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Nonlinear Model Order Reduction for Parametric Time Evolution Problems using the Method of Freezing

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Outline

- Reduced Basis Methods
- Freezing of Nonlinear Evolution Equations
- FrozenRB Approximation of Nonlinear Evolution Equations

Reduced Basis Methods

Parametric Model Order Reduction

Consider parametric problems

$$\Phi : \mathcal{P} \to V, \qquad s : V \to \mathbb{R}^S$$

where

- $\mathcal{P} \subset \mathbb{R}^P$ *compact* set (parameter domain).
- V solution state space, $\dim V \gg 0$ (theoretically $\dim V = \infty$).
- Φ maps parameters to solutions (*hard* to compute).
- s maps state vectors to quantities of interest.

Objective

Compute

$$s \circ \Phi : \mathbb{R}^P \to V \to \mathbb{R}^S$$

for *many* $\mu \in \mathcal{P}$ or *quickly* for unknown single $\mu \in \mathcal{P}$.

Reduced Basis Methods: Three Basic Ideas

Objective

Compute

$$s \circ \Phi : \mathbb{R}^P \to V \to \mathbb{R}^S.$$

When Φ, s sufficiently smooth, quickly computable low-dimensional approximation of $s \circ \Phi$ should exist.

- **Idea 1:** State space projection:
  - Define approximation $\tilde{\Phi} : \mathcal{P} \to \tilde{V}$, dim $\tilde{V} \ll$ dim V, via Galerkin projection.
  - Approximate $s \circ \Phi \approx s \circ \tilde{\Phi}$.
- **Idea 2:** Construct $\tilde{V}$ from solution snapshots $\Phi(\mu_1), \ldots, \Phi(\mu_k)$.
- **Idea 3:** Select $\mu_1, \ldots, \mu_k$ iteratively via greedy search over $\mathcal{P}$ using quickly computable surrogate $\eta(\tilde{\Phi}(\mu), \mu) \geq \|\Phi(\mu) - \tilde{\Phi}(\mu)\|$.

RB for Nonlinear Evolution Equations

Full order problem

$\Phi(\mu) = u_\mu \in V \subseteq L^2(0, T; V_h)$ is the solution of

$$\partial_t u_\mu(t) + \mathcal{L}_\mu(u_\mu(t)) = 0, \quad u_\mu(0) = u_0,$$

where $\mathcal{L}_\mu : \mathcal{P} \times V_h \to V_h$ is a nonlinear Finite Volume operator.

Reduced order problem

For given $V_N \subset V_h$, let $\tilde{\Phi}(\mu) := \tilde{u}_\mu \in \tilde{V} \subseteq L^2(0, T; V_N)$ be given by Galerkin proj. onto V_N, i.e.

$$\partial_t \tilde{u}_\mu(t) + P_N(\mathcal{L}_\mu(\tilde{u}_{\mu,N}(t))) = 0, \quad \tilde{u}_\mu(0) = P_N(u_0),$$

where $P_N : V_h \to V_N$ is orthogonal proj. onto V_N.

Empirical Operator Interpolation (a.k.a. DEIM, EIM)

Problem: Still expensive to evaluate

$$P_N \circ \mathcal{L}_\mu : V_N \longrightarrow V_h \longrightarrow V_N.$$

Solution:

- Use locality of finite volume operators:

  to evaluate M DOFs of $\mathcal{L}_\mu(u)$ we need $M' \leq C \cdot M$ DOFs of u.

- Approximate

  $$\mathcal{L}_\mu \approx \mathcal{I}_M[\mathcal{L}_\mu] := I_M \circ \mathcal{L}_{M,\mu} \circ R_{M'},$$

  where

$R_{M'}: V_h \to \mathbb{R}^{M'}$	restriction to M' DOFs needed for evaluation
$\mathcal{L}_{M,\mu}: \mathbb{R}^{M'} \to \mathbb{R}^M$	$\mathcal{L}_\mu$ restricted to M interpolation DOFs
$I_M: \mathbb{R}^M \to V_h$	linear combination with interpolation basis

- Use greedy algorithm to determine DOFs and interpolation basis from operator evaluations on appropriate solution trajectories.

Full Reduction

Reduced order problem (with EI)

$\tilde{\Phi}(\mu) := \tilde{u}_\mu \in \tilde{V} \subseteq L^2(0, T; V_N)$ is given by

$$\partial_t \tilde{u}_\mu(t) + \{(P_N \circ I_M) \circ \mathcal{L}_{M,\mu} \circ R_{M'}\}(\tilde{u}_{\mu,N}(t)) = 0, \quad \tilde{u}_\mu(0) = P_N(u_0).$$

Offline/Online decomposition

- Precompute the linear operators $P_N \circ I_M$ and $R_{M'}$ w.r.t. basis of V_N.
- Effort to evaluate $(P_N \circ I_M) \circ \mathcal{L}_{M,\mu} \circ R_{M'}$ w.r.t. this basis:

$$\mathcal{O}(MN) + \mathcal{O}(M) + \mathcal{O}(MN).$$

- Use POD-GREEDY algorithm for the construction of V_N.

