Westfälische Wilhelms-Universität Münster

Freezing of Evolution Equations for Improved Reduced Basis Approximation

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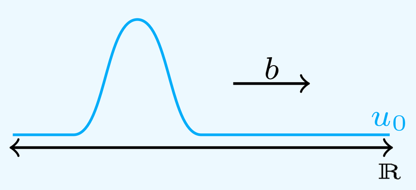
The Problem

Reduced basis methods can effectively approximate the solution manifolds of parameterized evolution problems by low-dimensional linear spaces, enabling fast online evaluation of the solution for arbitrary parameter values.

For convection dominated problems however, standard methods fail to adequately handle the variation of the solution manifold over time that is introduced by the convection.

Example

Consider the linear transport of a bump function on the real line:



$$\partial_t u(t,x) + b \cdot \partial_x u(t,x) = 0$$
$$u(0,x) = u_0(x)$$
$$x \in \mathbb{R}, t \in [0, T_{max}], u(t) \in V$$

- To approximate $u(t)$ by a linear subspace of V, we need $\sim T_{max}$ basis functions.
- However, we can describe $u(t)$ easily as $$u(t,x) = u_0(x - bt)$$ using only one basis function which is transformed by spatial shifts.

We present the method of freezing as a new ingredient for reduced basis schemes to approach this problem:
The linear reduced basis space is enlarged by allowing nonlinear transformations of its elements via the action of an arbitrary Lie group. Phase conditions choose appropriate transformations dynamically during time stepping.

Method of Freezing

Setting. Consider an evolution equation

$$\partial_t u(t) = \mathcal{L}(u(t)) \qquad (1)$$

with $u(t) \in V$ for some function space V.
Let G be a **Lie group** of transformations acting on V.

Basic Idea. Choose $v(t) \in V$, $g(t) \in G$ with

$$u(t) = g(t).v(t)$$

dynamics of u – large variation in time (→ $g(t)$); shape of u – small variation in time (→ $v(t)$)

substitute into (1) | assume action of G is smooth enough

$$\partial_t g(t).v(t) + g(t).\partial_t v(t) = \mathcal{L}(g(t).v(t))$$

multiply by $g(t)^{-1}$ | assume action is **invariant** i.e. $g^{-1}\mathcal{L}(g.v) = \mathcal{L}(v)$

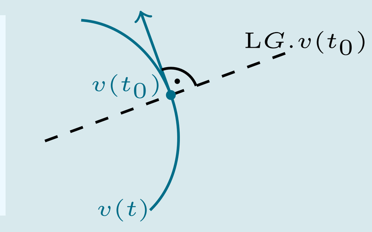
$$\partial_t v(t) = \mathcal{L}(v(t)) - \eta(t).v(t)$$
$$\eta(t) = g^{-1}\partial_t g(t).$$

$\eta(t) \in \mathrm{LG}$ (Lie algebra of G)
action of LG = differential of action of G

Phase Conditions. Determine $\eta(t)$ by adding an algebraic 'phase condition' which forces v to have minimal change.

A possible choice if V is a Hilbert space is to require the change of v to be orthogonal to the action of LG:

$$\Big(\mathcal{L}(v(t)) - \eta(t).v(t)\,,\,\xi.v(t)\Big) = 0 \quad \text{for all } \xi \in \mathrm{LG}$$



Denote such a phase condition by $\Phi(v(t), \eta(t)) = 0$.

Method of freezing. Instead of solving (1), solve the frozen partial differential algebraic equation **(PDAE)**

$$\partial_t v(t) = \mathcal{L}(v(t)) - \eta(t).v(t)$$
$$\Phi(v(t), \eta(t)) = 0$$

and the **reconstruction equation**

$$\partial_t g(t) = g(t)\eta(t).$$

The method of freezing was developed independently by Rowley et al. [1] and Beyn and Thümmler [2] for stability analysis of relative equilibria (e.g. travelling waves). See also [3].

