_{\mu \in \mathcal{S}_{train}} \textsc{Err-Est}(\textsc{ROM-Solve}(\mu), \mu)$
        $V_N \leftarrow \text{span}(V_N \cup \{\textsc{FOM-Solve}(\mu^*)\})$
    end while
    return V_N
end function

Err-Est

Use residual-based error estimate w.r.t. FOM (finite dimensional $\rightsquigarrow$ can compute dual norms).

RB Methods – Computing V_N

Weak greedy basis generation

function WEAK-GREEDY($\mathcal{S}_{train} \subset \mathcal{P}, \varepsilon$)
    $V_N \leftarrow \{0\}$
    while $\max_{\mu \in \mathcal{S}_{train}}$ ERR-EST(ROM-SOLVE(μ), μ) $> \varepsilon$ **do**
        $\mu^* \leftarrow \text{arg-max}_{\mu \in \mathcal{S}_{train}}$ ERR-EST(ROM-SOLVE(μ), μ)
        $V_N \leftarrow \text{span}(V_N \cup \{$FOM-SOLVE$(\mu^*)\})$
    end while
    return V_N
end function

ERR-EST

Use residual-based error estimate w.r.t. FOM (finite dimensional $\rightsquigarrow$ can compute dual norms).

- Use dual weighted residual approach for improved convergence w.r.t to output $y_N(\mu)$.

RB Methods – Online Efficiency

Parametric linear elliptic problem (reduced order model)

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Affine decomposition

Assume that a_μ can be written as

$$
a(u, v; \mu) = \sum_{q=1}^{Q} \theta_q(\mu) a_q(u, v).
$$

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Offline/Online splitting

By pre-computing

$$a_q(\varphi_i, \varphi_j),\ f(\varphi_i),\ g(\varphi_i)$$

for a reduced basis $\varphi_1, \ldots, \varphi_N$ of V_N, solving ROM becomes independent of dim V_h.

Example: RB Approximation of Li-Ion Battery Models

ulm university universität uulm

DLR Institute of Technical Thermodynamics

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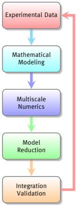
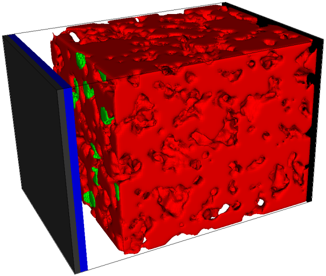
MULTIBAT

Fraunhofer ITWM

Westfälische Wilhelms-Universität Münster





MULTIBAT: Gain understanding of degradation processes in rechargeable Li-Ion Batteries through mathematical modeling and simulation at the pore scale.

FOM:

- 2.920.000 DOFs
- Simulation time: ≈ 15.5h

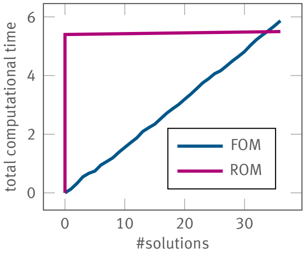
ROM:

- Snapshots: 3
- dim $V_N = 245$
- Rel. err.: $< 4.5 \cdot 10^{-3}$
- Reduction time: ≈ 14h
- Simulation time: ≈ 8m
- Speedup: 120

Localized Reduced Basis Methods for Elliptic Problems

Caveats

- Offline time too large in not-so-many-query scenarios?
- $\mathcal{P}$ too large?



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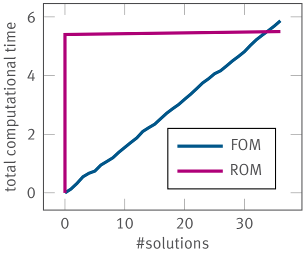

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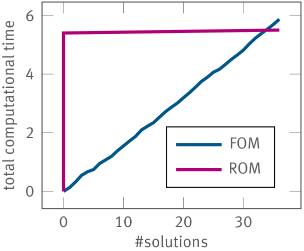