Error Control

Basic a posteriori error estimate

Assume that $\mathcal{L}_\mu$ is globally Lipschitz, i.e. $\|\mathcal{L}_\mu(u) - \mathcal{L}_\mu(v)\| \leq C_\mathcal{L} \cdot \|u - v\|$ for all u, v, μ. Then:

$$\|u_\mu(t) - \tilde{u}_\mu(t)\| \leq \left\{ \|u_\mu(0) - \tilde{u}_\mu(0)\| + \int_0^t \mathcal{R}_\mu(\tilde{u}_\mu)(s) ds \right\} \cdot e^{C_\mathcal{L} \cdot t},$$

where the residual $\mathcal{R}_\mu$ is given as

$$\mathcal{R}_\mu(\tilde{u}_\mu)(t) := \partial_t \tilde{u}_\mu(t) + \mathcal{L}_\mu(\tilde{u}_\mu(t)).$$

Residual approximation

$$\begin{aligned}
&\|\mathcal{R}_\mu(\tilde{u}_\mu)(t)\| \\
&\qquad \leq \|\partial_t \tilde{u}_\mu - \mathcal{I}_M[\mathcal{L}_\mu](\tilde{u}_\mu(t))\| + \|\mathcal{I}_M(\mathcal{L}_\mu)(\tilde{u}_\mu(t)) - \mathcal{L}_\mu(\tilde{u}_\mu(t))\| \\
&\qquad \approx \|\partial_t \tilde{u}_\mu - \mathcal{I}_M[\mathcal{L}_\mu](\tilde{u}_\mu(t))\| + \|\mathcal{I}_M[\mathcal{L}_\mu](\tilde{u}_\mu(t)) - \mathcal{I}_{M+K}[\mathcal{L}_\mu](\tilde{u}_\mu(t))\|.
\end{aligned}$$

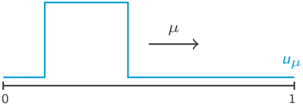
Freezing of Nonlinear Evolution Equations

Advection Dominated Problems

- Typically slow decay of Kolmogorov N-widths $d_N(\Phi(\mathcal{P}))$, but RB will only work well for rapid decay!

$$d_N(\Phi(\mathcal{P})) := \inf_{\substack{V_N \subseteq V_h \\ \dim V_N \leq N}} \sup_{\substack{u \in \Phi(\mathcal{P}) \\ t \in [0,T]}} \inf_{v \in V_N} \|u(t) - v\|.$$

- An issue, even for very simple examples:



$$\begin{aligned} &\partial_t u(t,x) + \mu \cdot \partial_x u_\mu(t,x) = 0 \\ &u_\mu(0,x) = u_0(x),\ u_\mu(0,t) = u_\mu(1,t) \\ &\mu, x, t \in [0,1] \end{aligned}$$

Here: $d_N(\Phi(\mathcal{P}) \subset L^2) \sim N^{-1/2}$ (note that Φ is not differentiable).

- **However:** Can describe solution easily by

$$u_\mu(t,x) = u_0(x - \mu \cdot t \mod 1).$$

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Nonlinear Approximation

- Write $u_\mu(t,x)$ as

$$u_\mu(t,x) = u_0(x - \mu \cdot t \mod 1) =: ((\mu \cdot t) \,.\, u_0)(x)$$

Nonlinear Approximation

- Write $u_\mu(t,x)$ as

$$u_\mu(t,x) = u_0(x - \mu \cdot t \mod 1) =: ((\mu \cdot t) \,.\, u_0)(x)$$

- **General idea:** Write $u_\mu(t,x)$ as

$$u_\mu(t,x) = g_\mu(t).v_\mu(t,x)$$

dynamics of u_μ
large variation in time

shape of u_μ
small variation in time

where $\mathcal{V}$ function space, $v_\mu(t) \in \mathcal{V}$ and $g_\mu(t)$ is element of Lie group G acting on $\mathcal{V}$.

- $v_\mu(t,x)$ should be easier to approximate than $u_\mu(t,x)$!

Method of Freezing [Beyn, Thümmler, 2004], [Rowley et. al., 2000, 2003]

- Consider Lie group G acting on $\mathcal{V}$ and evolution equation of the form:

$$\partial_t u_\mu(t) + \mathcal{L}_\mu(u_\mu(t)) = 0, \quad u_\mu(0) = u_0, \quad u_\mu(t) \in \mathcal{V}$$

- Substituting the *ansatz* $u_\mu(t) = g_\mu(t).v_\mu(t)$ leads to:

$$\partial_t v_\mu(t) + g_\mu(t)^{-1}.\mathcal{L}_\mu(g_\mu(t).v_\mu(t)) + \mathfrak{g}_\mu(t).v_\mu(t) = 0$$
$$\mathfrak{g}_\mu(t) = g_\mu(t)^{-1}\partial_t g_\mu(t).$$

Method of Freezing [Beyn, Thümmler, 2004], [Rowley et. al., 2000, 2003]

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$$\mathfrak{g}_\mu(t) = g_\mu(t)^{-1}\partial_t g_\mu(t).$$

- Have $\dim(G)$ additional degrees of freedom.
  $\rightarrow$ Add additional algebraic constraint (phase condition):

$$\Phi(v_\mu(t), \mathfrak{g}_\mu(t)) = 0.$$

- Further assume invariance of $\mathcal{L}_\mu$ under action of G:

$$h^{-1}.\,\mathcal{L}_\mu(h.w) = \mathcal{L}_\mu(w) \quad \forall h \in G, w \in \mathcal{V}.$$

Method of Freezing [Beyn, Thümmler, 2004], [Rowley et. al., 2000, 2003]

Definition (Method of Freezing)

With initial conditions $v_\mu(0) = u(0),\ g_\mu(0) = e$, solve:

$$\begin{aligned} \partial_t v_\mu(t) + \mathcal{L}_\mu(v_\mu(t)) + \mathfrak{g}_\mu(t).v_\mu(t) &= 0 \\ \Phi(v_\mu(t), \mathfrak{g}_\mu(t)) &= 0 \end{aligned}$$

