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# Adaptive Localized Reduced Basis Methods

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MM Mathematics Münster Cluster of Excellence

# Outline

1. Reduced Basis Methods for Elliptic Problems
2. Localized Reduced Basis Methods for Elliptic Problems
3. Localized Reduced Basis Domain Decomposition Methods for Elliptic Problems
   (and their convergence)

# Reduced Basis Methods for Elliptic Problems

# Reduced Basis Methods

## Parametric linear elliptic problem (full order model)

For given parameter $\mu \in \mathcal{P}$, find $u_h(\mu) \in V_h$ s.t.

$$\begin{aligned} a(u_h(\mu), v_h; \mu) &= f(v_h) \qquad\qquad \forall v_h \in V_h \\ y_h(\mu) &= g(u_h(\mu)) \end{aligned}$$

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## Parametric linear elliptic problem (reduced order model)

For given $V_N \subset V_h$, let $u_N(\mu) \in V_N$ be given by Galerkin proj. onto $V_N$, i.e.

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# RB Methods – Computing $V_N$

## Weak greedy basis generation

**function** $\textsc{Weak-Greedy}(\mathcal{S}_{train} \subset \mathcal{P}, \varepsilon)$
    $V_N \leftarrow \{0\}$
    **while** $\max_{\mu \in \mathcal{S}_{train}} \textsc{Err-Est}(\textsc{ROM-Solve}(\mu), \mu) > \varepsilon$ **do**
        $\mu^* \leftarrow \operatorname{arg-max}_{\mu \in \mathcal{S}_{train}} \textsc{Err-Est}(\textsc{ROM-Solve}(\mu), \mu)$
        $V_N \leftarrow \operatorname{span}(V_N \cup \{\textsc{FOM-Solve}(\mu^*)\})$
    **end while**
    **return** $V_N$
**end function**

## Resiudal-based error estimation

$$\textsc{Err-Est}(u_N(\mu), \mu) := \frac{1}{C(\mu)} \| f - a(u_N(\mu), \cdot\,; \mu) \|_{V_h'}$$

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- Use dual weighted residual approach for improved convergence w.r.t to output $y_N(\mu)$.

# RB Methods – Online Efficiency

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## Affine decomposition

Assume that $a(\cdot, \cdot; \mu)$ can be written as

$$a(u, v; \mu) = \sum_{q=1}^{Q} \theta_q(\mu) a_q(u, v).$$

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## Offline/Online splitting

By pre-computing

$$a_q(\varphi_i, \varphi_j),\ f(\varphi_i),\ g(\varphi_i)$$

for a reduced basis $\varphi_1, \ldots, \varphi_N$ of $V_N$, solving ROM becomes independent of dim $V_h$.

# Example: RB Approximation of Li-Ion Battery Models

ulm university universität ulm

DLR Institute of Technical Thermodynamics

MULTIBAT

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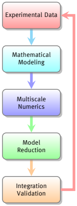
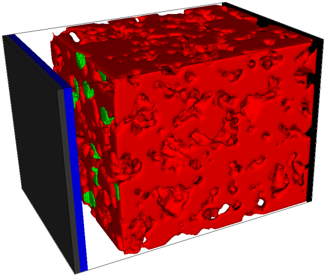


**MULTIBAT:** Gain understanding of degradation processes in rechargeable Li-Ion Batteries through mathematical modeling and simulation at the pore scale.

**FOM:**

- 2.920.000 DOFs
- Simulation time: ≈ 15.5h

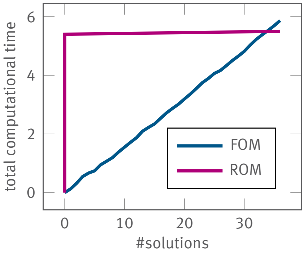
**ROM:**

- Snapshots: 3
- dim $V_N = 245$
- Rel. err.: $< 4.5 \cdot 10^{-3}$
- Reduction time: ≈ 14h
- Simulation time: ≈ 8m
- Speedup: 120

# Localized Reduced Basis Methods for Elliptic Problems

# Caveats

- Offline time too large in not-so-many-query scenarios?
- $\mathcal{P}$ too large?