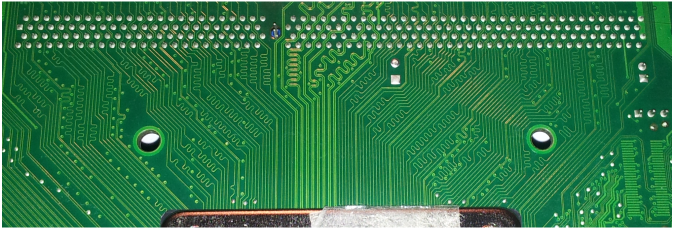

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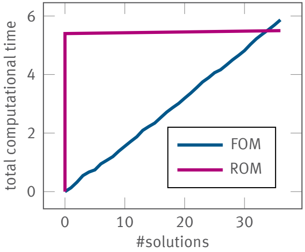

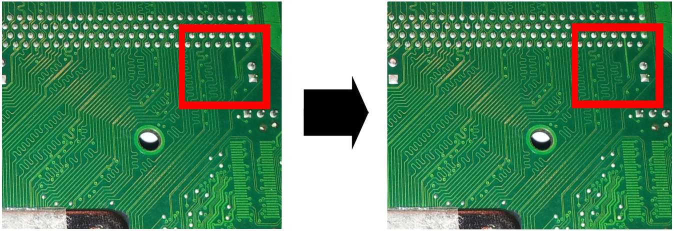

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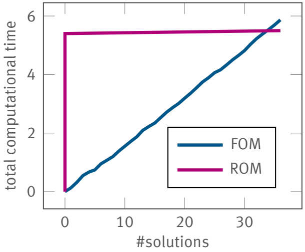
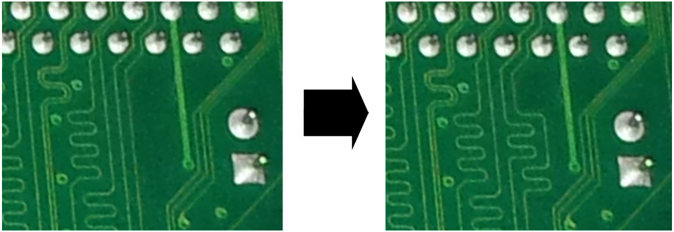




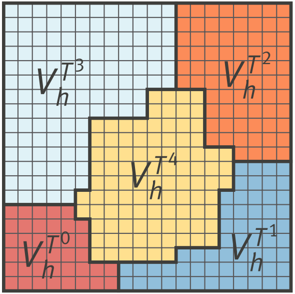
Localized RB Methods for Elliptic Problems

Idea of the **LRBMS**: given a finely-resolved grid τ_h [Albrecht et al., 2012]

- decompose approximation space into *local* spaces $V_h = \bigoplus_{T \in \mathcal{T}_H} V_h^T$
- associated with subdomains $T \in \mathcal{T}_H$

  independent local discretizations and approximation spaces (CG or DG)

  and global SWIPDG coupling [Ern, Stephansen, Zunino, 2009]



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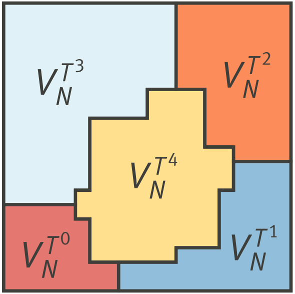
- build local reduced spaces $V_N^T \subset V_h^T$
- reduced *broken* space $V_N = \bigoplus_{T\in\mathcal{T}_H} V_N^T$



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- build local reduced spaces $V_N^T \subset V_h^T$
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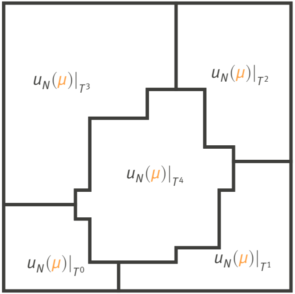
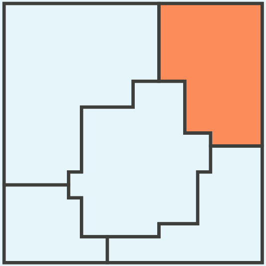
- larger V_N, but sparse ROM system matrices
- initialization of V_N^T:
  - empty
  - global solution snapshots
  - **local training**



Offline Initialization of V_N

Training algorithm (adapted from [Buhr, Engwer, Ohlberger, R, 2017]) for all $T \in \mathcal{T}_H$



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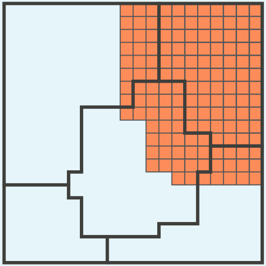

