Imaging and Inverse Problems of Partial Differential Equations

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X-ray Tomography (CT)

Emission Computed Tomography (SPECT, PET)

Optical Tomography -Near Infrared Imaging (NIR)

Electrical Impedance Tomography (EIT)

Seismic imaging

Synthetic Aperture Radar (SAR)

Ultrasound Tomography

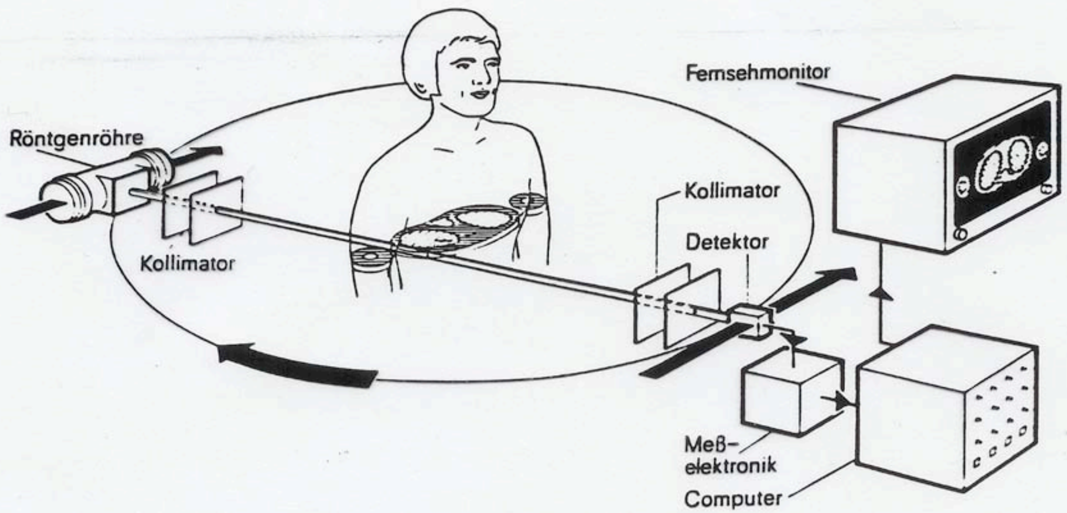
Goal: Unified treatment as inverse problem of partial differential equations

CT(Principle)

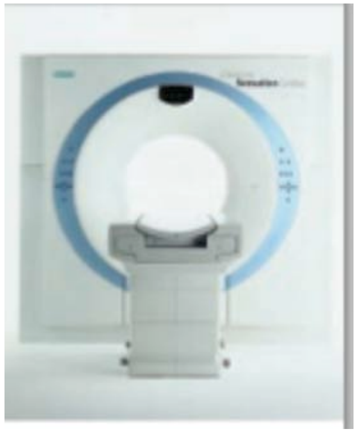
Computerized Tomography (CT):

Technique for Imaging a two-dimensional cross section of three-dimensional object

$$(\tau o \mu o s \text{ (greek)} = \text{slice})$$



Modern CT Scanners

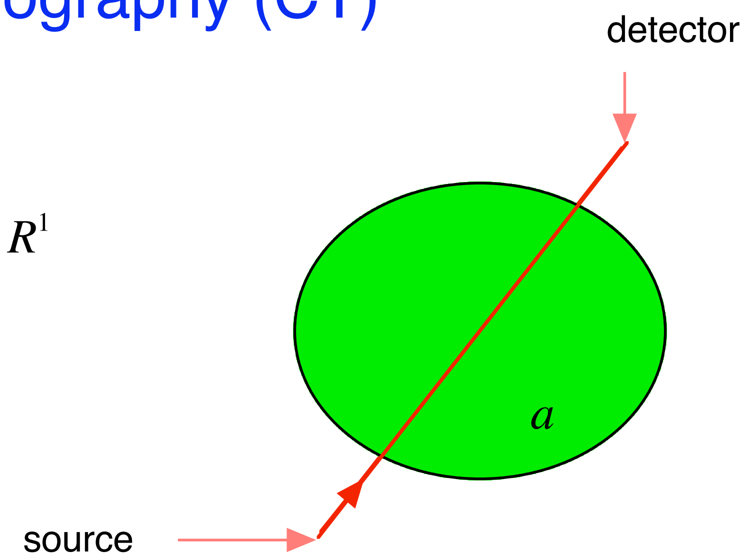


X-Ray Tomography (CT)

$a = a(x)$ absorption coefficient

$$(Ra)(\theta,s) = \int_{x\cdot\theta=s} a(x)dx,\ \theta \in S^1,\ s \in R^1$$

Radon transform



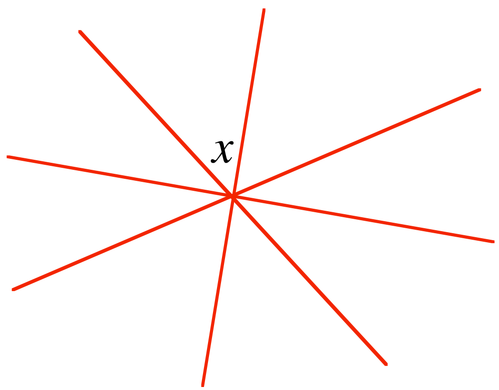
Radon's 1917 inversion formula:

$$f = R^* K g,\ g = Rf$$

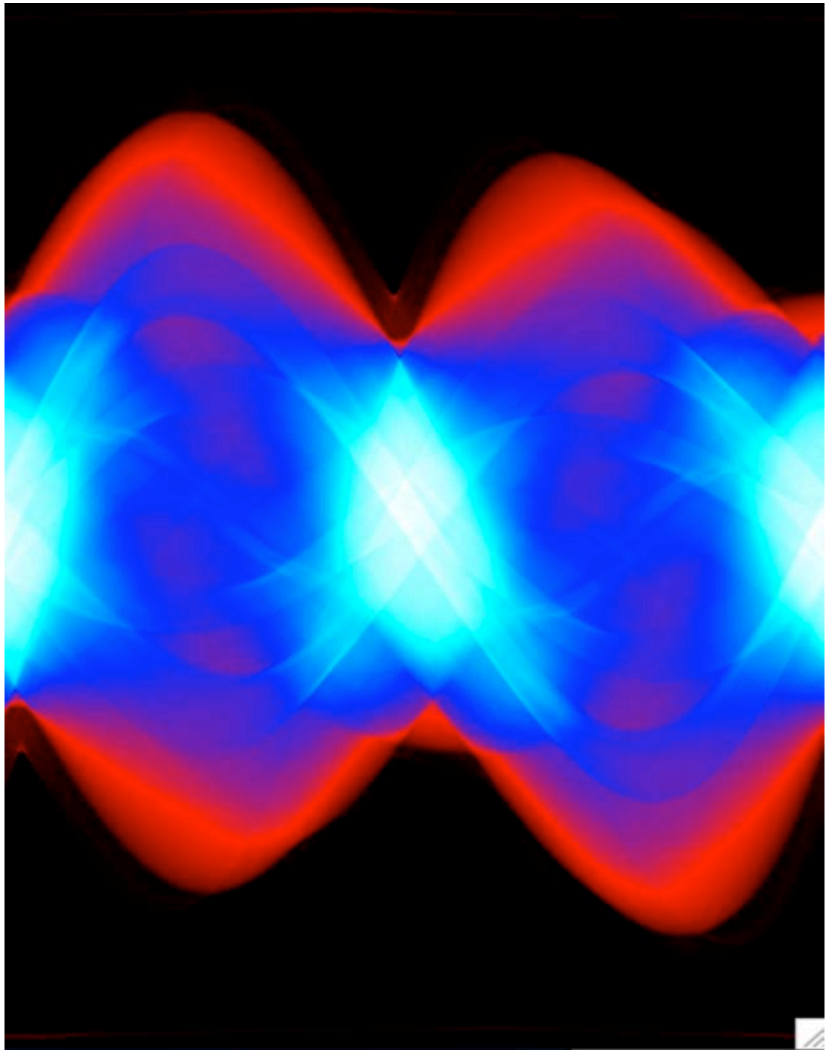
$$(R^* g)(x) = \int_{S^1} g(\theta, x\cdot\theta)d\theta$$