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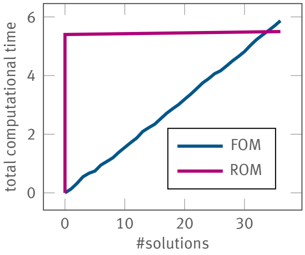
**Merge offline and online phase:**

- Use ERR-EST to detect when ROM is inaccurate.
- Adaptively enrich $V_N$ with FOM snapshot.



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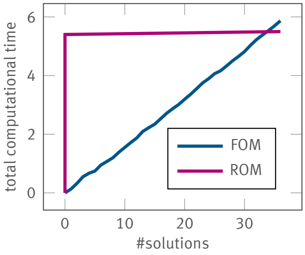

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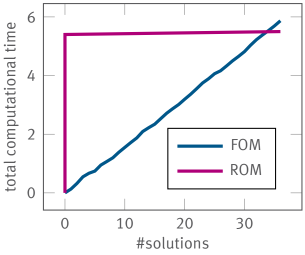


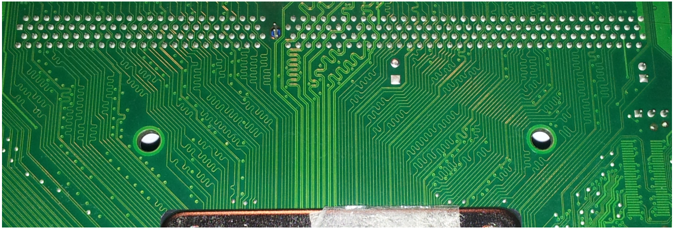

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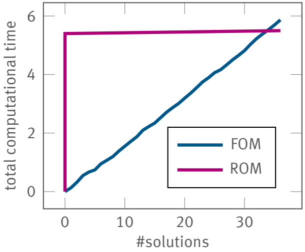


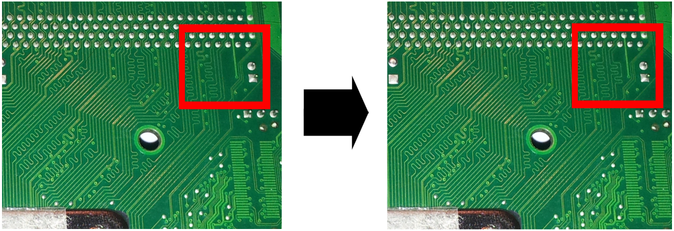

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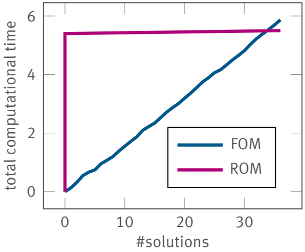
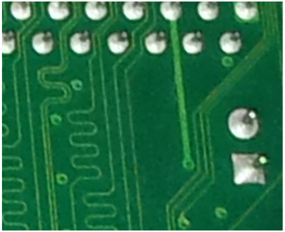
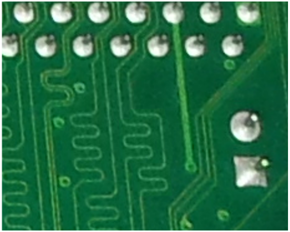


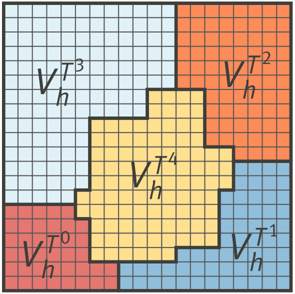





# Localized RB Methods for Elliptic Problems

Idea of the **LRBMS**: given a finely-resolved grid $\tau_h$ [ALBRECHT ET AL., 2012]

- decompose approximation space into *local* spaces $V_h = \bigoplus_{T \in \mathcal{T}_H} V_h^T$
- independent local discretizations and approximation spaces (CG or DG) associated with subdomains $T \in \mathcal{T}_H$ and global SWIPDG coupling [ERN, STEPHANSEN, ZUNINO, 2009]