Reduced Basis Methods

Problem

Quickly solve the time-dependent parameterized partial differential equation **(P²DE)**

$$\partial_t u(t;\mu) = \mathcal{L}(u(t;\mu);\mu)$$

$u(t;\mu) \in V, t \in [0, T_{max}]$ with initial data $u(0;\mu) = u_0(\mu)$ for differing parameters $\mu \in \mathcal{P} \subseteq \mathbb{R}^k$.

Assumption

There is an (expensive) discrete scheme, e.g.

$$dt^{-1}(u_h^{n+1}(\mu) - u_h^n(\mu)) = L_h(u_h^n(\mu);\mu)$$
$$u_h^0(\mu) = u_{h,0}(\mu),$$

providing detailed solutions in a high-dimensional discrete space V_h.

RB-Scheme

Solve

$$dt^{-1}(u_N^{n+1}(\mu) - u_N^n(\mu)) = L_N(u_N^n(\mu);\mu)$$
$$u_N^0(\mu) = u_{N,0}(\mu)$$

with solutions in a low-dimensional reduced basis space $V_N \subseteq V_h$. L_N is a projection of L_h onto V_N.

V_N is precomputed in an offline-phase by approximating a finite snapshot set $\{u_h^n(\mu_i)\}$. If $\mathcal{L}$ is nonlinear, use empirical operator interpolation [4] of L_N to achieve fast online evaluation of L_N.

