SUPPLEMENTARY NOTES

Teresa Esposito

Let $x \in \mathbb{R}^n$ and $r > 0$. Henceforth we denote by $B_r(x)$ the open ball of R^n with center x and radius r,

$$B_r(x) = \{y \in \mathbb{R}^n : \; |y - x| < r\}, \tag{0.1}$$

and by $\partial B_r(x)$ the boundary of $B_r(x)$,

$$\partial B_r(x) = \{y \in \mathbb{R}^n : \; |y - x| = r\}. \tag{0.2}$$

If its center is the origin, the ball of radius r is denoted by B_r and, similarly, its spheric surface by ∂B_r.

1. Coarea formula and polar coordinates

The following integration formula holds.

Theorem 1.1. *Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then, for each point $x_0 \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \left(\int_{\partial B_r(x_0)} f(x)\, dS(x) \right) dr, \tag{0.3}$$

where S denotes the surface measure on the boundary of $B_r(x_0)$.

Theorem 1.1 can be proved passing in polar coordinates in $\mathbb{R}^n$. Observe that the above theorem is a particular case of the following result.

Theorem 1.2 (Coarea formula)**.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and assume that for a.e. $r \in \mathbb{R}$ the level set $\{x \in \mathbb{R}^n : \; u(x) = r\}$ is a smooth, $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$. Suppose also $f : \mathbb{R}^n \to \mathbb{R}$ to be continuous und summable. Then*

$$\int_{\mathbb{R}^n} f(x)\, |\nabla u(x)|\, dx = \int_{-\infty}^{+\infty} \left(\int_{\{u=r\}} f(x)\, dS(x) \right) dr. \tag{0.4}$$

The *Coarea Formula* is a kind of "curvilinear" version of Fubini's Theorem and allows to convert n-dimensional integrals into integrals over the level surfaces of a suitable function.

Remark 1.3. *Theorem 1.1 follows from Theorem 1.2 by taking $u(x) = |x - x_0|$.*

2. Volume of the ball and measure of the spheric surface

In order to compute the volume $|B_r(x)|$ of the ball (0.1) and the measure $S(\partial B_r(x))$ of the spheric surface (0.2), let us introduce the so-called *Gamma function.* Let $t > 0$ and set

$$\Gamma(t) := \int_0^{+\infty} e^{-x} x^{t-1}\, dx. \tag{0.5}$$

Let us first check that the definition of the *Gamma function* (0.5) is well-posed.
Set $f(x) := e^{-x}x^{t-1}$; then, since $f(x) < x^{t-1}$ if $x > 0$ (being $e^{-x} < 1$) and $t-1 > -1$, we infer that f is summable at 0. On the other hand, since $\lim\limits_{x\to+\infty} x^{t+1}e^{-x} = 0$, there exists $M > 0$ such that $x^{t+1}e^{-x} < 1$ for all $x > M$ and, accordingly, $f(x) < 1/x^2$ for all $x > M$, which leads immediately to conclude the summability of f at $+\infty$. Thus, $\Gamma(t) < \infty$ for all $t > 0$.
We see that

(1) $\Gamma(1) = \displaystyle\int_0^{+\infty} e^{-x}\, dx = 1$

(2) $\Gamma(t+1) = \displaystyle\int_0^{+\infty} x^t e^{-x}\, dx = t\,\Gamma(t)$ for all $t > 0$.

These two properties show that the *Gamma function* extends to $(0,\infty)$ the factorial of a number; indeed, $\forall n \in \mathbb{N}$

$$\Gamma(n+1) = n\,\Gamma(n) = n\,(n-1)\,\Gamma(n-1) = \cdots = n!\,\Gamma(1) = n!\,.$$

Another expression of the *Gamma function* is given by

$$\Gamma(t) = 2^{1-t}\int_0^{+\infty} e^{-\frac{y^2}{2}} y^{2t-1}\, dy, \tag{0.6}$$

and it is obtained by applying the change of variables $x = y^2/2$ in (0.5).
Denoted by $Q_1 = [0,+\infty)\times[0,+\infty)$ the first quadrant of the plane, from (0.6) we deduce easily, applying first Fubini's theorem and then passing in polar coordinates, that

