Theorem 66 (Weak maximum principle). *Let $u \in H^1(\Omega)$ satisfy $Lu = 0$ in the weak sense, then*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+, \qquad \inf_{\Omega} u \le \inf_{\partial\Omega} u^-.$$

Proof. For all $v \ge 0$ with $uv \ge 0$ we have $\int_\Omega \nabla v^T A \nabla u + b \cdot \nabla u v \,\mathrm{d}x = -\int_\Omega c u v \,\mathrm{d}x \le 0$. If $b = 0$, the choice $v = (u - \sup_{\partial\Omega} u^+)^+$ yields

$$\lambda \int_\Omega |\nabla v|^2 \,\mathrm{d}x \le 0$$

and thus the first result (the second follows analogously). The case $b \ne 0$ has to be done differently, see homework. □

Theorem 67 (Uniqueness of weak solution). *A weak solution to (29), if it exists, is unique.*

Proof. Let u_1, u_2 be two solutions, then $w = u_1 - u_2$ satisfies $Lw = 0$ in Ω, $w = 0$ on $\partial\Omega$, in a weak sense and thus $w \equiv 0$. □

The existence will be based on the following important two abstract tools.

Theorem 68 (Riesz representation theorem). *Let $f : H \to \mathbb{R}$ be a bounded linear functional on a Hilbert space H, then there exists $u \in H$ with $\|u\|_H = \|f\|$ such that $f(v) = (u, v)_H$ for all $v \in H$.*

Bemerkung 69. *A bounded linear functional or operator is a linear mapping T from a normed vector space V into another one W such that $\|Tu\|_W \le C\|u\|_V$ for a constant C and all $u \in V$. This is equivalent to T being continuous:*

$\Rightarrow$ *Let $u_k \to u$ in V, then $\|Tu_k - Tu\|_W = \|T(u_k - u)\|_W \le C\|u_k - u\|_V \to 0$.*

$\Leftarrow$ *Assume there exists $u_k \in V$ with $\|u_k\|_V = 1$, but $\|Tu_k\|_W \to \infty$. Thus, $v_k := \frac{u_k}{\|Tu_k\|_W} \to 0$ in V with $\|Tv_k\|_W = 1$, but this contradicts the continuity of T.*

Proof of Thm. 68. Let $u \in H$ such that $f(u) = 1$ and let $\hat{u} \in \ker(f)$ be its orthogonal projection onto $\ker(f)$. Define $v = u - \hat{u}$; we will show $f = (\frac{v}{\|v\|_H^2}, \cdot)_H$. Indeed, for $w \in H$, $w = w - f(w)v + f(w)v$. Thus,

$$\left(\frac{v}{\|v\|_H^2}, w\right)_H \overset{w - f(w)v \in \ker(f) \text{ and } v \perp \ker(f)}{\Longrightarrow} \left(\frac{v}{\|v\|_H^2}, f(w)v\right)_H = f(w).$$

□

Theorem 70 (Lax–Milgram theorem). *Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ a bounded, coercive bilinear form (i. e. $B(u, v)$ is linear in u and v with $|B(u, v)| \le \alpha\|u\|_H\|v\|_H$ and $B(u, u) \ge \beta\|u\|_H^2$ for two constants $\alpha, \beta > 0$ and all $u, v \in H$). Then there exists a bounded linear operater $A : H \to H$ with bounded inverse such that $B(u, v) = (Au, v)_H$ for all $u, v \in H$.*

Proof.
1. $B(u, \cdot)$ is a bounded linear functional on $H \overset{Thm. 68}{\Rightarrow}$ there exists $v \in H$ with $B(u, \cdot) = (v, \cdot)_H$
2. define $Au = v$, then A is clearly linear

3. $\|Au\|_H^2 = (Au, Au)_H = B(u, Au) \le \alpha\|u\|_H\|Au\|_H$ so that $\|Au\|_H \le \alpha\|u\|_H$ (i.e. A is bounded)

4. $\beta\|u\|_H^2 \le B(u,u) = (Au,u)_H \le \|Au\|_H\|u\|_H$ so that $\|Au\|_H \ge \beta\|u\|_H$ (i.e. A^{-1}, if it exists, is bounded)

5. A is injective due to $\|Au - Av\|_H = \|A(u-v)\|_H \ge \beta\|u-v\|_H$

6. range(A) is a closed subspace of H

7. range(A) $= H$ so that A^{-1} exists: Let $0 \ne u \in \text{range}(A)^\perp$, then $0 = (Au,u)_H = B(u,u) \ge \beta\|u\|_H^2 > 0$, a contradiction.