$$(Kg)(s) = \frac{1}{4\pi^2}\int \frac{g'(t)}{s-t}dt$$

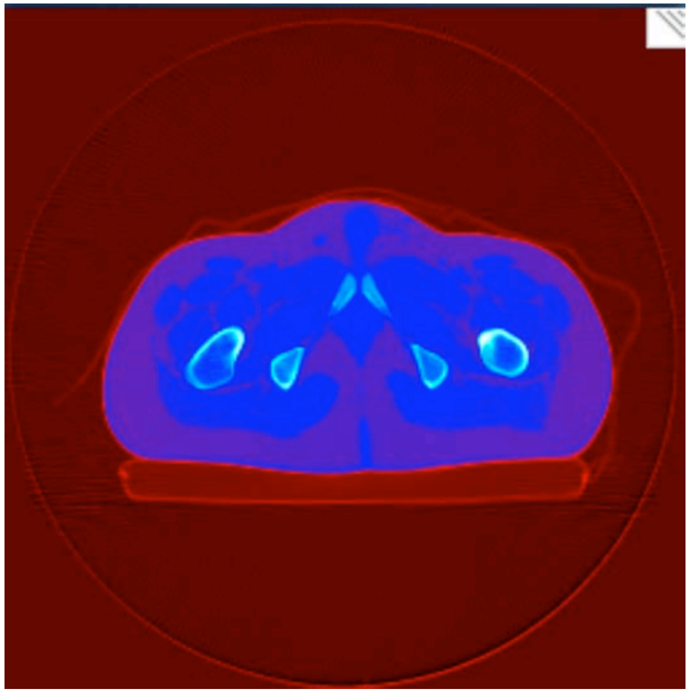
R^* = adjoint of R = backprojection



Data (Sinogram)

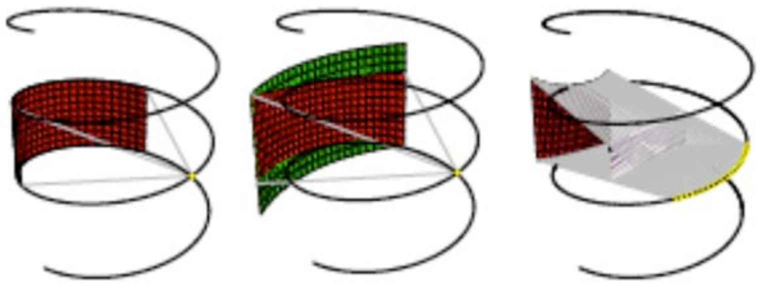


Tomogram

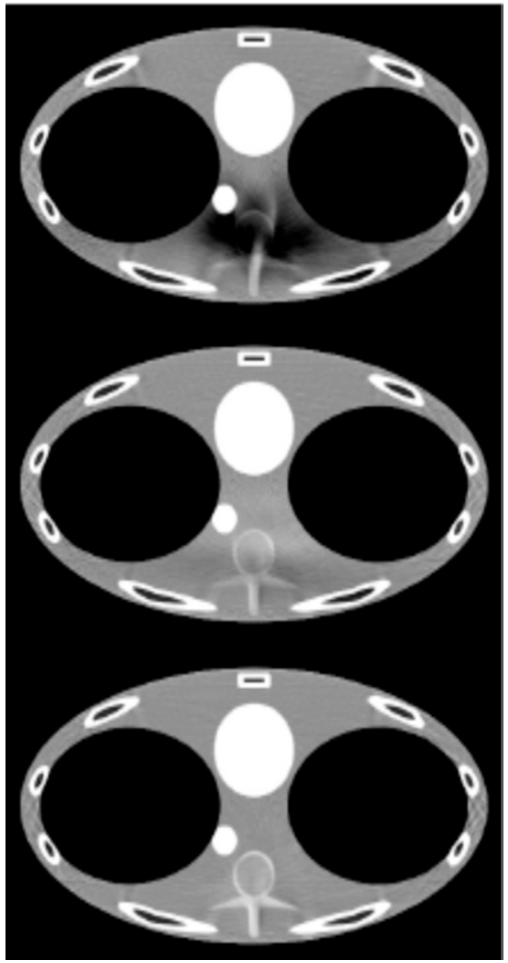


3D cone beam reconstruction in CT

Algorithm based on interpolation

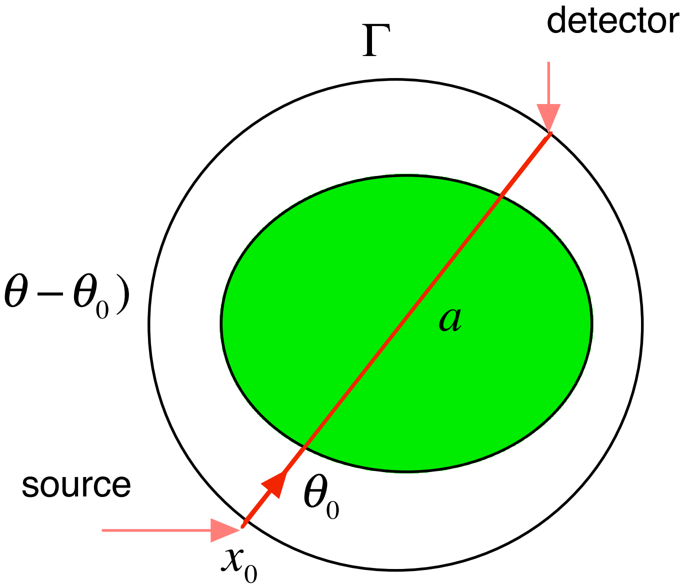


Katsevich algorithm



CT as an inverse problem of the transport equation

Introduce particle density $u(x,\theta)$ at x in direction θ:



