- *energy* $E[u] = \int_\Omega L(\nabla u(x), u(x), x)\,\mathrm{d}x$ defined for (weakly) differentiable functions $u : \Omega \to \mathbb{R}$
- *Gâteaux derivative* of E in direction $v : \Omega \to \mathbb{R}$:

$$\partial_u E[u](v) = \tfrac{\mathrm{d}}{\mathrm{d}t} E[u + tv] = \int_\Omega L_p(\nabla u, u, x) \cdot \nabla v + L_z(\nabla u, u, x) v \,\mathrm{d}x$$

Consider the minimisation problem

$$\min_{u:\Omega\to\mathbb{R}} E[u] \text{ subject to } u = g \text{ on } \partial\Omega$$

and assume it has a smooth minimiser u^*. Then for any $v \in C_c^\infty(\Omega)$, 0 must be a minimiser of $t \mapsto E[u^* + tv]$, i. e. $\partial_u E[u^*](v) = 0$. Integration by parts yields

$$0 = \int_\Omega v\big(-\mathrm{div} L_p(\nabla u^*, u^*, x) + L_z(\nabla u^*, u^*, x)\big)\,\mathrm{d}x$$

for all $v \in C_c^\infty(\Omega)$, i. e. u^* satisfies the PDE

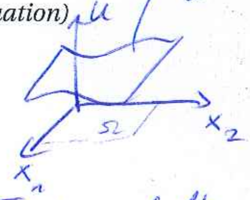
$$0 = -\mathrm{div} L_p(\nabla u(x), u(x), x) + L_z(\nabla u(x), u(x), x) \text{ in } \Omega \qquad \text{with } u = g \text{ on } \partial\Omega\,.$$

Beispiel 75.

Lagrangian $L(p, z, x)$	*energy* $E[u]$	*PDE*	
$\frac{1}{2}\lvert p\rvert^2$	$\int_\Omega \frac{1}{2}\lvert\nabla u\rvert^2\,\mathrm{d}x$	$\Delta u = 0$	*(Laplace's equation)*
$\frac{1}{2}p^T A(x) p - z f(x)$	$\int_\Omega \frac{1}{2}\nabla u^T A \nabla u - f u\,\mathrm{d}x$	$-\mathrm{div}(A\nabla u) = f$	*(generalised Poisson's equation)*
$\frac{1}{2}\lvert p\rvert^2 - F(z)$	$\int_\Omega \frac{1}{2}\lvert\nabla u\rvert^2 - F(u)\,\mathrm{d}x$	$-\Delta u = F'(u)$	*(nonlinear Poisson's equation)*
$\sqrt{1+\lvert p\rvert^2}$	$\int_\Omega \sqrt{1+\lvert\nabla u\rvert^2}\,\mathrm{d}x$	$\mathrm{div}\frac{\nabla u}{\sqrt{1+\lvert\nabla u\rvert^2}} = 0$	*(minimal surface equation)*

Graph von u

u



Flächeninhalt $= \int_\Omega \sqrt{1+|\nabla u|^2}\,dx$

We will now introduce a few functional analytic tools and then prove existence of minimisers (and thus solutions to the PDEs) for a broad class of nonlinear energies.

Definition 76 (Dual space). *The dual space X' to a Banach space X is the space of bounded linear functionals $f : X \to \mathbb{R}$ on X.*

Definition 77 (Weak convergence). *A sequence $x_k \in X$ is said to converge weakly against $x \in X$, $x_k \rightharpoonup x$, if $f(x_k) \to f(x)$ for any $f \in X'$.*
A sequence $f_k \in X'$ is said to converge weakly- against $f \in X'$, $f_k \overset{*}{\rightharpoonup} f$, if $f_k(x) \to f(x)$ for any $x \in X$.*
Obviously, convergence in X or X' implies weak or weak- convergence, respectively.*

Theorem 78 (Weak-* compactness). *The unit ball (and thus any bounded subset) in a separable Banach space X is weakly-* sequentially compact, i. e. any sequence contains a convergent subsequence.*

Proof. See e. g. Alt, "Lineare Funktionalanalysis", p. 229. □

Definition 79 (Reflexivity). *A Banach space X is called reflexive, if it can be identified with its bidual $(X')'$ (i. e. there is an isometric isomorphism between X and X'').*
Note: By the previous theorem, if X is separable, bounded sequences in X contain weakly convergent subsequences.

Theorem 80 (Reflexivity of Sobolev spaces). *Let Ω be open, bounded, $\partial\Omega$ Lipschitz, $k \in \mathbb{N}_0$. $W^{k,p}(\Omega)$ for $p \in (1, \infty)$ is separable and reflexive.*

Proof. See e. g. Alt, "Lineare Funktionalanalysis", p. 234. □

Bemerkung 81. *Knowing the reflexivity of $L^p(\Omega)$ for $p \in (1,\infty)$, it is now easy to see via Hölder's inequality that* ~~$(L^p(\Omega))'' =$~~ $L^p(\Omega) \subset (L^{p^*}(\Omega))'$, ~~*i. e.* $(L^p(\Omega))' \subset L^{p^*}(\Omega)$~~. *Likewise,* $(L^p(\Omega))' \supset L^{p^*}(\Omega)$ *via Hölder's inequality* ~~*so that*~~ In fact, by Radon-Nikodym Theorem,

$$(L^p(\Omega))' = L^{p^*}(\Omega).$$

Theorem 82. *Assume*

- $L(p,z,x)$ *is convex in* p,
- $L(p,z,x)$ *is* ~~*lower semi-continuous in z,*~~ smooth
- $L(p,z,x) \geq \alpha|p|^q - \beta$ *for some* $\alpha, \beta > 0$, $q \in (1,\infty)$.