□

Theorem 71 (Existence of weak solutions). *Let Ω be bounded with Lipschitz boundary and $f \in L^2(\Omega)$, A, b, c bounded. There exists a weak solution $u \in H^1(\Omega)$ of (29).*

Proof. Setting $\tilde{u} = u - g$, we seek $\tilde{u} \in H_0^1(\Omega)$ with $B(\tilde{u}, v) = F(v) := \int_\Omega (f - b\cdot\nabla g - cg)v - \nabla v^T A\nabla g\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$.

1. F is a bounded linear functional on $H_0^1(\Omega)$ by Hölder's inequality $\overset{\text{Thm. 68}}{\Rightarrow}$ there exists $R(F) \in H_0^1(\Omega)$ with $F(v) = (R(F), v)_{H_0^1(\Omega)}$ $\forall v \in H_0^1(\Omega)$

2. $B(\cdot,\cdot)$ is a bounded bilinear form on $H_0^1(\Omega)$.

3. If $b = 0$, $B(v,v) \ge \lambda\|\nabla v\|_{L^2(\Omega)}^2 \ge c\|v\|_{H_0^1(\Omega)}^2$ by Poincaré's inequality, i. e. B is coercive, and we can directly apply the Lax–Milgram theorem: there exists some operator A with bounded inverse s. t. $B(u,v) = (Au, v)_{H_0^1(\Omega)}$ for all $u, v \in H_0^1(\Omega)$, thus $\tilde{u} = A^{-1}R(F)$ satisfies $B(\tilde{u}, v) = (R(F), v)_{H_0^1(\Omega)}$ for all $v \in H_0^1(\Omega)$.
   If $b \ne 0$ one needs a modification, see homework.

□

Having established existence and uniqueness of a weak solution, we can now analyse its regularity.

Theorem 72 (Inner regularity). *Let Ω be bounded with Lipschitz boundary, $f \in L^2(\Omega)$, $A \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{n\times n})$, $b \in L^\infty(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$. Let $u \in H^1(\Omega)$ be the weak solution of (29). For any $\Omega' \subset\subset \Omega$ there exists a constant $C > 0$ such that*

$$\|u\|_{H^2(\Omega')} \le C(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)})$$

and hence $u \in H^2(\Omega')$.

Proof.

1. For $i \in \{0,\dots,n\}$, $h \in \mathbb{R}$ define the finite difference operator $\Delta_i^h : \Delta_i^h u = \frac{u(\cdot+h)-u(\cdot)}{h}$. It is not difficult to check $Du \in L^2(\Omega) \Leftrightarrow \exists\kappa > 0 : \sum_{i=1}^n \|\Delta_i^h u\|_{L^2(\Omega)} < \kappa$ for all $|h|$ small enough. Also note $\Delta_i^h \nabla = \nabla\Delta_i^h$.

2. Let $2|h| < \text{dist}(\text{supp}\, v, \partial\Omega)$. (30) implies

$$\int_\Omega \nabla v^T \Delta_i^h (A\nabla u)\,\mathrm{d}x = -\int_\Omega \nabla(\Delta_i^{-h} v)^T A\nabla u\,\mathrm{d}x$$
$$= \int_\Omega (\Delta_i^{-h} v) b\cdot\nabla u + c(\Delta_i^{-h} v)u - f(\Delta_i^{-h} v)\,\mathrm{d}x$$

or equivalently, using $\Delta_i^h(A\nabla u)(x) = A(x+he_i)\Delta_i^h(\nabla u)(x) + \Delta_i^h(A(x))\nabla u(x)$,

$$\int_\Omega \nabla v^T A(x+he_i)\Delta_i^h \nabla u \,\mathrm{d}x = \int_\Omega -\nabla v^T \Delta_i^h A \nabla u + \Delta_i^{-h} v b\cdot\nabla u + c\Delta_i^{-h} v u - f\Delta_i^{-h} v \,\mathrm{d}x$$
$$\leq \text{const.}(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)})\|\nabla v\|_{L^2(\Omega)}. \qquad (31)$$