Freezing of the two-dimensional Burgers Equation

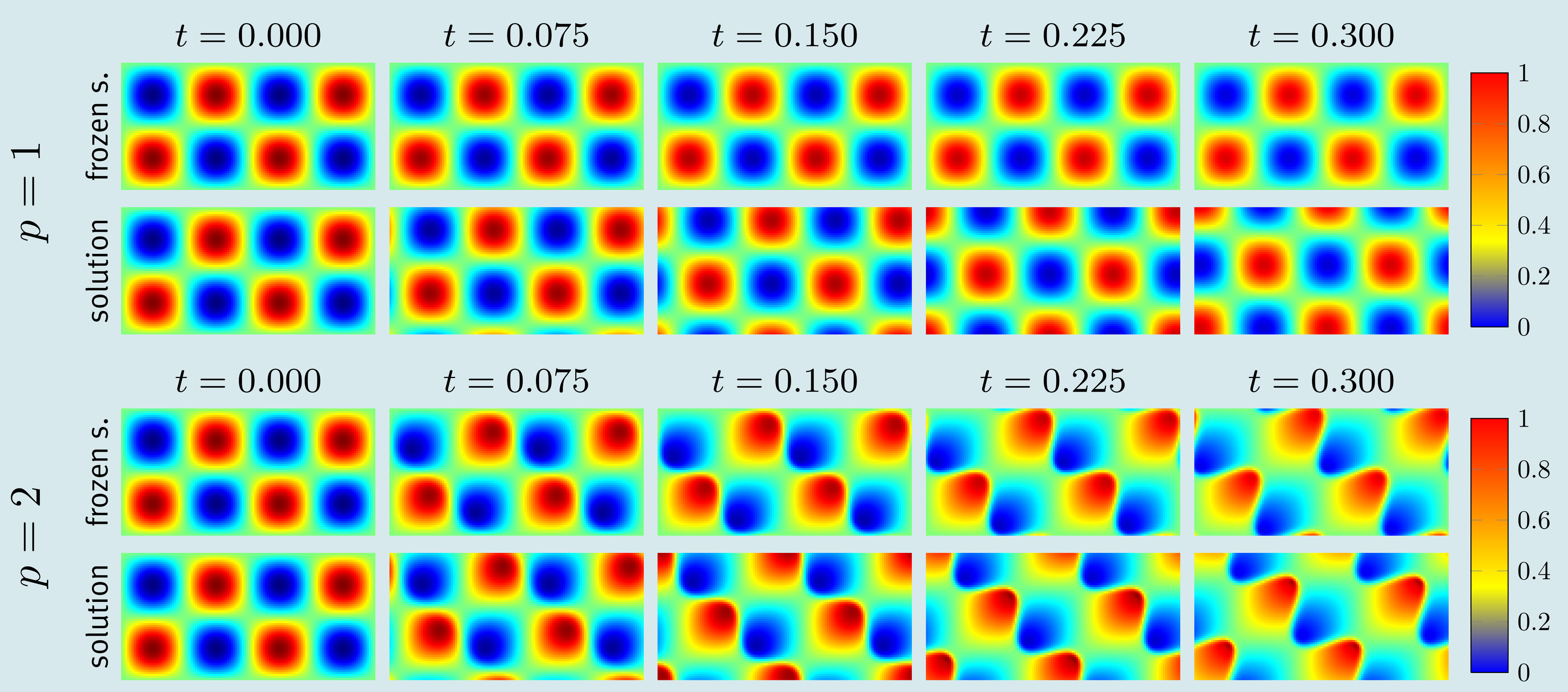
Problem

$$\partial_t u(t,x,y) = -b \cdot \nabla u(t,x,y)^p, \quad b = (1,1)^T$$
$$u(0,x,y) = 0.5 \cdot (1 + \sin(2\pi x)\sin(2\pi y))$$

domain: $[0,2] \times [0,1]$, periodic boundary conditions

Frozen PDAE ($\mathbb{R}^2$ acting by shifts of the domain)

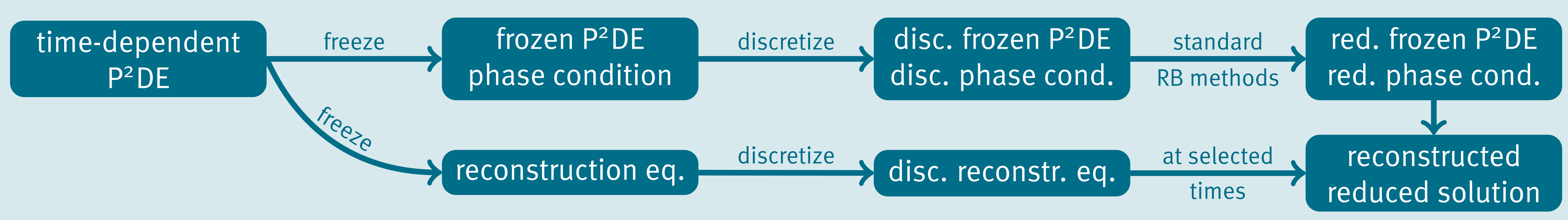
$$\partial_t v = -b \cdot \nabla v^p + \eta_x \partial_x v + \eta_y \partial_y v$$
$$\begin{bmatrix} (\partial_x v, \partial_x v) & (\partial_y v, \partial_x v) \\ (\partial_x v, \partial_y v) & (\partial_y v, \partial_y v) \end{bmatrix} \cdot \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix} = \begin{bmatrix} (b \cdot \nabla v^p, \partial_x v) \\ (b \cdot \nabla v^p, \partial_y v) \end{bmatrix}$$



Discretization

- finite volumes
- Lax-Friedrichs flux
- 120×60 cells
- time-stepping ($u_h^n \rightsquigarrow u_h^{n+1}$):
  1. evaluate $L_h(u_h^n)$
  2. solve phase condition for η_h^n
  3. compute u_h^{n+1} with forward Euler

