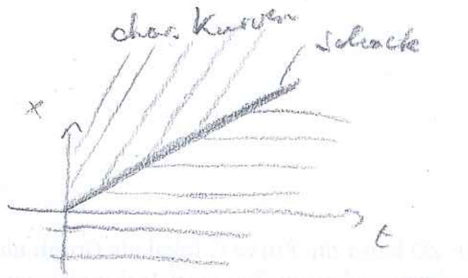
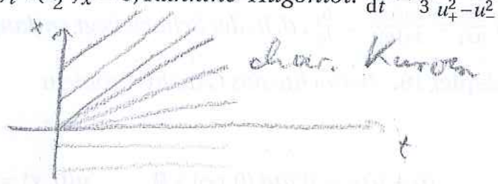


- $u(t,x) = \begin{cases} 0 \text{ für } x \le t/2 \\ 1 \text{ sonst} \end{cases}$

(Schock zu Erhaltungsform $u_t + (\frac{u^2}{2})_x = 0$; Rankine–Hugoniot: $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\frac{u_+^2}{2} - \frac{u_-^2}{2}}{u_+ - u_-} = \frac{u_+ + u_-}{2}$),

- $u(t,x) = \begin{cases} 0 \text{ für } x \le 2t/3 \\ 1 \text{ sonst} \end{cases}$

(Schock zu Erhaltungsform $(\frac{u^2}{2})_t + (\frac{u^3}{3})_x = 0$; Rankine–Hugoniot: $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{3}\frac{u_+^3 - u_-^3}{u_+^2 - u_-^2} = \frac{2}{3}\frac{u_+^2 + u_- u_+ + u_-^2}{u_+ + u_-}$),



- $u(t,x) = \begin{cases} 0 & \text{für } x \le 0 \\ x/t & \text{für } 0 < x < t \\ 1 & \text{sonst} \end{cases}$

(Verdünnungswelle).

Welche Erhaltungsform korrekt ist, hängt vom modellierten physikalischen Problem ab. Auch ob ein Schock oder stattdessen eine Verdünnungswelle korrekt sind, hängt vom modellierten Problem ab. Man kann für eine vorgegebene Erhaltungsform eine eindeutige Lösung selektieren durch Hinzunahme einer zusätzlichen Bedingungen, z. B.

1. *Entropiebedingung: Wir legen eine bestimmte Funktion von $f(u)$ fest, die über den Schock zunehmen soll, d. h. $[f(u)]_-^+ \ge 0$. Die Wahl $f(u) = u$ verbietet obige Schocks.*
2. *Viskosität: Wir betrachten $0 = F(\nabla u, u, x)$ als Grenzfall für $\varepsilon \to 0$ von $\varepsilon \Delta u = F(\nabla u, u, x)$, welches eine eindeutige Lösung besitzt.*
3. *Kausalität: Ist eine Variable die Zeit, so fließt physikalisch Information entlang der charakteristischen Kurven in Richtung wachsender Zeit. Es soll Information in den Schock hinein und nicht aus ihm heraus fließen (dies verbietet obige Schocks).*

Viscosity solutions

Consider the nonlinear first-order problem

$$H(x, u(x), \nabla u(x)) = 0 \text{ in } \Omega \tag{15}$$

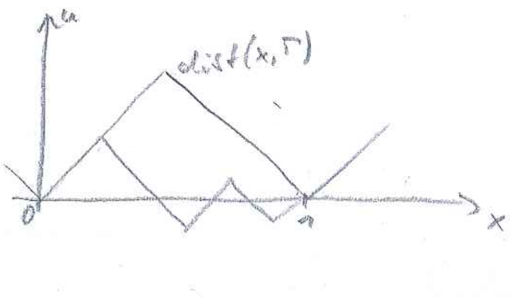
with boundary conditions on $\partial\Omega$, where $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and H is convex in ∇u. (15) is also called a Hamilton–Jacobi equation, and we changed the notation from F to H since this is conventionally used in this context.

"Solutions" are typically
- neither unique
- nor differentiable everywhere

Beispiel 20. *The* eikonal equation

$$0 = H(x, u, p) = |p| - 1$$

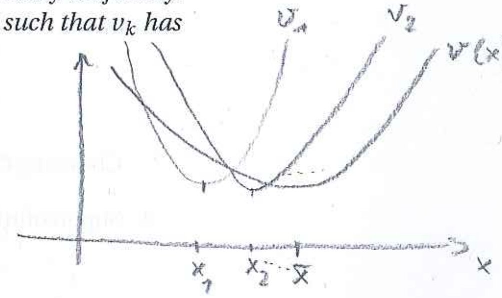
is often used to compute the distance $u(x) = \operatorname{dist}(x, \Gamma)$ from a given set $\Gamma \subset \mathbb{R}^n$. However, already for $n = 1$ and boundary data $u = 0$ on $\Gamma = \{0, 1\}$ there is no continuously differentiable solution u, but many functions u satisfying (15) *almost everywhere.*