- For every $\mu \in \mathcal{S}_{train} \subset \mathcal{P}$:
  - Solve training problem on oversampling subdomain $T^\delta \supset T$:

$$a(\varphi_{h,0}(\mu), v_h; \mu) = f(v_h) \qquad \text{in } T^\delta$$
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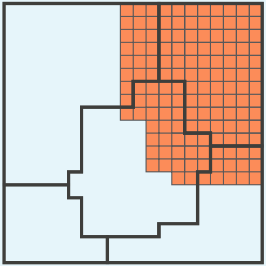

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  - For $1 \le k \le K$, solve training problem:

$$a(\varphi_{h,k}(\mu), v_h; \mu) = 0 \qquad \text{in } T^\delta$$
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    for K random Dirichlet data functions g_k on ∂T^δ.

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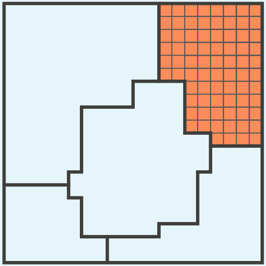

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- Initialize local RB space on T as
$$V_N^T := \operatorname{span} \bigcup_{\mu \in \mathcal{S}_{train}} \{\varphi_{h,0}(\mu)|_T, \ldots, \varphi_{h,K}(\mu)|_T\}.$$

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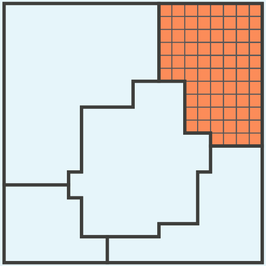

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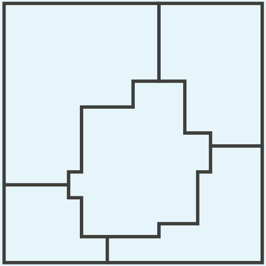
$$V_N^T := \operatorname{span} \bigcup_{\mu \in \mathcal{S}_{train}} \{\varphi_{h,0}(\mu)|_T, \ldots, \varphi_{h,K}(\mu)|_T\}.$$

- Use greedy algorithm for large $\mathcal{S}_{train}$.

Online-Adaptive Enrichment of V_N

Enrichment algorithm — for some $\mu \in \mathcal{P}$



- compute reduced solution $u_N(\mu)$
- estimate error $n_{h,N}(\mu)$
- if $n_{h,N}(\mu) > \Delta$, start intermediate local enrichment phase:
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Online-Adaptive Enrichment of V_N

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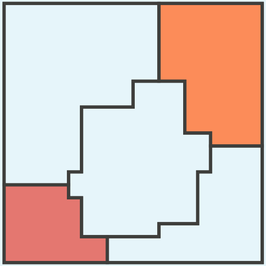

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  - mark subdomains for enrichment: $\mathcal{X} = \texttt{mark}(\mathcal{T}_H)$

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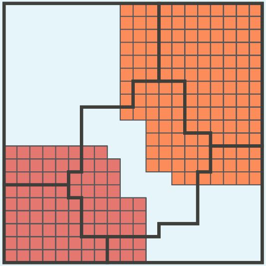

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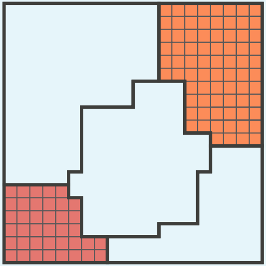

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\end{aligned}
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  - extend local reduced basis for all $T \in \mathcal{X}$:

$$
V_N^T := \operatorname{span} V_N^T \cup \{\varphi_h(\mu)|_T\}
$$

Online-Adaptive Enrichment of V_N

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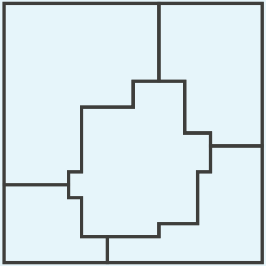

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  - update reduced quantities
  - compute updated reduced solution $u_N(\mu)$ and $\eta_{h,N}(\mu)$