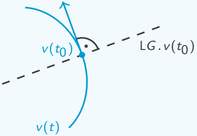
frozen PDAE

$$\mathfrak{g}_\mu(t) = g(t)_\mu^{-1} \partial_t g_\mu(t)$$

reconstruction equation

Orthogonality phase condition

$$\begin{aligned} \Phi(v, \mathfrak{g}) = 0 \ &:\Longleftrightarrow \partial_t v(t) \perp LG.v(t) \\ &\Longleftrightarrow (\mathcal{L}(v) + \mathfrak{g}.v,\ \mathfrak{h}.v) = 0 \quad \forall \mathfrak{h} \in LG \end{aligned}$$



Example: 2D-Shifts

Consider $G = \mathbb{R}^2$, L$G = \mathbb{R}^2$ acting via

$$\begin{aligned} g.u(x) &:= u(x-g), \quad x \in \mathbb{R}^2 \\ \mathfrak{g}.u &= -\mathfrak{g} \cdot \nabla u \end{aligned}$$

The Method of Freezing for 2D-shifts

Solve

$$\partial_t v_\mu(t) + \mathcal{L}_\mu(v_\mu(t)) - \mathfrak{g}_\mu(t) \cdot \nabla v_\mu(t) = 0$$
$$\left[(\partial_{x_i} v_\mu,\, \partial_{x_j} v_\mu)\right]_{i,j} \cdot \left[\mathfrak{g}_\mu\right]_j = \left[(\mathcal{L}_\mu(v_\mu),\, \partial_{x_i} v_\mu)\right]_i$$

and

$$\partial_t g_\mu(t) = \mathfrak{g}_\mu(t)$$

with initial conditions $v_\mu(0) = u(0)$, $g_\mu(0) = (0,0)^T$.

Test Problem

2D Burgers-type problem

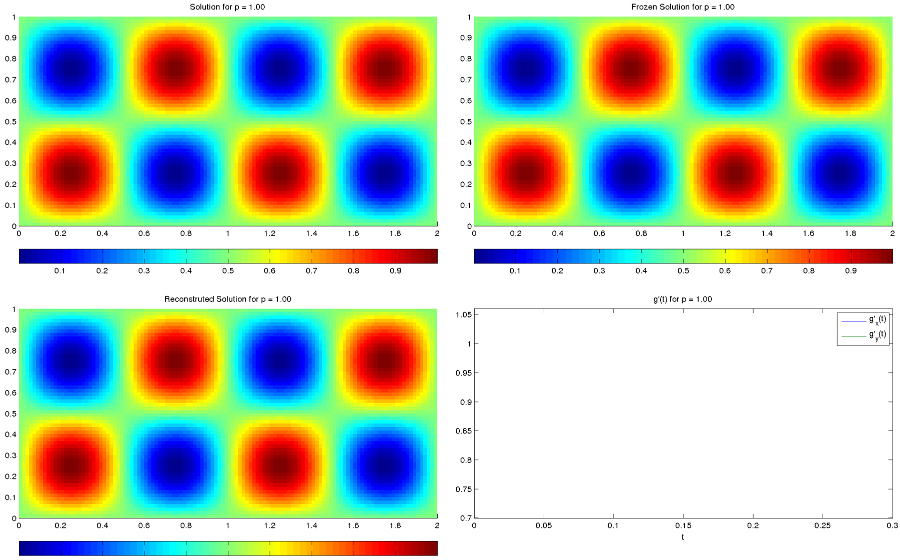
Solve on $\Omega = [0,2] \times [0,1]$:

$$\partial_t u + \nabla \cdot (\vec{v} \cdot u^\mu) = 0$$
$$u(0, x_1, x_2) = 1/2(1 + \sin(2\pi x_1)\sin(2\pi x_2))$$

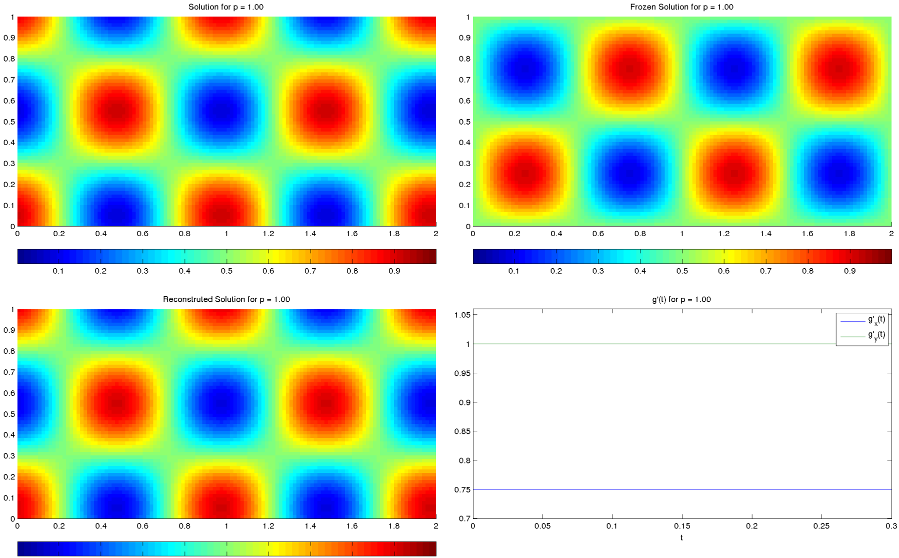
for $t \in [0, 0.3]$, $\vec{v} \in \mathbb{R}$ with periodic boundary conditions and $\mu \in \mathcal{P} = [1, 2]$.