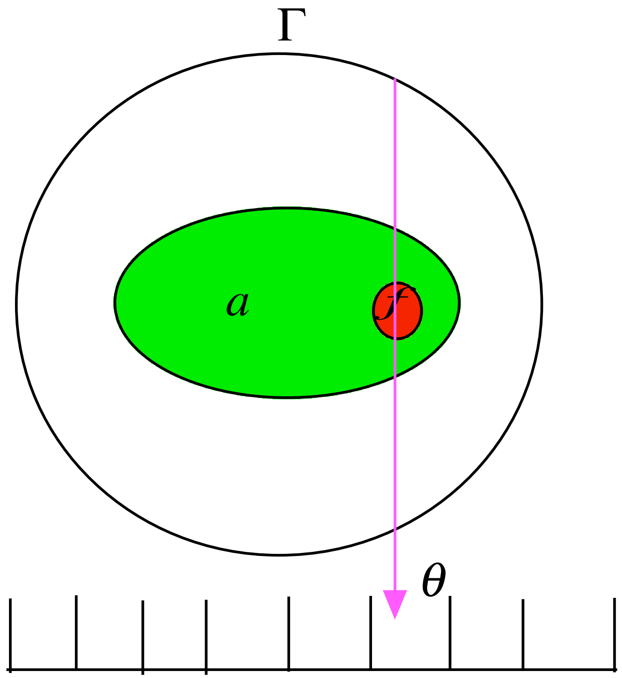
$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = \delta(x - x_0)\delta(\theta - \theta_0)$$
$$u(x,\theta) = 0,\ x \in \Gamma,\ \theta \cdot \nu_x \le 0$$

Inverse problem: Determine a from $u(x,\theta),\ x,x_0 \in \Gamma,\ \theta = (x - x_0)/|x - x_0|$

$$u(x,\theta) = H((x - x_0)\cdot\theta)\delta((x - x_0)\cdot\theta^{\perp})\delta(\theta - \theta_0)\exp\{-\int_{x_0}^{x} a\,ds\}$$

Single Particle Emission Computed Tomography (SPECT)

$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = f(x)$$
$$u(x,\theta) = 0,\ x \in \Gamma,\ \theta \cdot \nu_x \le 0$$



Inverse problem 1:

Find f from $u(x,\theta),\ x \in \Gamma,\ \theta \in S^1$, a known!

Uniquely solvable by Novikov's inversion formula for the attenuated Radon transform R_a

$$(R_a f)(\theta,s) = \int_{x\cdot\theta=s} f(x)\exp\{-\int_0^\infty a(x+s'\theta)ds'\}dx$$

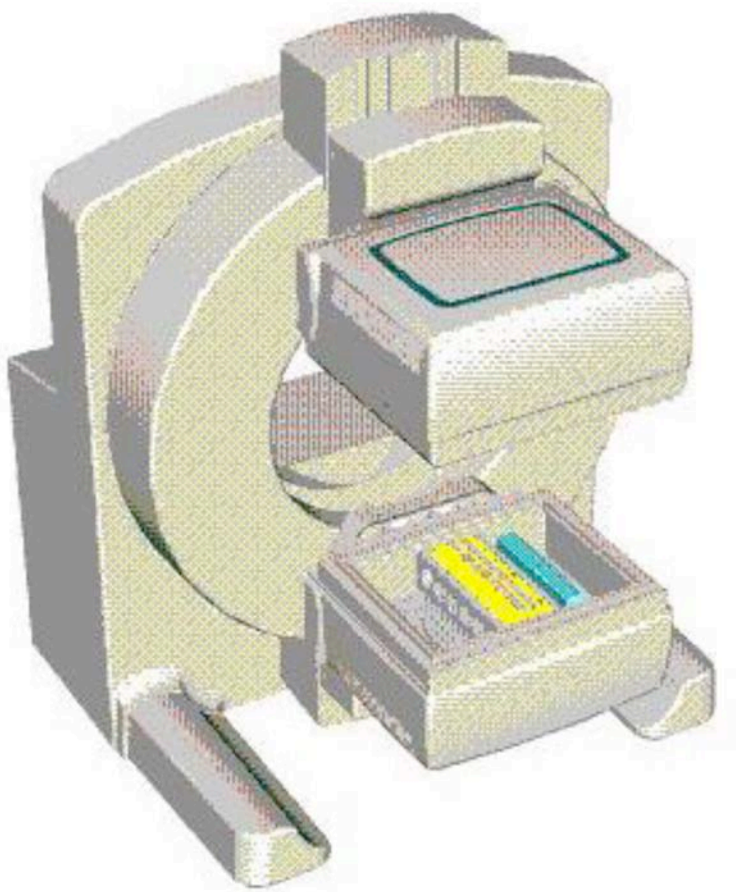
Inverse problem 2:

Find f and a from $u(x,\theta),\ x \in \Gamma,\ \theta \in S^1$

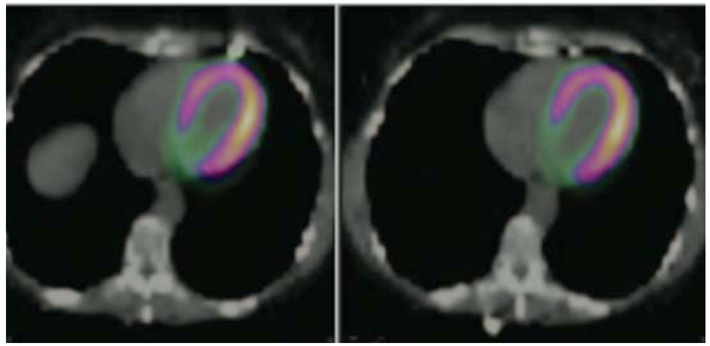
Nonlinear inverse problem, not uniquely solvable

$$u(x,\theta) = \int_{-\infty}^{0} f(x+s\theta)\exp\{-\int_s^0 a(x+s'\theta)ds'\}ds$$

Spect Scanner

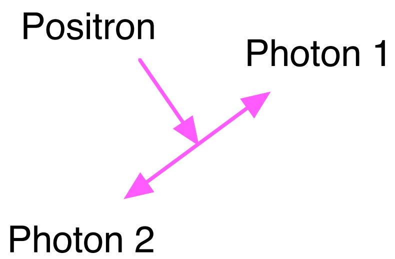


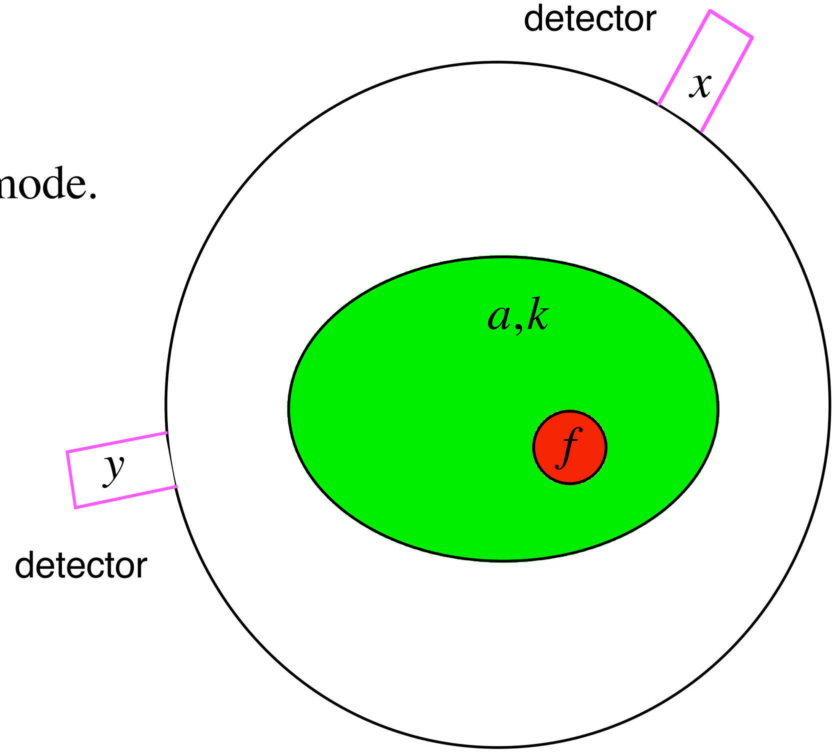
SPECT Images



Positron Emission Tomography (PET)

