Hausaufgabe 6 (Abgabe bis Mittwoch, 21. Mai, 12 Uhr)

1. Show the equivalence of

   - u is 1-Lipschitz, i. e. $|u(x) - u(y)| \leq |x - y|\ \forall x, y \in \Omega \subset \mathbb{R}^n$,
   - u is a viscosity subsolution of $|\nabla u| - 1 = 0$.

   (4 pt)

2. In 2D we can define characteristic curves as curves on which Cauchy data does not determine a unique solution. Derive for the semilinear second order equation

   $$a(x)u_{x_1x_1}(x) + 2b(x)u_{x_1x_2}(x) + c(x)u_{x_2x_2}(x) = f(x, u(x), u_x(x))$$

   that the slope of a characteristic curve satisfies

   $$a(x)(\tfrac{\mathrm{d}x_2}{\mathrm{d}x_1})^2 - 2b(x)(\tfrac{\mathrm{d}x_2}{\mathrm{d}x_1}) + c(x) = 0\,.$$

   (2.5 pt)

   Show the following characterisation:

   - The PDE is hyperbolic, if each point is intersected by two distinct characteristic curves.
   - The PDE is parabolic, if each point is intersected by exactly one characteristic curve.
   - The PDE is elliptic, if there are no characteristic curves.

   (1 pt)

3. Show that the classification into hyperbolic, parabolic, and elliptic equations is independent of the coordinate choice, i. e. it remains the same if we choose new coordinates $X(x)$. (3 pt)

4. Classify the following equations:

   - $u_{tt} - u_{xx} = 0$ (wave equation)
   - $\Delta u = 0$ (Laplace's equation)
   - $u_t + \Delta u = 0$ (heat equation)
   - $3u_{x_1x_1} + u_{x_2x_2} + 4u_{x_2x_3} + 4u_{x_3x_3} = 0$

   (2 pt)

5. Derive a mean value formula for functions u satisfying $\Delta u + u = 0$ by deriving and solving a second order ordinary differential equation for $U(r) = \int_{B_r(x)} u(y)\,\mathrm{d}y$. (Hint: Start by computing $U'(r)$ and $U''(r)$.) (2.5 pt)