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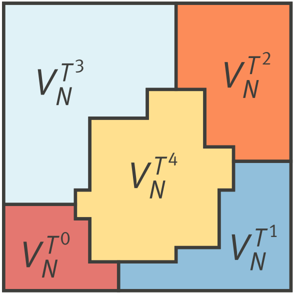
- build local reduced spaces $V_N^T \subset V_h^T$ using local computations only
- reduced *broken* space $V_N := \oplus_{T \in \mathcal{T}_H} V_N^T$



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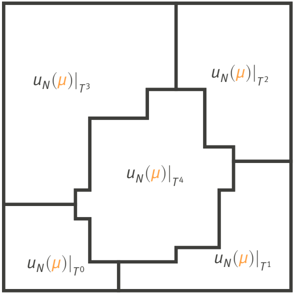
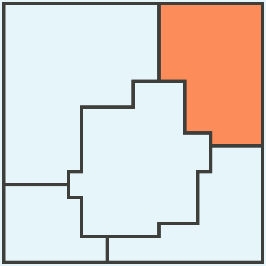
- reduced *broken* space $V_N := \oplus_{T \in \mathcal{T}_H} V_N^T$
- larger $V_N$, but sparse ROM system matrices
- expect $\max_{T \in \tau_h} \dim V_N^T < \dim V_{N,global}$ when influence of $\mu$ is localized.



# Training algorithm (adapted from [Buhr, Engwer, Ohlberger, R, 2017])

## Offline phase — for all $T \in \mathcal{T}_H$



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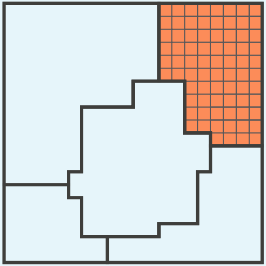

- For every $\mu \in \mathcal{S}_{train} \subset \mathcal{P}$:
  - Solve training problem

$$a(\varphi_{h,0}(\mu), v_h; \mu) = f(v_h) \qquad \text{in } T$$

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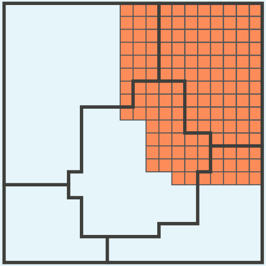

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    for $K$ random Dirichlet data functions $g_k$ on $\partial T^\delta$.

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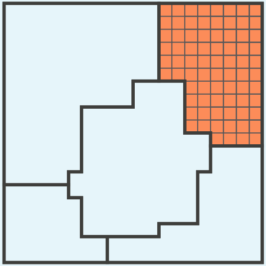

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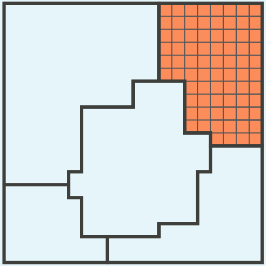

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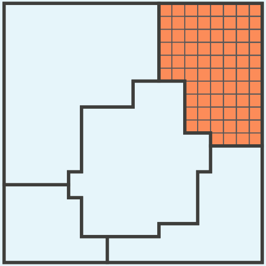

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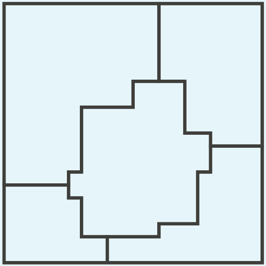
- Boundary training $\cong$ truncated randomized SVD of transfer operator [Buhr, Smetana, 2018].
- Optimal approximation space for unknown boundary data on oversampling boundary.