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 2\left(\int_0^{+\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_0^{+\infty} e^{-\frac{y^2}{2}}\, dy\right)\\ &= 2\iint_{Q_1} e^{-(x^2+y^2)/2}\, dx\, dy = 2\int_0^{\pi/2} d\vartheta \int_0^{\infty} \varrho\, e^{-\frac{\varrho^2}{2}}\, d\varrho = \pi;\end{aligned}$$

hence $\Gamma(1/2) = \sqrt{\pi}$.
Let ω_n and σ_n denote the volume of the unit ball B_1 of $\mathbb{R}^n$ and the measure of the spheric surface of B_1, respectively; clearly

$$|B_r(x)| = r^n \omega_n, \qquad S(\partial B_r(x)) = r^{n-1}\sigma_n. \tag{0.7}$$

Theorem 2.1. *Let $n \geq 2$. Then, $\sigma_n = n\omega_n$.*

Proof. Applying (0.3) with $f \equiv 1$, we get immediately

$$\begin{aligned} \omega_n &= \int_{B_1} dx = \int_0^1 \left(\int_{\partial B_\varrho} dS \right) d\varrho = \int_0^1 S(\partial B_\varrho)\, d\varrho \\ &= \sigma_n \int_0^1 \varrho^{n-1}\, d\varrho = \frac{\sigma_n}{n}. \end{aligned}$$

□

Finally we give the expression of the volume ω_n of the unit ball, for all n.

Theorem 2.2. *Let $n \geq 1$. Then*

$$\omega_n = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)}. \tag{0.8}$$

Proof. Being $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$, (0.8) is verified for $n = 1$ and $n = 2$:

$$\omega_1 = \frac{\pi^{1/2}}{(1/2)\Gamma(1/2)} = 2, \qquad \omega_2 = \frac{\pi}{1 \cdot \Gamma(1)} = \pi.$$

Then, let us prove (0.8) for $n \geq 3$ by induction on n; suppose (0.8) is true for $n-2$, with $n \geq 3$, and let us show that it is true for n.
Take $x \in B_1$; then we can write $x = (x', x'')$, with $x' = (x_1, x_2)$ and $x'' = (x_3, \ldots, x_n)$ such that

$$x' \in D_1 = \{(x_1, x_2) \in \mathbb{R}^2 \colon\ x_1^2 + x_2^2 < 1\}$$

and

$$\begin{aligned} x'' \in (B_1)_{x'} &= \{x'' \in \mathbb{R}^{n-2} \colon\ (x', x'') \in B_1\} \\ &= \{(x_3, \ldots, x_n) \in \mathbb{R}^{n-2} \colon\ x_3^2 + \cdots + x_n^2 < 1 - x_1^2 - x_2^2\}. \end{aligned}$$

Thus, appealing to Fubini's theorem and using the induction hypotesis, we have

$$\begin{aligned}
\omega_n &= \int_{B_1} dx = \int_{D_1} dx' \int_{(B_1)_{x'}} dx'' \\
&= \int_{D_1} \omega_{n-2}\Big(1 - x_1^2 - x_2^2\Big)^{(n-2)/2} \, dx_1 \, dx_2 \\
&= \omega_{n-2} \int_0^{2\pi} d\vartheta \int_0^1 (1-\varrho)^{(n-2)/2} \, \varrho \, d\varrho \\
&= \frac{2\pi}{n} \, \omega_{n-2} = \frac{2\pi}{n} \, \frac{\pi^{(n-2)/2}}{\frac{n-2}{2} \, \Gamma\Big(\frac{n-2}{2}\Big)} \\
&= \frac{2\pi}{n} \, \frac{\pi^{(n-2)/2}}{\Gamma(n/2)} = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)}.
\end{aligned}$$

□

References

[1] N. Fusco, *Note integrative per il corso di equazioni alle derivate parziali.*