3. Taking $v = \eta^2\Delta_i^h u$ for a smooth cutoff function $\eta \in C_0^\infty(\Omega;[0,1])$, $\eta = 1$ on Ω',

$$\lambda\int_\Omega |\eta\nabla\Delta_i^h u|^2 \,\mathrm{d}x \leq \int_\Omega \eta^2\Delta_i^h \nabla u^T A(x+he_i)\Delta_i^h\nabla u \,\mathrm{d}x$$
$$\overset{(31)}{\leq} \text{const.}(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)})(\|\eta^2\nabla\Delta_i^h u\|_{L^2(\Omega)} + \|2\eta\Delta_i^h u\nabla\eta\|_{L^2(\Omega)}).$$

Using Young's inequality $\alpha\beta \leq \frac{\varepsilon\alpha^2}{2} + \frac{\beta^2}{2\varepsilon}$ for any $\alpha,\beta,\varepsilon > 0$ as well as $(\alpha+\beta)^2 \leq 2\alpha^2 + 2\beta^2$,

$$\lambda\|\eta\nabla\Delta_i^h u\|_{L^2(\Omega)}^2 \leq \tfrac{1}{2\varepsilon}\text{const.}^2(\|u\|_{H^1(\Omega)}+\|f\|_{L^2(\Omega)})^2 + \tfrac{\varepsilon}{2}(\|\eta^2\nabla\Delta_i^h u\|_{L^2(\Omega)} + \|2\eta\Delta_i^h u\nabla\eta\|_{L^2(\Omega)})^2$$
$$\leq \text{const.}(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} + \|2\eta\Delta_i^h u\nabla\eta\|_{L^2(\Omega)})^2 + \varepsilon\|\eta^2\nabla\Delta_i^h u\|_{L^2(\Omega)}^2.$$

Subtracting $\varepsilon\|\nabla\Delta_i^h u\|_{L^2(\Omega)}^2$ on both sides and noting $\|2\eta\Delta_i^h u\nabla\eta\|_{L^2(\Omega)} \leq \text{const.}\|\nabla u\|_{L^2(\Omega)}$, we get

$$\|\nabla\Delta_i^h u\|_{L^2(\Omega')} \leq \|\eta\nabla\Delta_i^h u\|_{L^2(\Omega)} \leq \text{const.}(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}),$$

which implies $\|D^2 u\|_{L^2(\Omega')} \leq \text{const.}(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)})$.

□

Bemerkung 73. *If in the above proof we use finite difference approximations of higher derivatives, we obtain*

$$A \in C^{k,1}(\overline{\Omega}), b, c \in C^{k-1,1}(\overline{\Omega}), f \in H^k(\Omega) \quad\Rightarrow\quad u \in H^{k+2}(\Omega').$$

Hence, if A, b, c, f are infinitely smooth, then also $u \in C^\infty(\Omega)$.

Bemerkung 74. *If the boundary data is smooth, one can even show smoothness of u on all of Ω,*

$$A \in C^{k,1}(\overline{\Omega}), b, c \in C^{k-1,1}(\overline{\Omega}), f \in H^k(\Omega), \partial\Omega \in C^{k+2}, g \in H^{k+2}(\Omega)$$
$$\Rightarrow \quad u \in H^{k+2}(\Omega) \textit{ with } \|u\|_{H^{k+2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)} + \|g\|_{H^{k+2}(\Omega)}).$$

(See e. g. Gilbarg & Trudinger, "Elliptic PDEs of 2nd Order", p. 187.)

Variational approach and nonlinear equations

Solving a PDE is often equivalent to minimising an energy. In particular in physics, PDEs are often just a consequence of an energy minimisation principle.

Setting:

- *Lagrangian* $L : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$, $(p, z, x) \mapsto L(p, z, x)$ (assumed smooth for simplicity, with derivatives L_p, L_z, L_x)