Idea to obtain well-posedness: add "viscosity"-term $-\varepsilon\Delta u_\varepsilon$ and consider $\varepsilon \to 0$

$$-\varepsilon\Delta u_\varepsilon + H(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) = 0 \tag{16}$$

Lemma 21. *Let $\Omega \subset \mathbb{R}^n$ open, $\bar{x} \in \Omega$, $v, v_k \in C^1(\Omega)$ with $v_k \to_{k\to\infty} v$ locally uniformly. If v has a strict local minimum in $\bar{x}$, then there is a sequence $x_k \to \bar{x}$ such that v_k has a local minimum in x_k.*



Proof. Wlog. $\bar{x} = 0, v(\bar{x}) = 0$.

- Define $\omega_\rho = \min_{\partial B_\rho(0)} v > 0$ for $0 < \rho < r$.
- Define $\rho_k = \inf\{\rho \mid \frac{\omega_\rho}{4} \geq \sup_{B_r(0)} |v_k - v|\}$, then $\rho_k \to_{k\to\infty} = 0$:
  Indeed, assume $\rho_{k_j} \geq \delta > 0$ for a sequence $k_j \to_{j\to\infty} \infty$,
  then $\frac{\omega_\delta}{4} < \sup_{B_r(0)} |v_{k_j} - v| \to_{j\to\infty} 0$, contradiction.

case a) $\rho_k = 0 \implies v_k = v \implies x_k = \bar{x}$ is local minimiser of v_k.

case b) $\rho_k > 0 \implies \min_{\partial B_{\rho_k}(0)} v_k \overset{\text{def.}\,\rho_k}{\geq} \min_{\partial B_{\rho_k}(0)} v - \frac{\omega_{\rho_k}}{4} \overset{\text{def.}\,\omega_{\rho_k}}{=} \frac{3\omega_{\rho_k}}{4} \geq \frac{\omega_{\rho_k}}{4} \overset{\text{def.}\,\rho_k}{\geq} v_k(0) \implies v_k$ has a local minimum in some $x_k \in B_{\rho_k}(0)$.

It follows $x_k \to_{k\to\infty} 0$. □

Assumption:
- solution u_ε of (15) is $C^2(\Omega)$
- $u_\varepsilon \to u$ locally uniformly

Consider: comparison function $\phi \in C^2(\Omega)$ with $\phi \leq u$ s. t. $u - \phi$ has a strict minimum in $\bar{x} \in \Omega$

Lemma 21 implies that there is a sequence $\varepsilon_k \to_{k\to\infty} 0$, $x_k \to_{k\to\infty} \bar{x}$ such that $u_{\varepsilon_k} - \phi$ has a local minimum at x_k, i. e.

$$\begin{aligned}
\nabla(u_{\varepsilon_k}(x_k) - \phi(x_k)) &= 0 && \text{(derivative 0)}\\
D^2(u_{\varepsilon_k}(x_k) - \phi(x_k)) &\geq 0 && \text{(positive semi-definite Hessian)}\\
\Delta u_{\varepsilon_k}(x_k) - \Delta\phi(x_k) &\geq 0 && \text{from previous eq.}
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= -\varepsilon_k \Delta u_{\varepsilon_k}(x_k) + H(x_k, u_{\varepsilon_k}(x_k), \nabla u_{\varepsilon_k}(x_k))\\
&\leq -\varepsilon_k \Delta\phi(x_k) + H(x_k, u_{\varepsilon_k}(x_k), \nabla\phi(x_k)) \to_{k\to\infty} H(\bar{x}, u_{\varepsilon_k}(\bar{x}), \nabla\phi(\bar{x})),
\end{aligned} \tag{17}$$

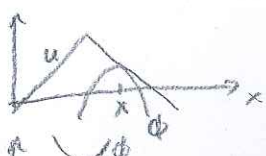
i. e.

$$H(\bar{x}, u_{\varepsilon_k}(\bar{x}), \nabla\phi(\bar{x})) \geq 0 \qquad \text{whenever } u - \phi \text{ has a strict local minimum in } \bar{x}.$$

This motivates the following.