Detectors x, y work in coincidence mode.
Sources emit particles pairwise in
oposite directions:

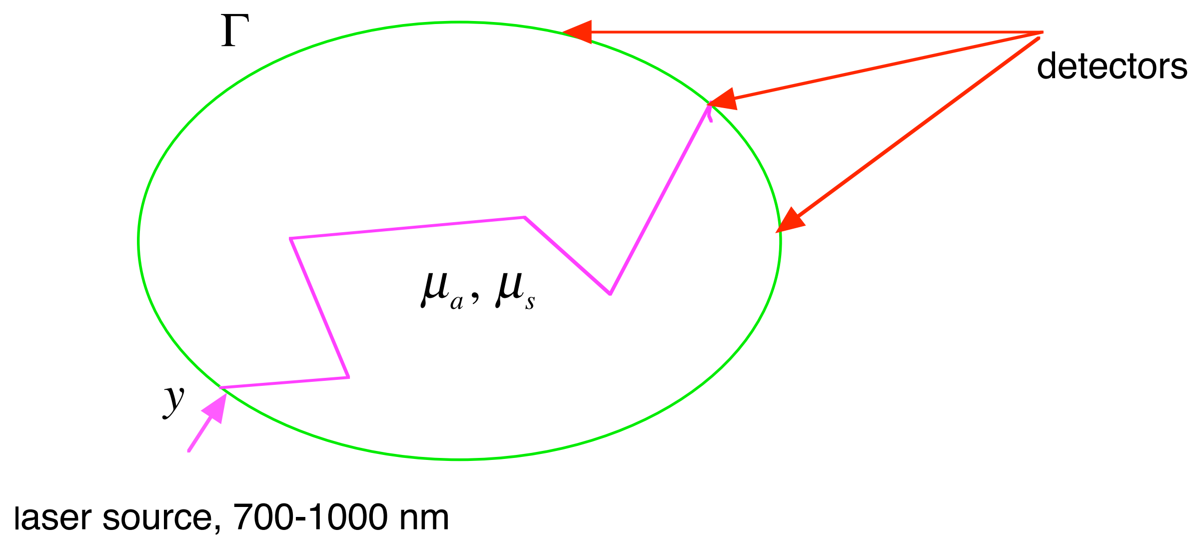




$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = \int_{S^1} k(x,\theta,\theta')u(x,\theta')d\theta' + f(x),$$

$k(x,\theta,\theta') =$ probability that a particle arriving at x with direction θ continues its journey in direction θ'

Optical Tomography - Near Infraread Imaging (NIR)



$$\theta \cdot \nabla u(x,\theta) + (\mu_a(x) + \mu_s(x))u(x,\theta) = \mu_s(x)\int_{S^1} k(x,\theta,\theta')d\theta' + \delta(x-y)$$

Inverse problem: Find μ_a, μ_s from measurements of u(x,θ), x,y $\in \Gamma$

Scattering by far exceeds transport, mean free path < 0.01 mm! - Switch to diffusion approximation!

Optical Tomography in Diffusion Approximation

Put $u(x) = \frac{1}{|S^1|}\int_{S^1} u(x,\theta)d\theta$

$$-\nabla\cdot\big(D(x)\nabla u(x)\big) + (\mu_a(x) + i\frac{\omega}{c})u(x) = 0 \qquad D = 1/3(\mu_a + \mu_s')$$

$$u(x) + 2D(x)\frac{\partial u(x)}{\partial \nu} = g^-(x) = \text{source} \qquad \frac{\partial u(x)}{\partial \nu} = g^+(x) = \text{measurement}$$

Numerically this problem is of the following form:

Suppose we have p sources, $j = 1,\ldots,p$. Put

$$R_j(f) = \frac{\partial u_j}{\partial \nu}, f = (D,\mu_s').$$

Then we have to solve the nonlinear system

$$R_j(f) = g_j^+, j = 1,\ldots,p.$$

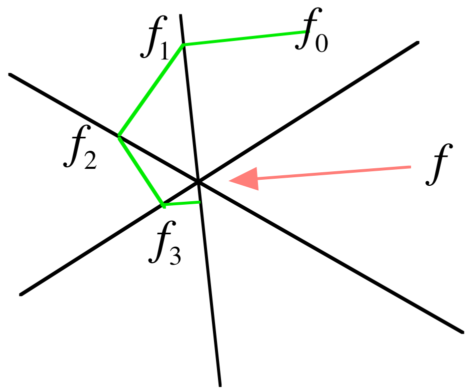
Kaczmarz' Method (Nonlinear)