# Adaptive Enrichment of $V_N$

## Online phase — for some $\mu \in \mathcal{P}$



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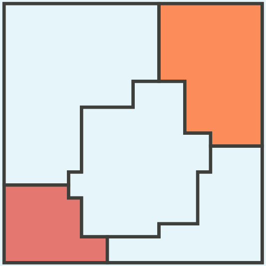

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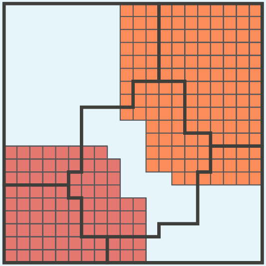

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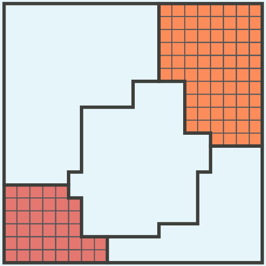

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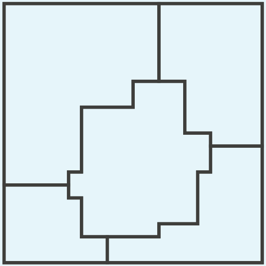

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  - compute updated $u_N(\mu)$ and $\text{Err-Est}(u_N(\mu), \mu)$

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- iterate until $\textsc{Err-Est}(u_N(\mu), \mu) \leq \Delta$, return $u_N(\mu)$

$\dim V_h(T^\delta)$

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&\leq \sup_{v_h \in V_h} \inf_{\substack{v_k \in W_k \\ v_1 + \ldots + v_K = v_h}} \sum_{k=1}^{K} \frac{|||\mathcal{R}_N(\cdot\,; \mu)|||_{W_k^{-1}, \mu} \cdot |||v_k|||_\mu}{|||v_h|||_\mu} \\
&\leq \left( \sum_{k=1}^{K} |||\mathcal{R}_N(\cdot\,; \mu)|||^2_{W_k^{-1}, \mu} \right)^{1/2} \cdot \underbrace{\sup_{v_h \in V_h} \inf_{\substack{v_k \in W_k \\ v_1 + \ldots + v_K = v_h}} \frac{\left( \sum_{k=1}^{K} |||v_k|||_\mu^{\,2} \right)^{1/2}}{|||v_h|||_\mu}}_{c_{pu,\mu}}
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**Theorem** [Buhr, Engwer, Ohlberger, R, 2017]

$$\textsc{Err-Est}(u_N(\mu), \mu) := c_{pu,V_N,\mu} \cdot \left( \sum_{k=1}^{K} \|\mathcal{R}_N(\cdot\,;\mu)\|^2_{W_k^{-1},\mu} \right)^{1/2}$$

is a locally efficient upper bound for the model reduction error.
Given shape regular $\mathcal{T}_H$, the constant $c_{pu,V_N,\mu}$ is independent of $h$ and $H := \min_{T\in\mathcal{T}_H} \operatorname{diam} T$.

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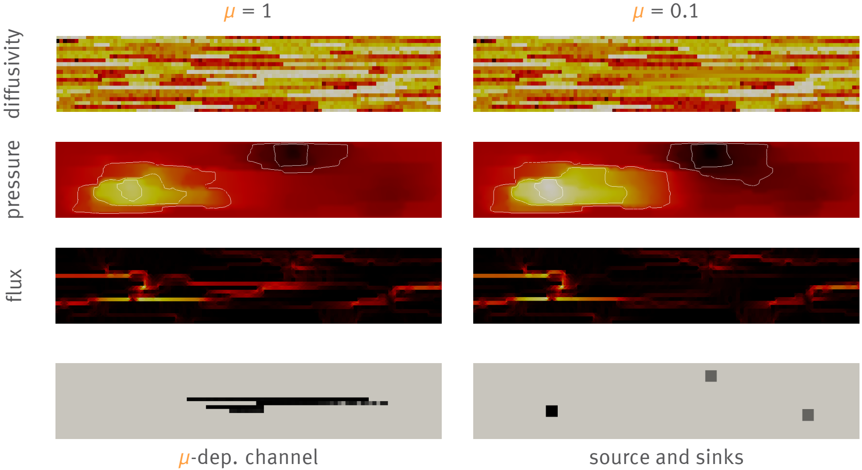
- To estimate $\|\mathcal{R}_N(\,\cdot\,;\mu)\|_{W_k^{-1},\mu}$ use, e.g.,

$$\|\mathcal{R}_N(\,\cdot\,;\mu)\|_{W_k^{-1},\mu} \leq 1/C(\mu)\|\mathcal{R}_N(\,\cdot\,;\mu)\|_{W_k^{-1}}$$

  with appropriate $\|\cdot\|$.