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  - update reduced quantities
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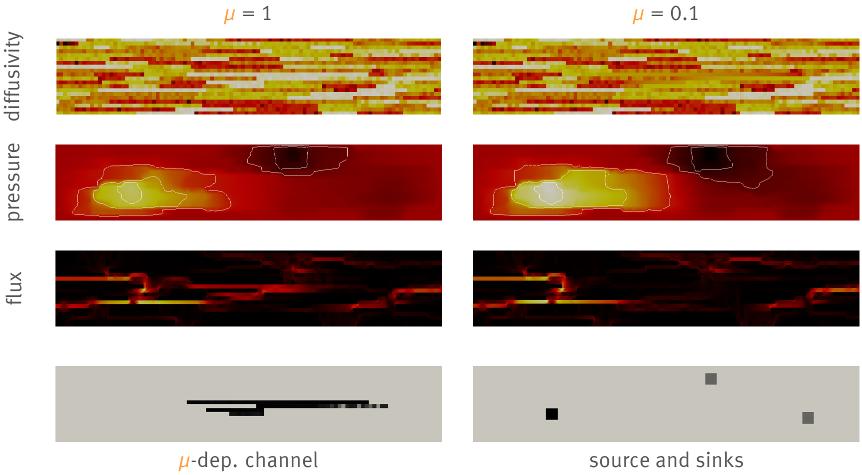
- iterate until $\eta_{h,N}(u_{\mu,N}) \leq \Delta$, return $u_N(\mu)$

$\dim V_h(T^\delta)$

LRBMS with online enrichment: Example SPE10



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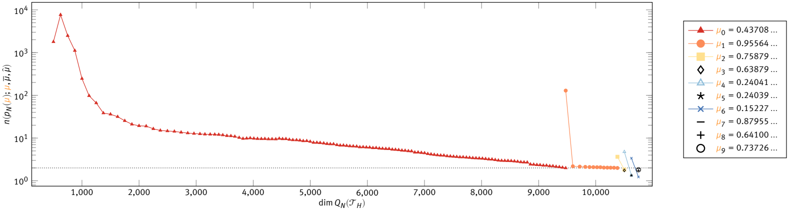
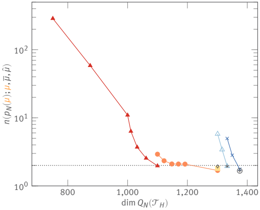
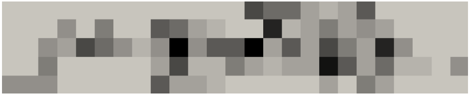


Convergence history of LRBMS with initially empty V_N



LRBMS initialized with 2 solution snapshots



Distribution of local basis size after online enrichment.

Related Approaches (incomplete)

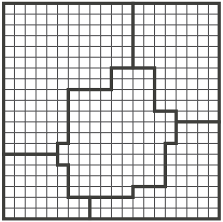
- Reduced Basis Element Method
  [Maday, Ronquist, 2002]

- Port-Reduced Static Condensation Reduced Basis Element Method
  [Eftang, Patera, 2013]

- Generalized Multiscale Finite Element Methods
  [Efendiev, Galvis, Hou 2013]

- Reduced Basis Hybrid Method
  [Iapichino, Quarteroni, Rozza, Volkwein, 2014]

- ArbiLoMod, a Simulation Technique Designed for Arbitrary Local Modifications
  [Buhr, Engwer, Ohlberger, R, 2017]



Localized Reduced Basis Domain Decomposition Methods for Elliptic Problems

Questions

- Where should be enriched?
- How fast will enrichment converge?
- Which training method to combine with enrichment?
- How to balance training and enrichment?

Goal: Minimize total number of local V_h-dependent computations/communication events.

Connections with Domain Decomposition Methods

- Local enrichment function $\varphi_h(\mu)|_T$

$$a(\varphi_h(\mu), v_h; \mu) = f(v_h) \qquad \text{in } T^\delta$$
$$\varphi_h(\mu) = u_N(\mu) \qquad \text{on } \partial T^\delta$$

corresponds to subdomain solution in Restricted Additive Schwarz (RAS) method.