- Finite volume (Lax-Friedrichs) space discretization on 240 x 120 grid.
- Explicit Euler time-stepping (200 time steps).
- Same problem as in [Drohmann, Haasdonk, Ohlberger, 2012].
- (The following videos are actually computed on a 120 x 60 grid.)

Frozen vs. Non-frozen Solution ($\mu = 1, \vec{v} = (0.75, 1)^T$)



Frozen vs. Non-frozen Solution $(\mu = 1, \vec{v} = (0.75, 1)^T)$



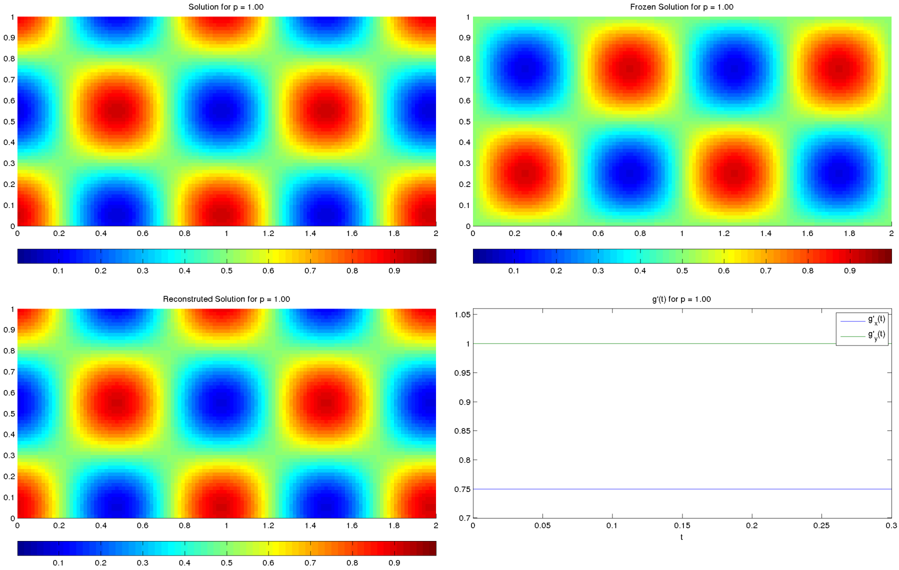
Solution for p = 1.00

Frozen Solution for p = 1.00

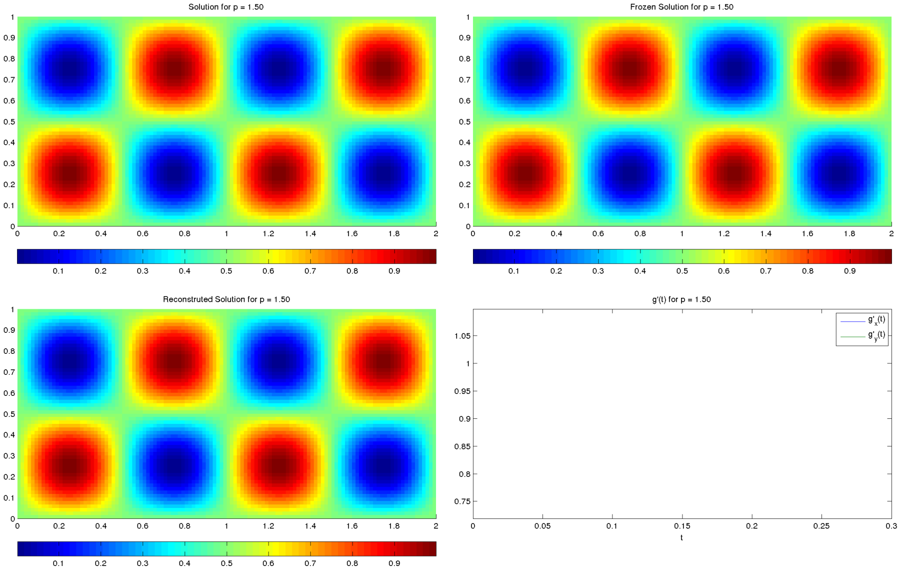
Reconstruted Solution for p = 1.00