If $g \in W^{1,q}(\Omega)$, *then* E *has a minimiser in* $\{u \in W^{1,q}(\Omega) \mid u = g \text{ on } \partial\Omega\}$.

Proof. "Direct method of the calculus of variations"

1. Neither is $E \equiv \infty$ nor unbounded from below.

2. Consider a minimising sequence u_k with $E[u_k] \to \inf_u E[u]$ monotonically.

3. Show compactness of the sequence, i. e. existence of a (in some particular sense) converging subsequence $u_k \to u^*$ for some u^*.
   $E[u] \geq \alpha\|\nabla u\|^q_{L^q(\Omega)} - \beta|\Omega|$, hence there is $C > 0$ with $\|\nabla u_k\|_{L^q(\Omega)} < C$ for all k.
   By Poincaré's inequality, $\|u_k\|_{W^{1,q}(\Omega)} \leq C$ for some (different) constant $C > 0$.
   Since $W^{1,q}(\Omega)$ is reflexive, a subsequence u_k converges weakly against a $u^* \in W^{1,q}(\Omega)$.

   $u_k - g$ hat 0-Randwerte
   $\Rightarrow \|u_k - g\|_{L^p} \leq \|\nabla(u_k - g)\|_{L^p}$
   $\Rightarrow \|u_k\|_{L^p} \leq \underbrace{\|\nabla u_k\|_{L^p} + \|\nabla g\|_{L^p} + \|g\|_{L^p}}_{\leq \tilde{C}}$
   $\Rightarrow \|u_k\|^p_{W^{1,p}} \leq \|u_k\|^p_{L^p} + \|\nabla u_k\|^p_{L^p} \leq \tilde{C}$

4. Show lower semicontinuity of E along the converging sequence, i. e. $E[u] \leq \liminf_{k\to\infty} E[u_k]$.
   Due to the compact embedding $W^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$, $u_k \to u^*$ strongly in $L^q(\Omega)$ for a subsequence and thus even pointwise a. e. after extracting another subsequence. By Egoroff's Theorem we can even find for each $\varepsilon > 0$ a measurable set $\Omega_\varepsilon \subset \Omega$ with $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ and $u_k \to u^*$ uniformly on Ω_ε.
   Wlog., $L \geq 0$. By convexity of L in p,

$$\begin{aligned}&\liminf_{k\to\infty} \int_\Omega L(\nabla u_k, u_k, x)\,\mathrm{d}x \\ &\qquad \geq \liminf_{k\to\infty} \left(\int_\Omega L(\nabla u^*, u_k, x)\,\mathrm{d}x + \int_\Omega L_p(\nabla u^*, u_k, x) \cdot (\nabla u_k - \nabla u^*)\,\mathrm{d}x \right).\end{aligned}$$

   We have

$$\begin{aligned}\liminf_{k\to\infty} \int_\Omega L(\nabla u^*, u_k, x)\,\mathrm{d}x &\geq \liminf_{k\to\infty} \int_\Omega \inf_{j\geq k} L(\nabla u^*, u_j, x)\,\mathrm{d}x \\ &\overset{(A)}{=} \int_\Omega \liminf_{k\to\infty} L(\nabla u^*, u_k, x)\,\mathrm{d}x \overset{(B)}{\geq} \int_\Omega L(\nabla u^*, u^*, x)\,\mathrm{d}x = E[u^*]\end{aligned}$$

by Beppo Levi's monotone convergence theorem (A) and the lower semi-continuity of L (B) (the fact $\liminf_k \int \Omega f_i \,\mathrm{d}x \geq \int_\Omega \liminf_k f_i \,\mathrm{d}x$ is also called "Fatou's lemma"). Furthermore,

$$\begin{aligned}
&\int_\Omega L_p(\nabla u^*, u_k, x) \cdot (\nabla u_k - \nabla u^*)\,\mathrm{d}x \\
&\geq \int_{\Omega_\varepsilon} L_p(\nabla u^*, u^*, x)\cdot(\nabla u_k - \nabla u^*)\,\mathrm{d}x + \int_{\Omega_\varepsilon} (L_p(\nabla u^*, u_k, x) - L_p(\nabla u^*, u^*, x))\cdot(\nabla u_k - \nabla u^*)\,\mathrm{d}x
\end{aligned}$$

of which the first term converges to zero due to the weak convergence of u_k and the second due to Hölder's inequality and the uniform convergence. Letting $\varepsilon \to 0$ yields the result.

if L_p cont. in z!

□

Parabolic PDEs

Heat equation

Hyperbolic PDEs

Wave equation