# LRBMS with online enrichment: Example SPE10 [Ohlberger, Schinder, 2015]



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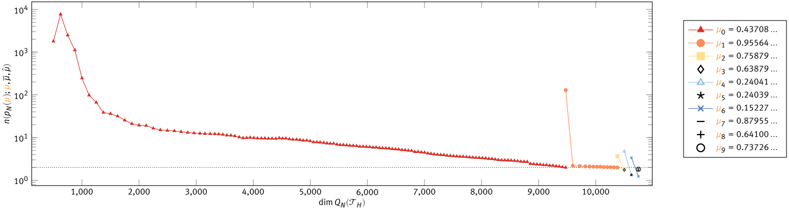
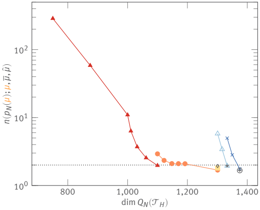


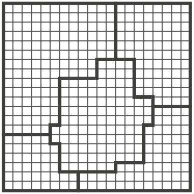

Convergence history of LRBMS with initially empty $V_N$




LRBMS initialized with 2 solution snapshots

Distribution of local basis size after online enrichment.

# Some Related Approaches



- Reduced basis element Method
  [Maday, Ronquist, 2002]

- Generalized multiscale finite element method
  [Efendiev, Galvis, Hou, 2013]

- Port-reduced static condensation Reduced basis element Method
  [Eftang, Patera, 2013]

- Reduced basis hybrid Method
  [Iapichino, Quarteroni, Rozza, Volkwein, 2014]

- ArbiLoMod, a Simulation Technique Designed for Arbitrary Local Modifications
  [Buhr, Engwer, Ohlberger, R, 2017]

# Localized Reduced Basis Domain Decomposition Methods for Elliptic Problems

# Questions

- How fast does enrichment converge?
- How to balance the effort for training and enrichment?
- Which training method to choose?

# Connections with Domain Decomposition Methods

- Local enrichment function $\varphi_h(\mu)|_T$

$$\begin{aligned} a(\varphi_h(\mu), v_h; \mu) &= f(v_h) && \text{in } T^\delta \\ \varphi_h(\mu) &= u_N(\mu) && \text{on } \partial T^\delta \end{aligned}$$

corresponds to subdomain solution in Restricted Additive Schwarz (RAS) method.

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**enrichment + Galerkin projection onto $V_N$**

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- Moreover:

**offline training of $V_N$**

$\cong$

**construction of multiscale coarse space**

e.g. DtN [Nataf et al., 2011], GenEO [Spillane et al., 2014], SHEM [Gander, Loneland, Rahman, 2015]

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3. Build local RB spaces $V_N^T$ from AS+CG solutions.
4. Use localized estimator to only enrich $V_N$ when needed:

$$\textsc{Err-Est}(u_N(\mu), \mu) := c_{pu, V_N, \mu} \cdot \left( \sum_{T \in \mathcal{T}_H} \| \mathcal{R}_N(\cdot\,; \mu) \|^2_{(V_h^T)^{-1}, \mu} \right)^{1/2}$$

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**Note: $c_{pu,V_N,\mu}$ is the stability constant of the space decomposition appearing in the abstract Schwarz framework. A good coarse space will yield an efficient error bound *and* convergence of enrichment in a fixed number of iterations.**

# Thank you for your attention!

Ohlberger, Schindler, *Error Control for the Localized Reduced Basis Multiscale Method with Adaptive On-Line Enrichment*, SISC, 37(6) (2015).

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