$$R_j(f) = g_j,\ j = 1,\dots p.$$

We compute approximations $f_j,\ j = 1,2,\dots$ to f according to

$$f_j = f_{j-1} + \alpha (R_j'(f_{j-1}))^*(g_j - R_j(f_{j-1}))$$

take this subscripts mod p

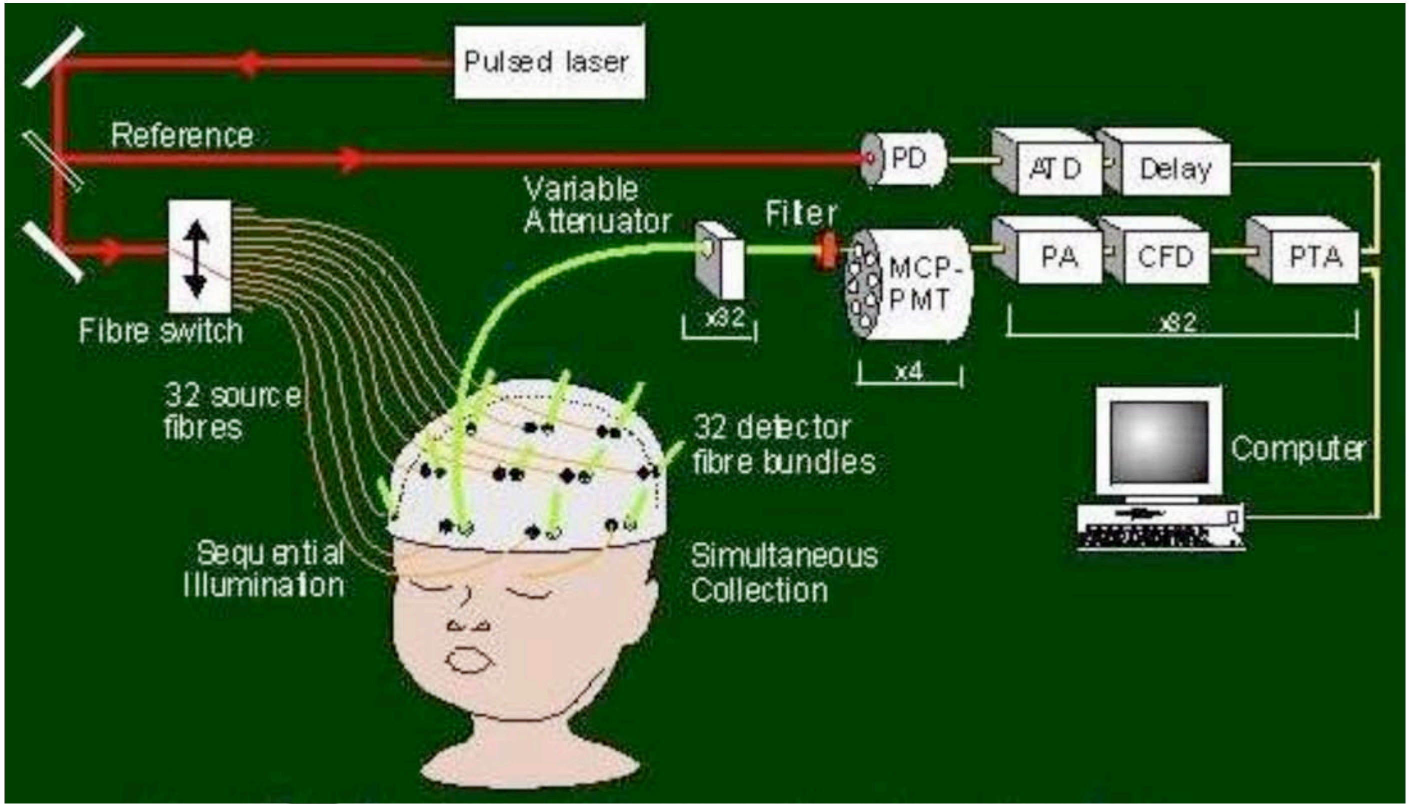


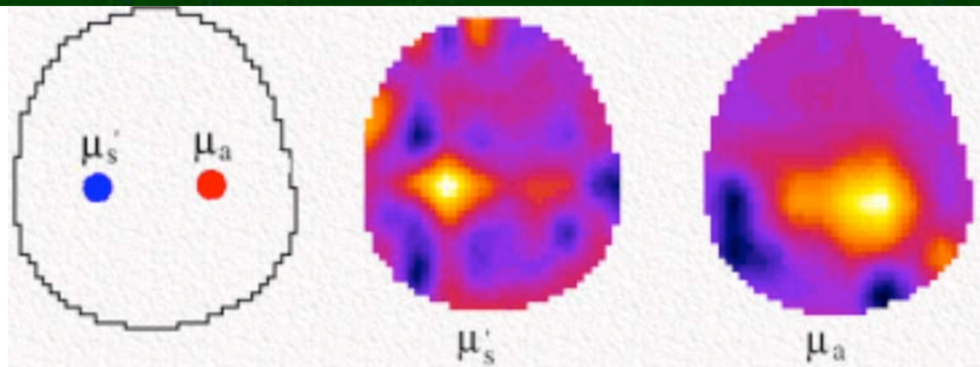
Compute the operator $(R_j'(f))^*$ by adjoint differentiation:

$$R_j'(f)^* r = (-\nabla u_j \cdot \nabla \bar{z},\ -u_j \bar{z})^T$$

$$-\nabla \cdot (D \nabla z) + (\mu_a + i\frac{\omega}{c}) z = 0 \text{ in } \Omega,\ z = \bar{r} \text{ on } \Gamma$$

The Monstir Optical Imaging System (Neonatal Head)





Optical Mamography

Small Animal Imaginç

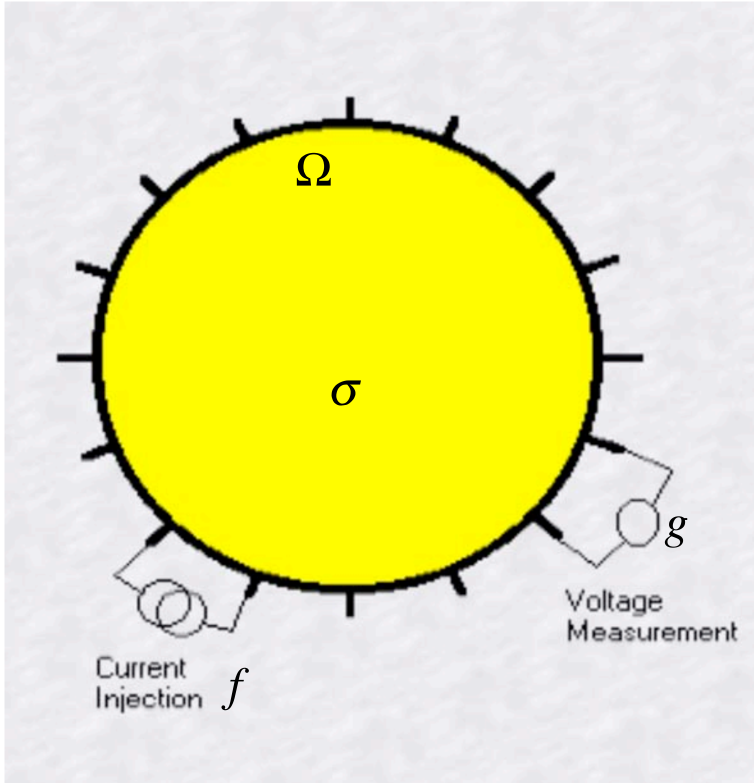
Electrical Impedance Tomography (EIT)