Connections with Domain Decomposition Methods

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- Moreover:

offline training of V_N

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construction of coarse space

e.g. DtN [Nataf et al., 2011], GenEO [Spillane et al., 2014], SHEM [Gander, Loneland, Rahman, 2015]

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4. Use RB estimator $n_{h,N}(u_N(\mu);\mu)$ to locally enrich V_N^T whith AS corrections where needed:

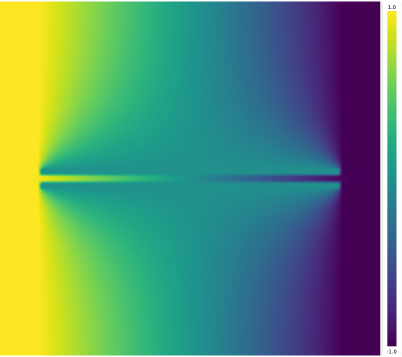
$$n_{h,N}(u_N(\mu);\mu)^2 := C(\mu)^2 \sum_{T\in\mathcal{T}_H} \left(\sup_{v_h \in V_h^T} \frac{f(v_h) - a(u_N(\mu), v_h; \mu)}{\|v_h\|} \right)^2$$

   where, with C_{stab} the stability constant of decomposition $V_h = V_N^0 + \sum_{T\in\mathcal{T}_H} V_h^T$:

$$C(\mu) \leq C_{inf\text{-}sup}(\mu) \cdot C_{stab}$$

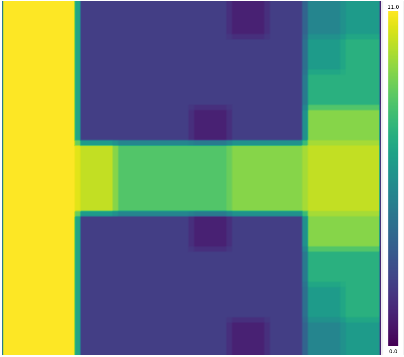
Simple Experiment (without μ, local non-parametric changes)

Solution (contrast: 10^5)



- 10 × 10 subdomains
- 4 elements overlap
- 6 GenEO basis functions per domain

Number of local solutions (max=11)



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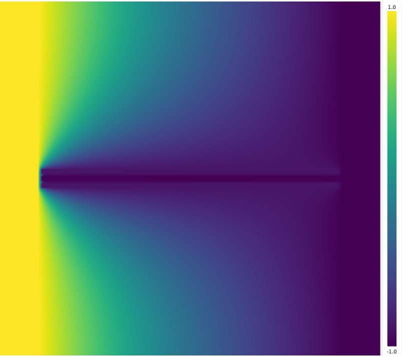

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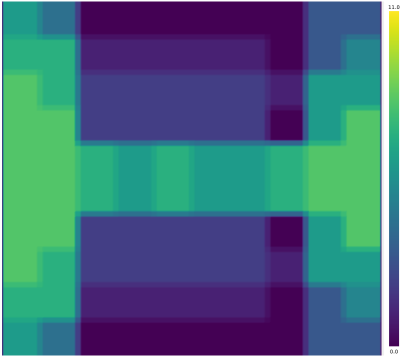

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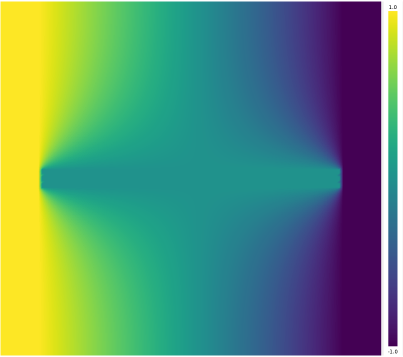

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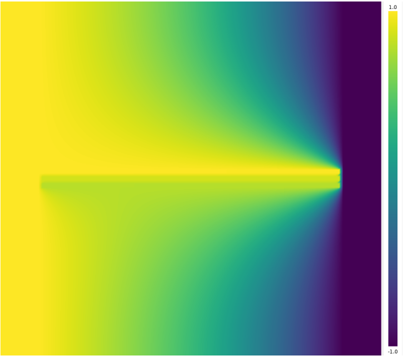

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Some Remarks

- Communication of V_h-dependet data only with neighbors of enriched subdomains.
- localized enrichment ≅ flexible multi-preconditioned projected CG with full orthogonalization.
- More iterations but less work.

	iterations	local solutions
PCG	118	11800
PCG + RB solution as initial value	84	8400
enrich localized (keep solutions in V_N^T)	**38**	**1803**
enrich everywhere (keep solutions in V_N^T)	36	3600
enrich localized (keep updates in V_N^T)	33	1718
enrich everywhere (keep updates in V_N^T)	29	2900

Thank you for your attention!

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