g'(t) for p = 1.00

Frozen vs. Non-frozen Solution $(\mu = 1, \vec{v} = (0.75, 1)^T)$



Frozen vs. Non-frozen Solution $(\mu = 1.5, \vec{v} = (0.75, 1)^T)$



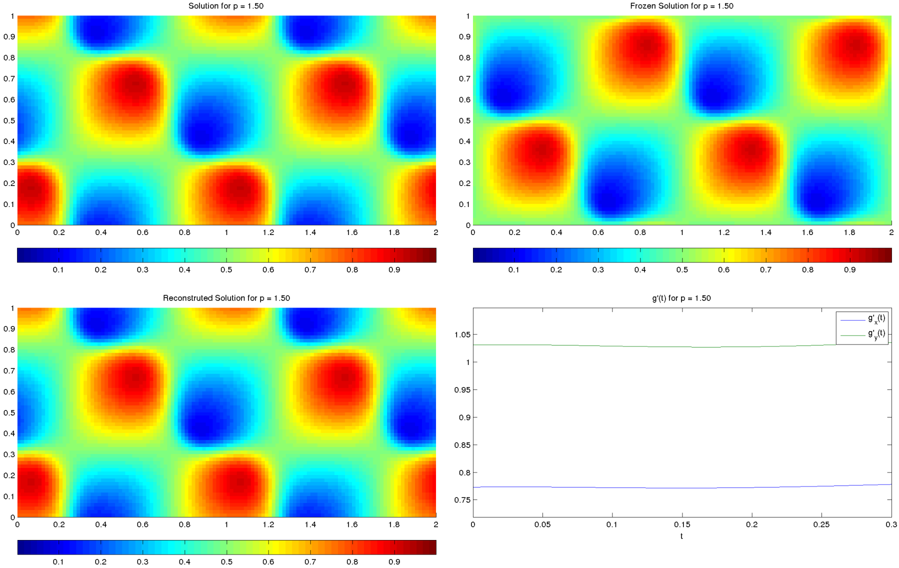
Solution for p = 1.50

Frozen Solution for p = 1.50

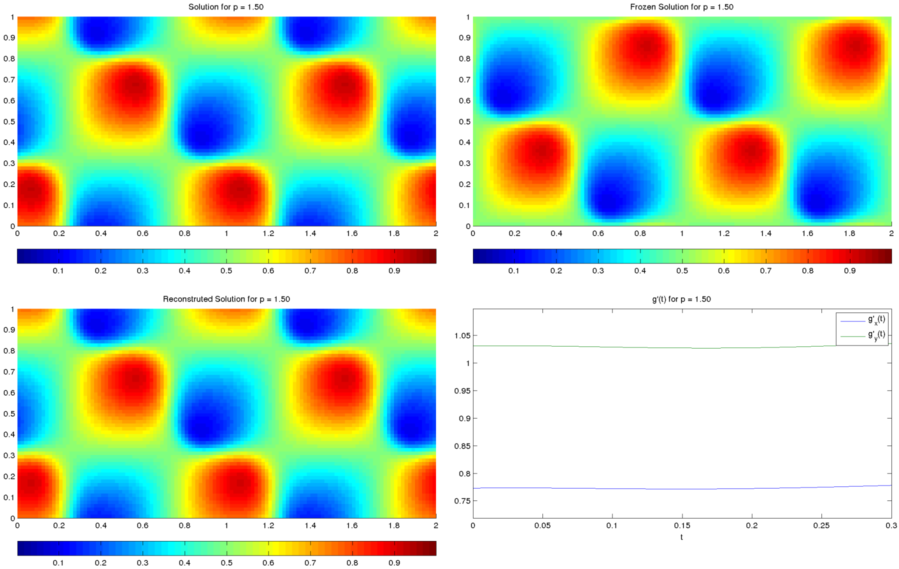
Reconstruted Solution for p = 1.50

g'(t) for p = 1.50

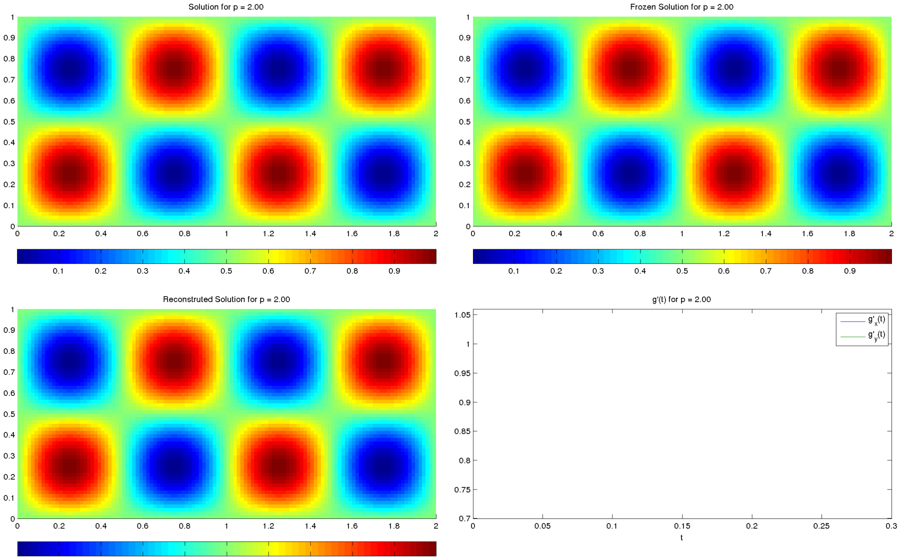
Frozen vs. Non-frozen Solution ($\mu = 1.5, \vec{v} = (0.75, 1)^T$)



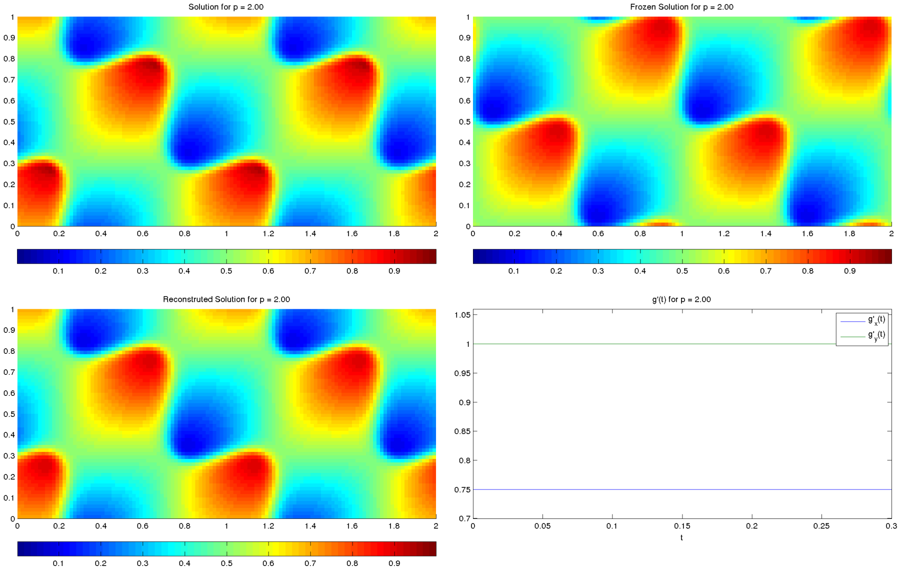
Frozen vs. Non-frozen Solution $(\mu = 1.5, \vec{v} = (0.75, 1)^T)$