$$\nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega$$

$$\frac{\partial u}{\partial \nu} = f \text{ prescribed on } \partial\Omega$$

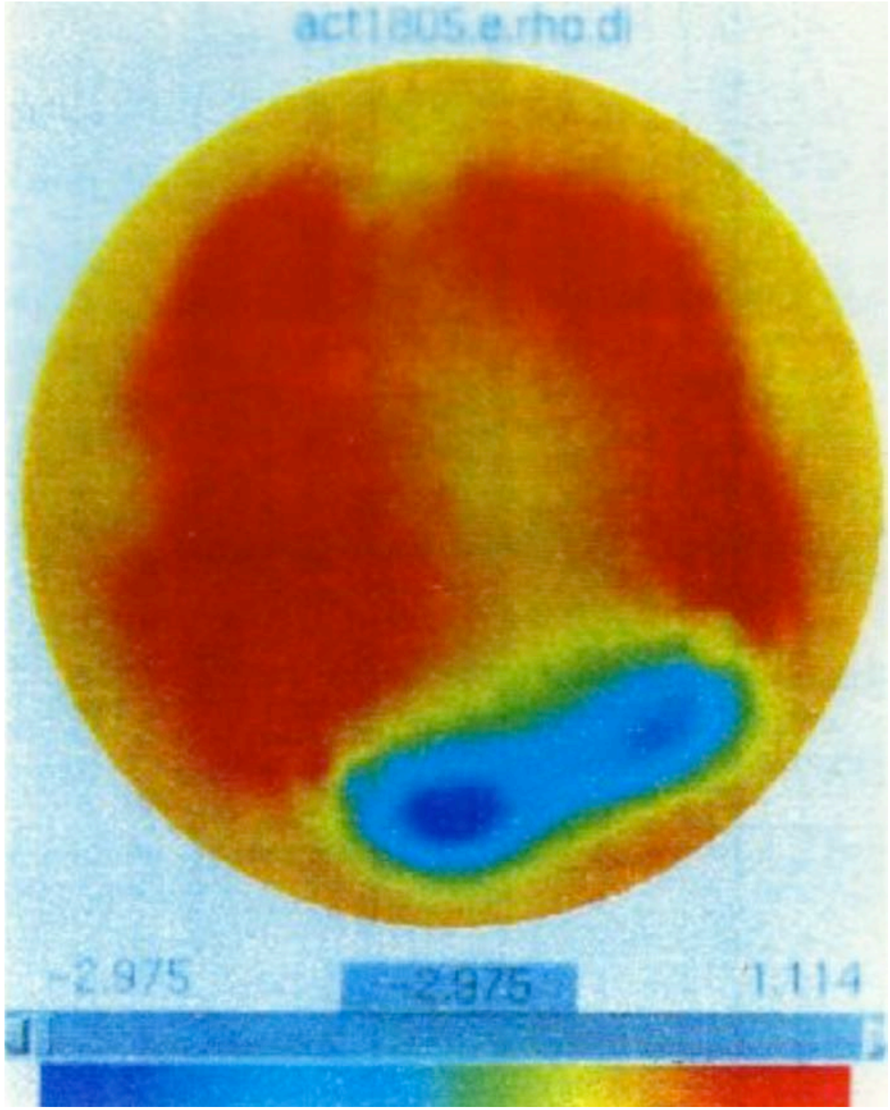
$$u = g \text{ measured on } \partial\Omega$$

$$\sigma = \sigma(x) \text{ conductivity}$$



Inverse problem: Find σ from many pairs f, g.

EIT Image Lungs and Heart



Seismic Imaging

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)(\Delta u + q(t)\delta(x-s))$$

$u = 0,\ t < 0$

c speed of sound, s source

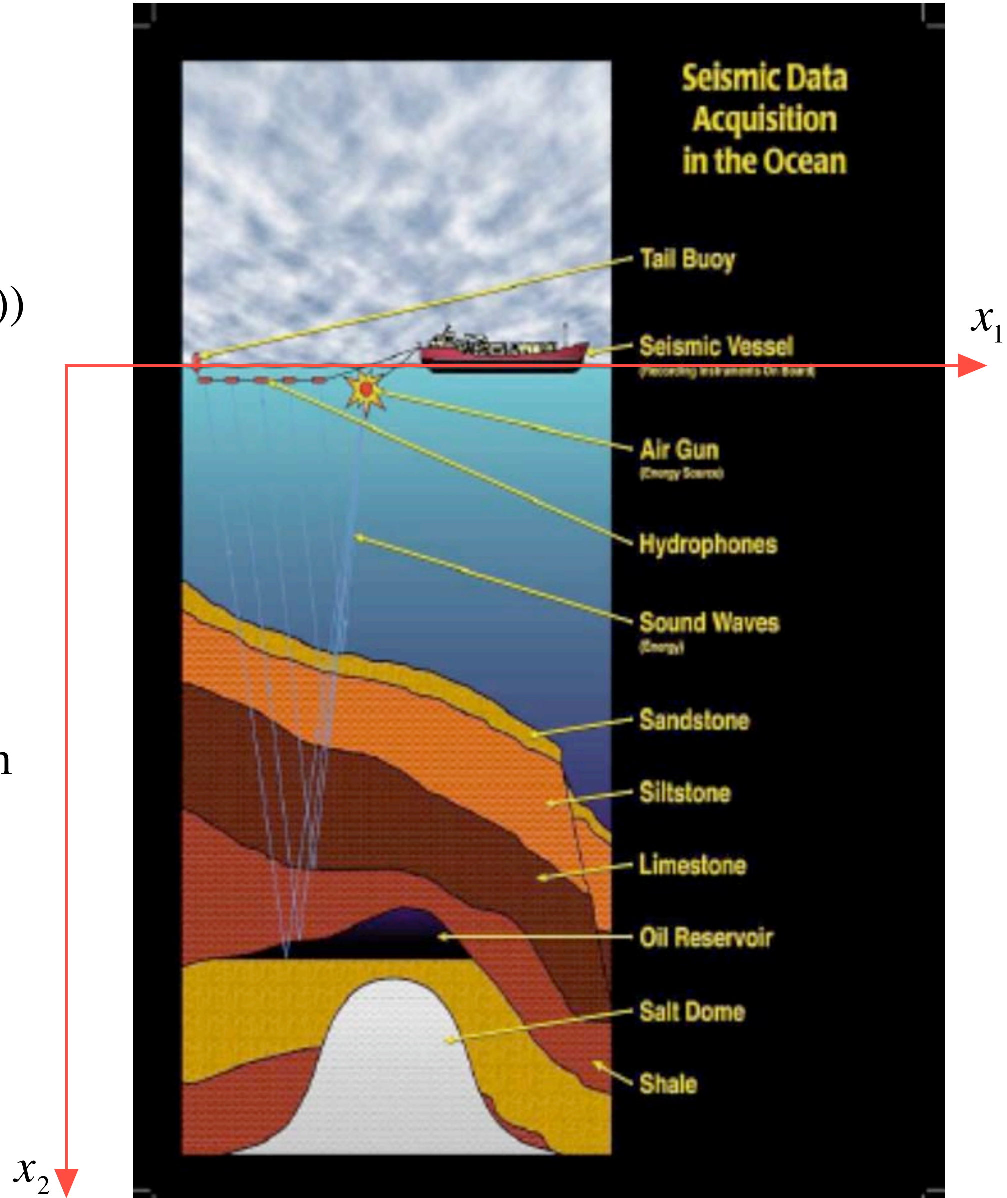
q source wavelet

(common source gather)