Definition 22 (Viscosity solution)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $H : \Omega \times \mathbb{R} \times \mathbb{R}^n$ continuous.*

1. *$u \in C(\Omega)$ is called a* viscosity supersolution, *if $H(x, u(x), \nabla\phi(x)) \geq 0$ for all $\phi \in C^1(\Omega)$ s. t. $u - \phi$ has a minimum in x.*



2. *$u \in C(\Omega)$ is called a* viscosity subsolution, *if $H(x, u(x), \nabla\phi(x)) \leq 0$ for all $\phi \in C^1(\Omega)$ s. t. $u - \phi$ has a maximum in x.*

3. *$u \in C(\Omega)$ is called a* viscosity solution, *if it is a viscosity super- and subsolution.*

Lemma 23 (Example: distance function). *For $\Gamma \subset \mathbb{R}^n$ closed, $u(x) = \operatorname{dist}(x,\Gamma)$ is a viscosity solution of $|\nabla u| - 1 = 0$ in $\mathbb{R}^n \setminus \Gamma$.*

Proof. 1. Subsolution: Let $u - \phi$ have a local maximum in x, i. e.

$$
\begin{aligned}
& (u-\phi)(x+z) - (u-\phi)(x) \le 0 \\
\Leftrightarrow\quad & \phi(x+z) - \phi(x) \ge u(x+z) - u(x) \ge -|z| \\
\Rightarrow\quad & \tfrac{\phi(x+s\zeta)-\phi(x)}{s} \ge -|\zeta| \text{ and thus } \nabla\phi(x)\cdot\zeta \ge -|\zeta| \ \forall \zeta \in \mathbb{R}^n .
\end{aligned}
$$

Choosing $\zeta = -\frac{\nabla\phi(x)}{|\nabla\phi(x)|}$ yields $|\nabla\phi(x)| \le 1$ or $|\nabla\phi(x)| - 1 \le 0$.

2. Supersolution: Let $u - \phi$ have a local minimum in x, i. e.

$$
\begin{aligned}
& (u-\phi)(x+z) - (u-\phi)(x) \ge 0 \\
\Leftrightarrow\quad & \phi(x+z) - \phi(x) \le u(x+z) - u(x)
\end{aligned}
$$

Let $\bar{x} \in \Gamma$ with $u(x) = |x - \bar{x}|$ and choose $z = -s\dfrac{x-\bar{x}}{|x-\bar{x}|}$

$$
\begin{aligned}
\Rightarrow\quad & \frac{\phi(x + s\frac{\bar{x}-x}{|\bar{x}-x|}) - \phi(x)}{s} \le \frac{|\bar{x}-x| - s - |\bar{x}-x|}{s} = -1 \\
\Rightarrow\quad & -\nabla\phi(x)\cdot\tfrac{\bar{x}-x}{|\bar{x}-x|} \ge 1 \quad\Rightarrow\quad |\nabla\phi(x)| \ge 1 \text{ or } |\nabla\phi(x)| - 1 \ge 0 .
\end{aligned}
$$

□

Bemerkung 24. *In general, $u = -\operatorname{dist}(x,\Gamma)$ is not a viscosity solution of $|\nabla u| - 1 = 0$: Let $u_{\min} = u(\bar{x})$ be a local minimum of $-\operatorname{dist}(x,\Gamma)$ and $\phi \equiv u_{\min}$ so that $u - \phi$ has a local minimum in $\bar{x}$. Then $|\nabla\phi| - 1 = -1 < 0$, i. e. u is no viscosity supersolution.*

Note: $u = -\operatorname{dist}(x,\Gamma)$ is a viscosity solution of $1 - |\nabla u| = 0$.

For simplicity, let us now consider Hamilton–Jacobi equations of the form

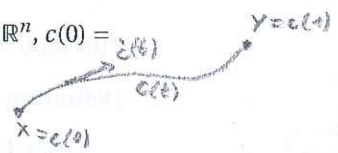
$$
H(x, \nabla u(x)) = 0 \tag{18}
$$

with H continuous, $H(x,\cdot) : \mathbb{R}^n \to \mathbb{R}$ uniformly convex, $\lim_{|p|\to\infty} H(x,p) = \infty$ uniformly in x (“coercivity”), and $H(x,0) \le 0$.

Definition 25 (Support function). *$L_x : \mathbb{R}^n \to \mathbb{R}$, $L_x(w) = \sup_{H(x,p)\le 0} w \cdot p$, is called* support function *of the set $\{p \in \mathbb{R}^n \,|\, H(x,p) \le 0\}$.*

Beispiel 26. *$H(x,p) = |p| - \frac{1}{v(x)}$ for a velocity $v(x)$ has the support function $L_x(w) = \frac{|w|}{v(x)}$.*

Definition 27 (Optical distance). *$\delta(x,y) = \inf\{\int_0^1 L_{c(t)}(\dot{c}(t))\,\mathrm{d}t \,|\, c : [0,1] \to \mathbb{R}^n,\ c(0) = x,\ c(1) = y\}$ is called the* optical distance *between x and y.*



Beispiel 28. *The optical distance $\delta(x,y)$ for $H(x,p) = |p| - \frac{1}{v(x)}$,*

$$
\delta(x,y) = \inf_{\substack{c:[0,1]\to\mathbb{R}^n \\ c(0)=x,\, c(1)=y}} \int_0^1 \frac{|\dot{c}(t)|}{v(c(t))}\,\mathrm{d}t
$$

describes the arrival time at y of a seismic wave starting from x and having local wave speed $v(x)$.