Frozen vs. Non-frozen Solution $(\mu = 2, \vec{v} = (0.75, 1)^T)$



Frozen vs. Non-frozen Solution $(\mu = 2, \vec{v} = (0.75, 1)^T)$



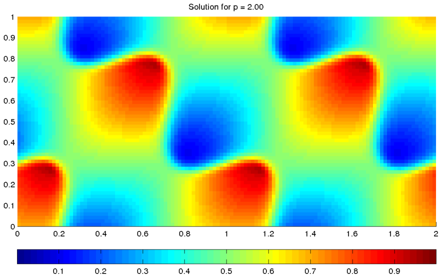
Solution for p = 2.00

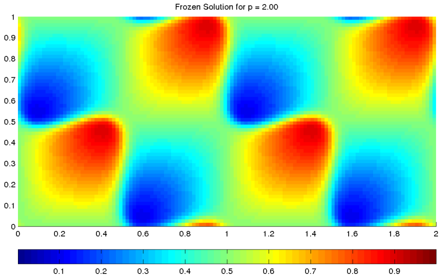
Frozen Solution for p = 2.00

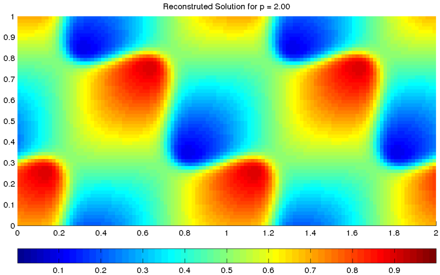
Reconstruted Solution for p = 2.00

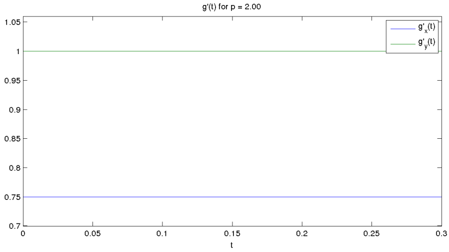
g'(t) for p = 2.00

Frozen vs. Non-frozen Solution $(\mu = 2, \vec{v} = (0.75, 1)^T)$

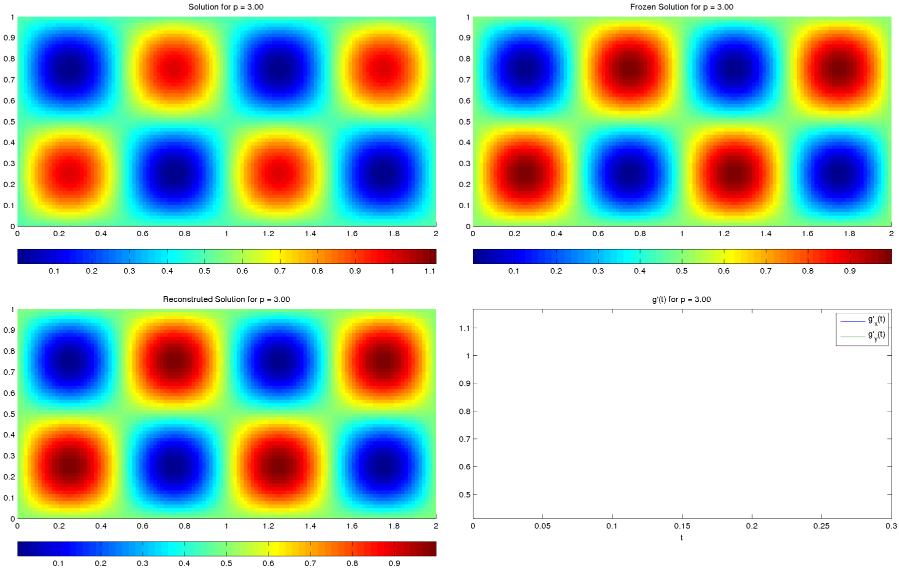

Solution for p = 2.00


Frozen Solution for p = 2.00

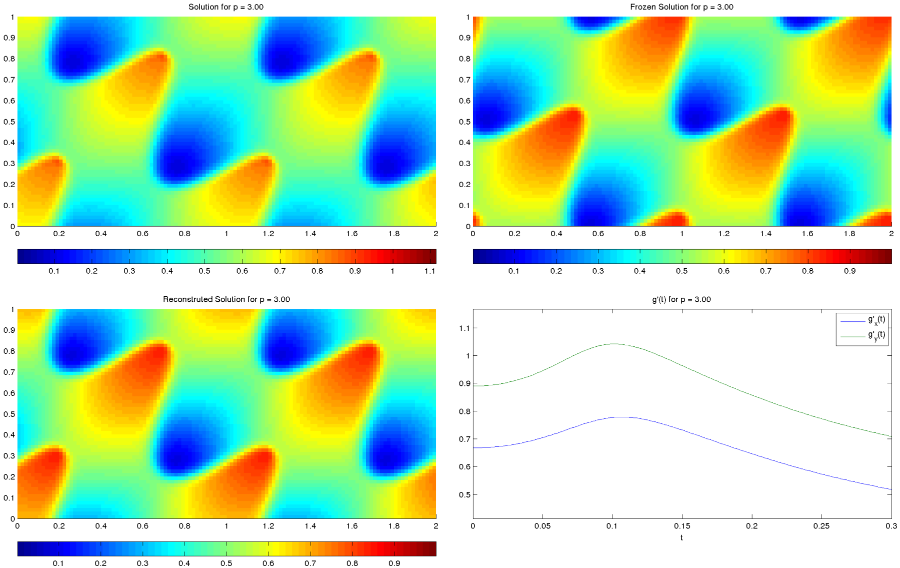

Reconstruted Solution for p = 2.00


g'(t) for p = 2.00