Inverse problem: Find c from the seismograms $g_s = R_s(c)$

$R_s(c)(x_1,t) = u(x_1,0,t),$

$x_1 \in R^1,\ 0 < t < T$



High Frequency Imaging

$$c = c_0 + c_1$$

smooth known background

small high frequency perturbation

Linearization: $R_s(c_0 + c_1) \approx R_s(c_0) + R_s'(c_0)\, c_1$

$$g_s - R_s(c_0) \approx R_s'(c_0)\, c_1$$

$$R_s'(c_0)^*(g_s - R_s(c_0)) \approx R_s'(\mathrm{c}_0)^* R_s'(c_0)\, c_1$$

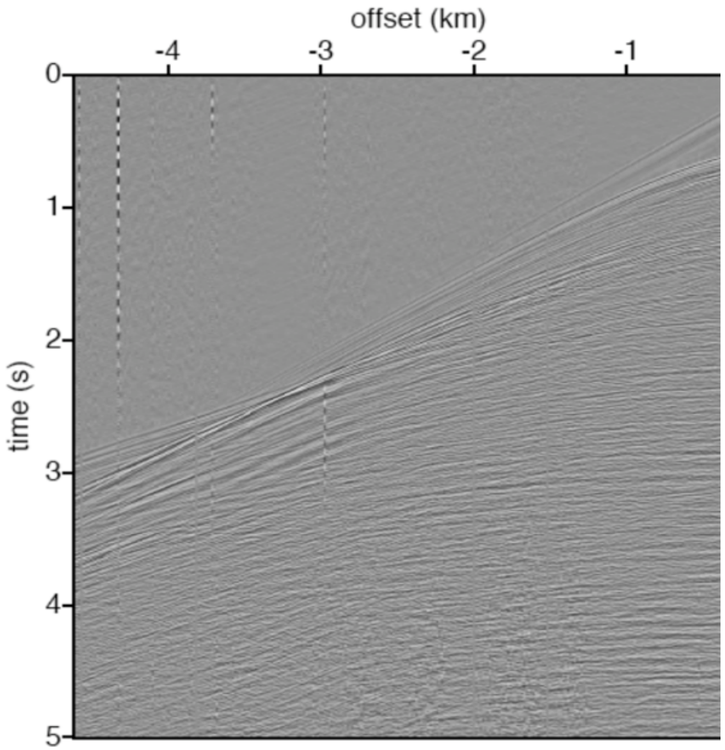
$$\sum_s R_s'(c_0)^*(g_s - R_s(c_0)) \approx \sum_s R_s'(\mathrm{c}_0)^* R_s'(c_0)\, c_1$$

reconstruction

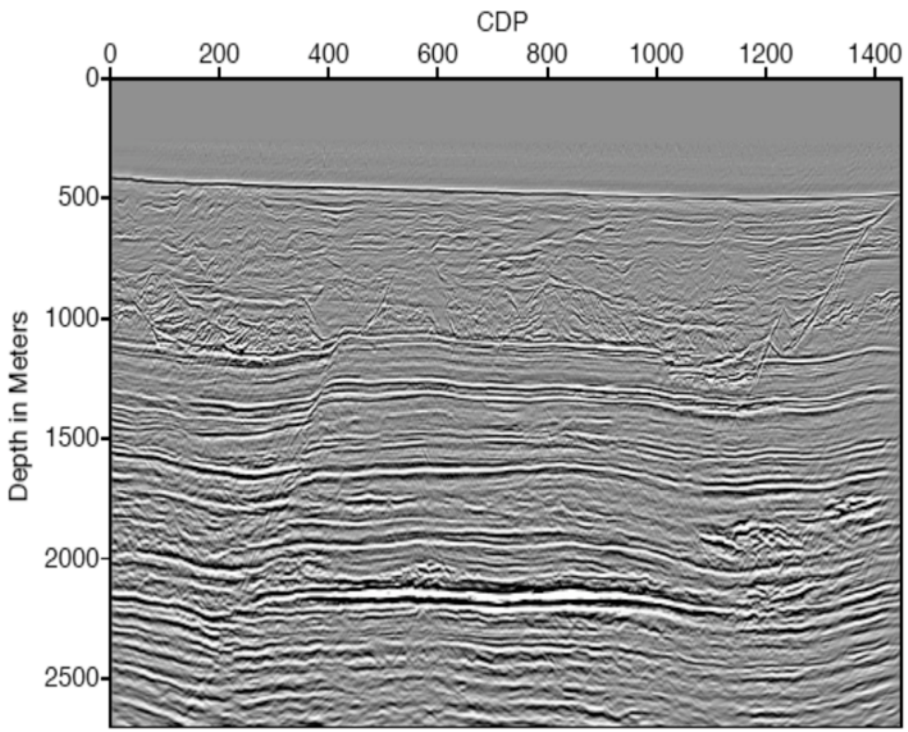
elliptic pseudodifferential operator,
preserves singular support

The reconstruction has the same singular support as the correct velocity!

Wave Equation Migration



seismogram



reconstructed velocity
= migrated seismogram

Kaczmarz' Method in Seismic Imaging

$$R_s(c) = u\big|_{x_2=0} = g_s = \text{ seismogram for source } s$$

For each source s

$$c \leftarrow c + \alpha (R_s'(c))^* (g_s - R_s(c))$$

Compute the adjoint by time reversal:

$$(R_s'(c))^* r)(x) = \int_0^T z(x,t) \frac{\partial^2 u(x,t)}{\partial t^2} dt$$

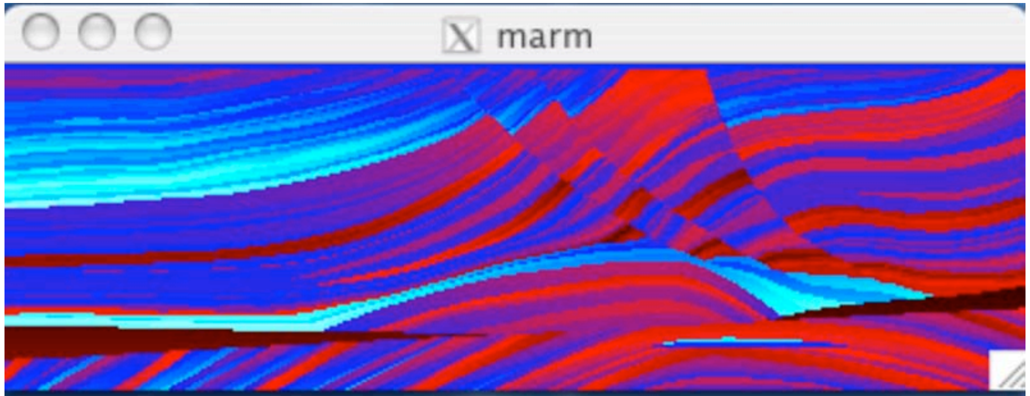
$$\frac{\partial^2 z}{\partial t} = c^2(x) \Delta z \text{ for } x_2 > 0$$

$$\frac{\partial z}{\partial x_2} = r \text{ on } x_2 = 0$$

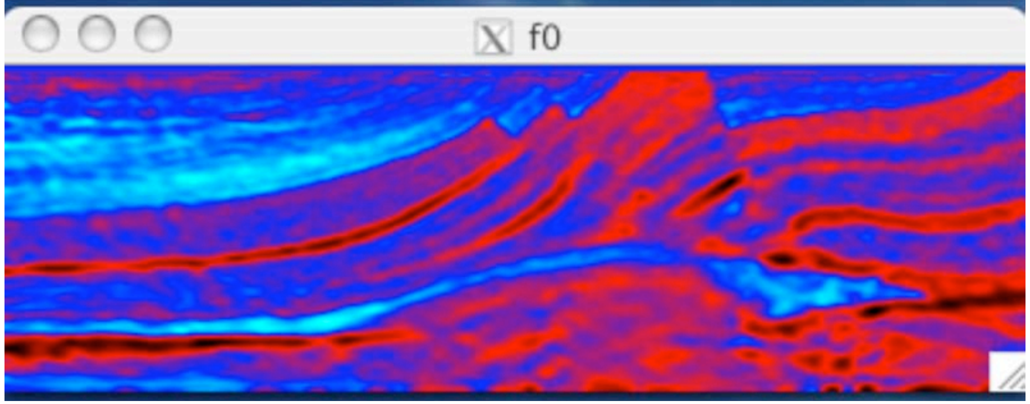
$$z = 0,\ t > T$$

Kaczmarz‘ method for the Marmousi Velocity Model

Original



Reconstruction



Works only for wavelets q that contain frequencies near zero -
unless we have transmission measurements.