Frozen-RB-Scheme



Application of the Frozen-RB-Scheme to the Burgers Problem

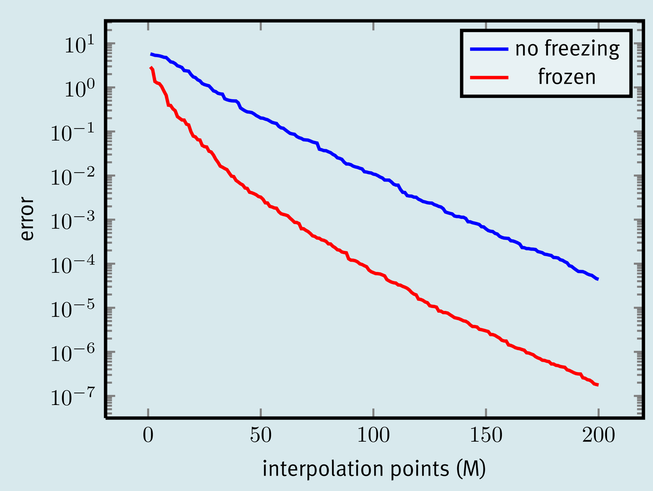


Figure 1: Error of empirical operator interpolation

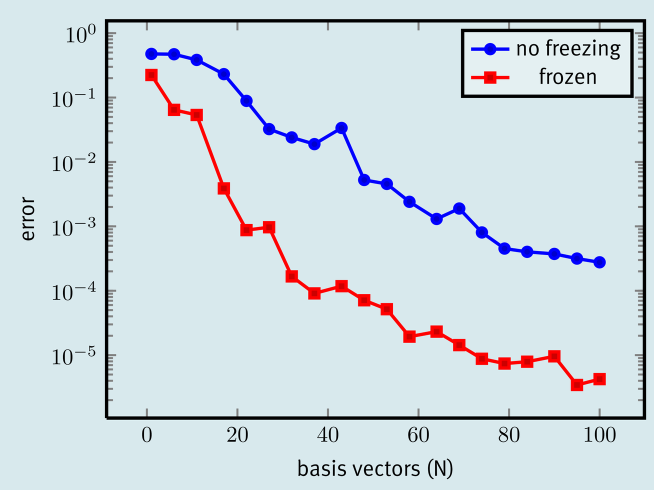


Figure 2: Error of RB-approximation

- parameter space: $p \in [1,2]$
- basis generation with greedy basis extension and proper orthogonal decomposition of trajectories
- empirical operator interpolation of $-b \cdot \nabla v^p$ and $-b \cdot \nabla v^p - \eta.v$
- $M/N = \begin{cases} 1.5 & \text{non-frozen scheme} \\ 1.6 & \text{frozen scheme} \end{cases}$
- same setting as in [4]

Figure 1 shows the maximal L^2-error of the empirical operator interpolation for the space operators on the snapshot set that is used to determine the interpolation points. Figure 2 shows the maximal L^2-approximation error of the fully reduced frozen and non-frozen schemes. The error is calculated by comparing the reduced solution to the detailed solution for randomly chosen parameter values.

Our results demonstrate that the Frozen-RB-scheme greatly succeeds in reducing the temporal variation of the problem's solution manifold. Both the error of the empirical operator interpolation as well as the approximation error for the reduced scheme are by orders of magnitude lower given the same amount of interpolation points / basis vectors.

References

[1] C. W. Rowley, I. G. Kevrekidis, J. E. Marsden, and K. Lust, *Reduction and reconstruction for self-similar dynamical systems*. Nonlinearity **16** (2003), no. 4, 1257–1275.

[2] W. J. Beyn and V. Thümmler, *Freezing solutions of equivariant evolution equations*. SIAM J. Appl. Dyn. Syst. **3** (2004), no. 2, 85–116.

[3] J. Rottmann-Matthes, *Computation and stability of patterns in hyperbolic-parabolic systems* (PhD dissertation, Universität Bielefeld). Shaker Verlag, Aachen, 2010.

[4] M. Drohmann, B. Haasdonk, and M. Ohlberger, *Reduced basis approximation for nonlinear parametrized evolution equations based on empirical operator interpolation*. SIAM J. Sci. Comput. **34** (2012), no. 2, A937–A969.

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