Frozen vs. Non-frozen Solution $(\mu = 3, \vec{v} = (0.75, 1)^T)$



Frozen vs. Non-frozen Solution $(\mu = 3, \vec{v} = (0.75, 1)^T)$



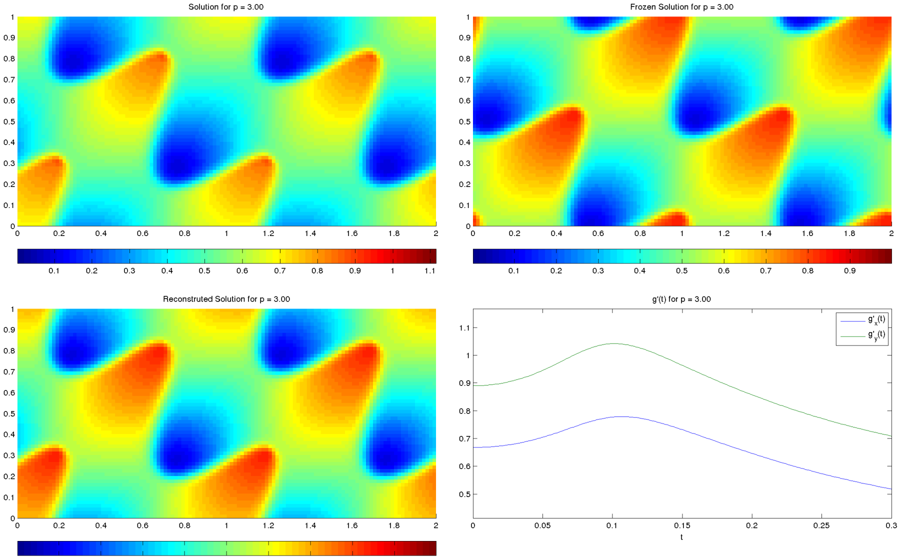
Solution for p = 3.00

Frozen Solution for p = 3.00

Reconstruted Solution for p = 3.00

g'(t) for p = 3.00

Frozen vs. Non-frozen Solution $(\mu = 3, \vec{v} = (0.75, 1)^T)$



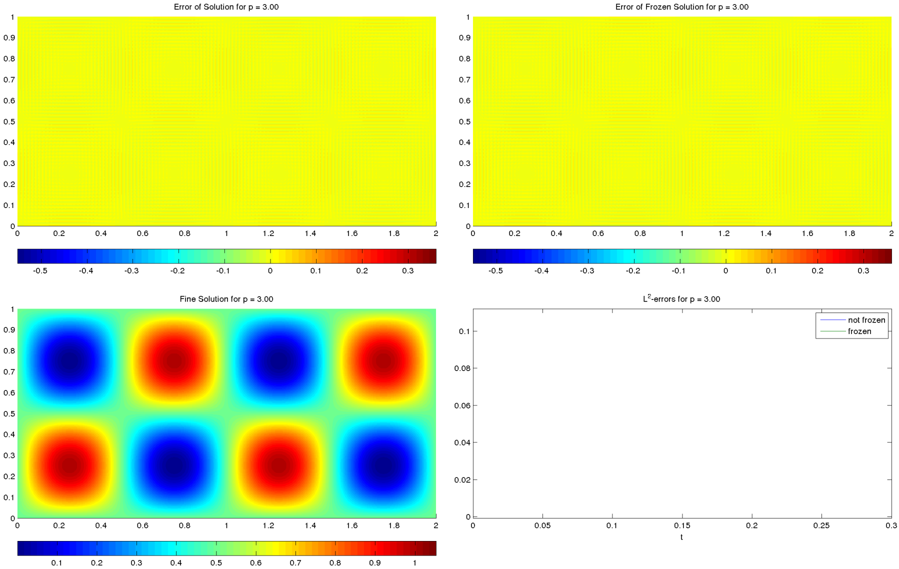
Solution for p = 3.00

Frozen Solution for p = 3.00

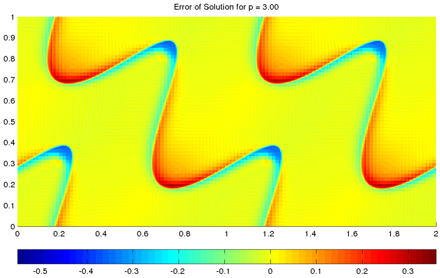
Reconstruted Solution for p = 3.00

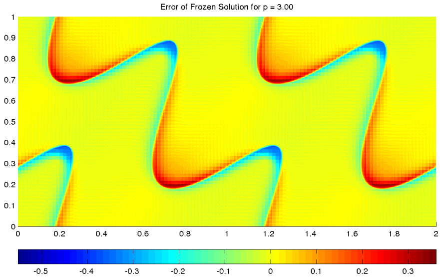
g'(t) for p = 3.00

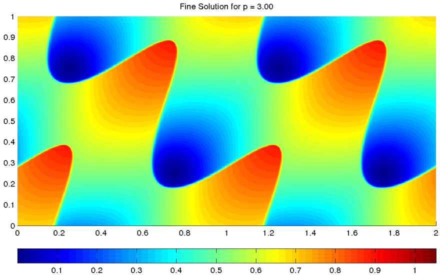
Errors of Frozen and Non-frozen Solution ($\mu = 3, \vec{v} = (0.75, 1)^T$)

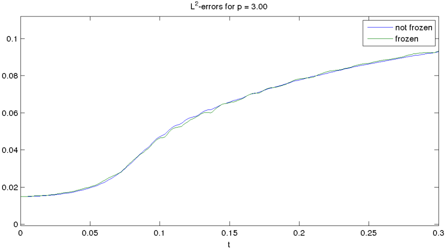


Errors of Frozen and Non-frozen Solution $(\mu = 3, \vec{v} = (0.75, 1)^T)$

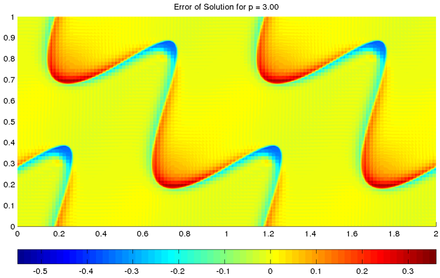


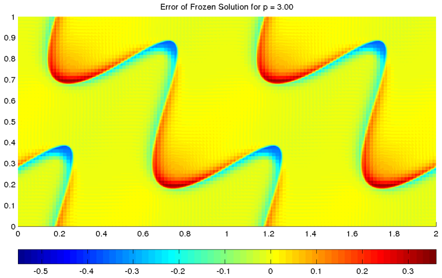


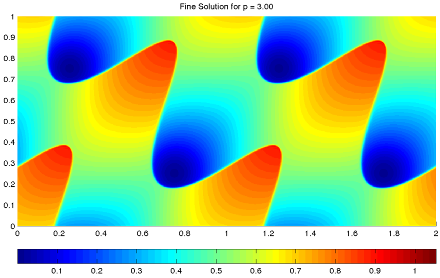


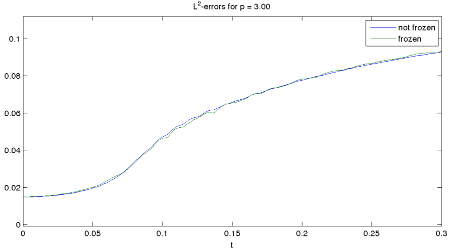


Errors of Frozen and Non-frozen Solution ($\mu = 3, \vec{v} = (0.75, 1)^T$)


Error of Solution for p = 3.00


Error of Frozen Solution for p = 3.00


Fine Solution for p = 3.00


L^2-errors for p = 3.00

FrozenRB Approximation of Nonlinear Evolution Equations

Combining RB with the Method of Freezing

FrozenRB-Scheme for 2D-shifts [Ohlberger, R, 2013]

Solve

$$\partial_t v_{\mu(t),N} + P_N \circ \mathcal{I}_M[\mathcal{L}_\mu](v_{\mu,N}(t)) - \mathfrak{g}_{\mu(t),N} \cdot (P_N \circ \nabla)(v_{\mu,N}(t)) = 0$$

$$\left[(\partial_{x_i} v_{\mu,N},\, \partial_{x_j} v_{\mu,N})\right]_{i,j} \cdot \left[\mathfrak{g}_{\mu,N}\right]_j = \left[(\mathcal{I}_M[\mathcal{L}_\mu](v_\mu),\, \partial_{x_i} v_{\mu,N})\right]_i$$

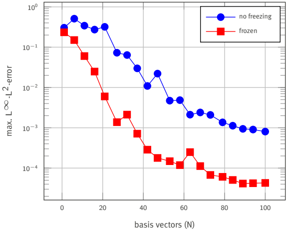
and

$$\partial_t g_\mu(t) = \mathfrak{g}_\mu(t)$$

with initial conditions $v_\mu(0) = u(0)$, $g_\mu(0) = (0,0)^T$.

- EI-GREEDY, POD-GREEDY algorithms for basis generation.
- Full offline/online decomposition.
- No additional evaluations of nonlinearity (small overhead).

Results for the Burgers Problem ($\vec{v} = (1,1)^T$)

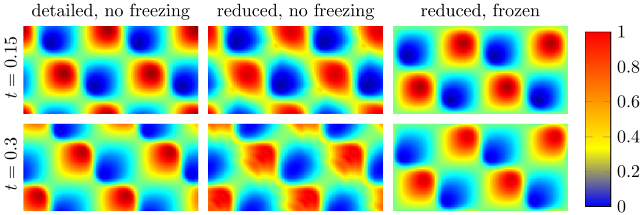


Left:

- 1.9 · N interpolation points.
- Test set: 100 random μ.

Bottom:

- dim $V_N = 20$, 38 interpolation points.
- $\mu = 1.5$.



Thank you for your attention!

Ohlberger, R, *Nonlinear reduced basis approximation of parameterized evolution equations via the method of freezing*, C. R. Math. Acad. Sci. Paris, 351 (2013).

Ohlberger, R, *Reduced Basis Methods: Success, Limitations and Future Challenges*, Proceedings of ALGORITMY 2016.

Get the code:
`http://stephanrave.de/code/frozenrb.tar.gz`

pyMOR – Model Order Reduction with Python
`http://www.pymor.org/`
arXiv:1506.07094