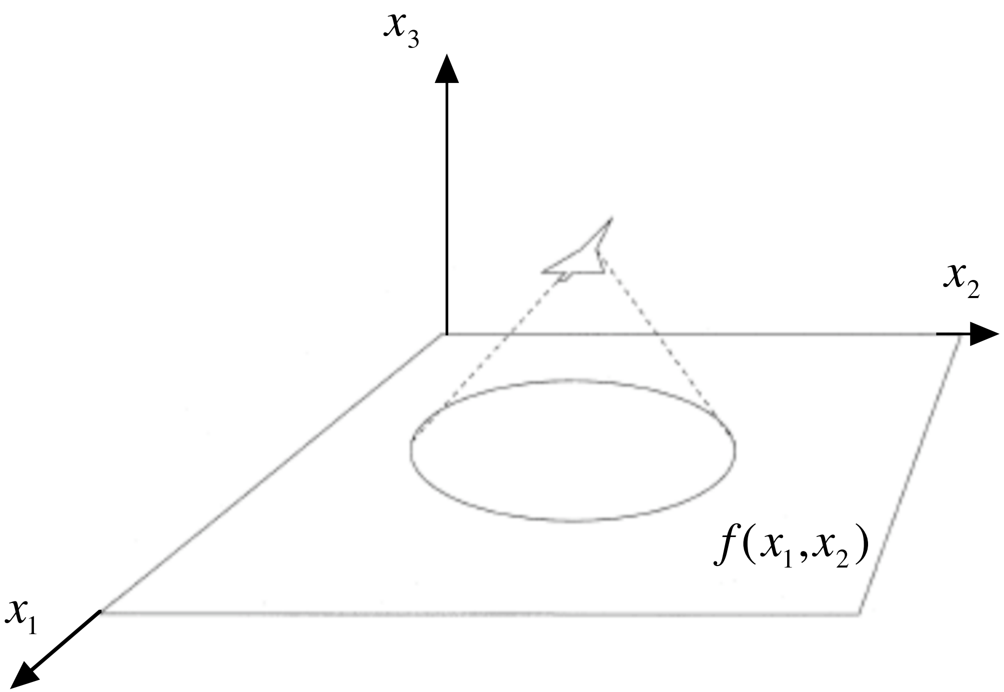
Synthetic Aperture Radar (SAR)

$$\frac{\partial^2 u}{\partial t^2} = c^2(\Delta u + q(t)\delta(x-y))$$

$$\frac{1}{c^2(x)} = \frac{1}{c_0^2} + f(x_1,x_2)\delta(x_3)$$

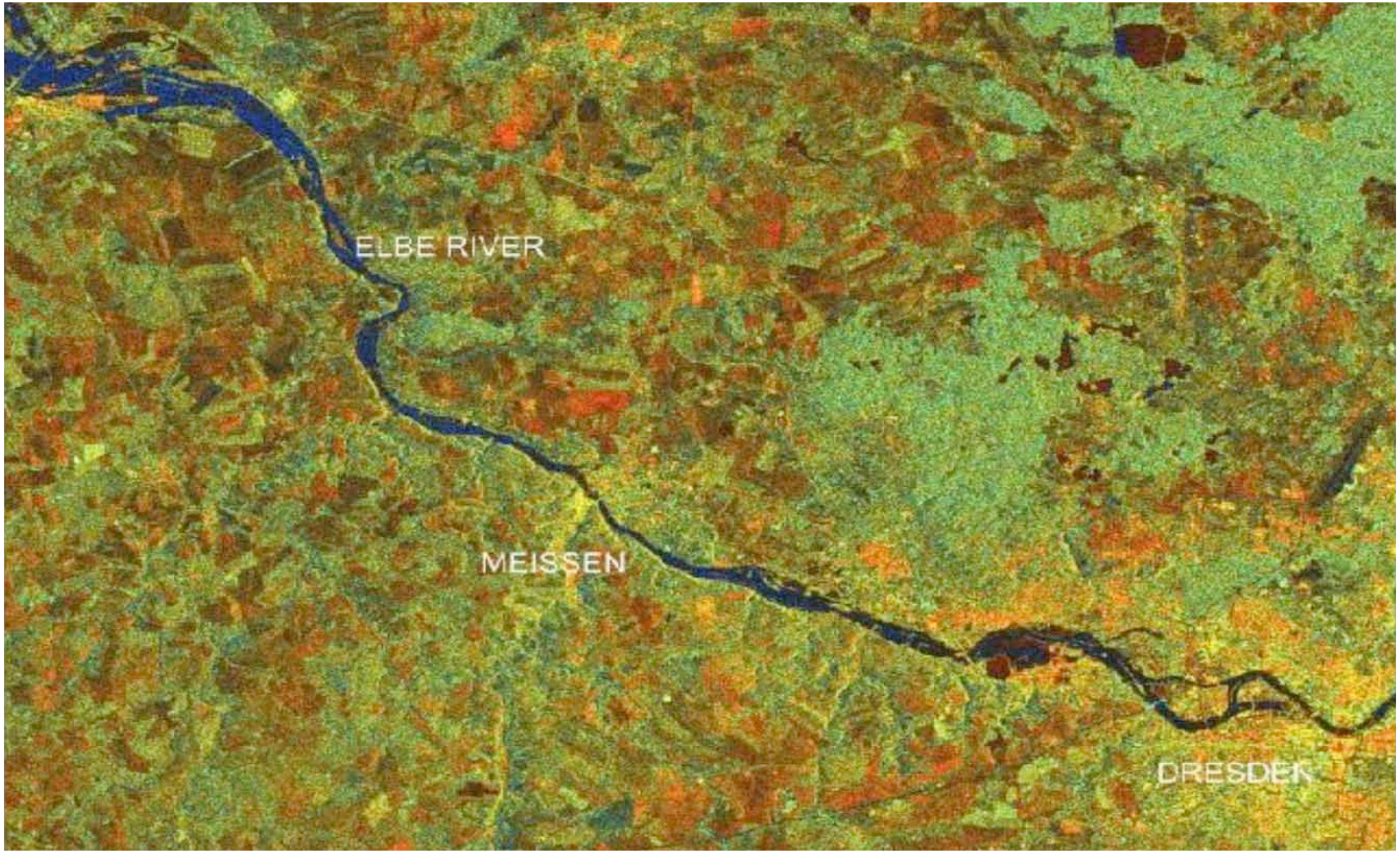
$$q(t) = Q(t)\exp(i\omega t)$$

f ground reflectivity function

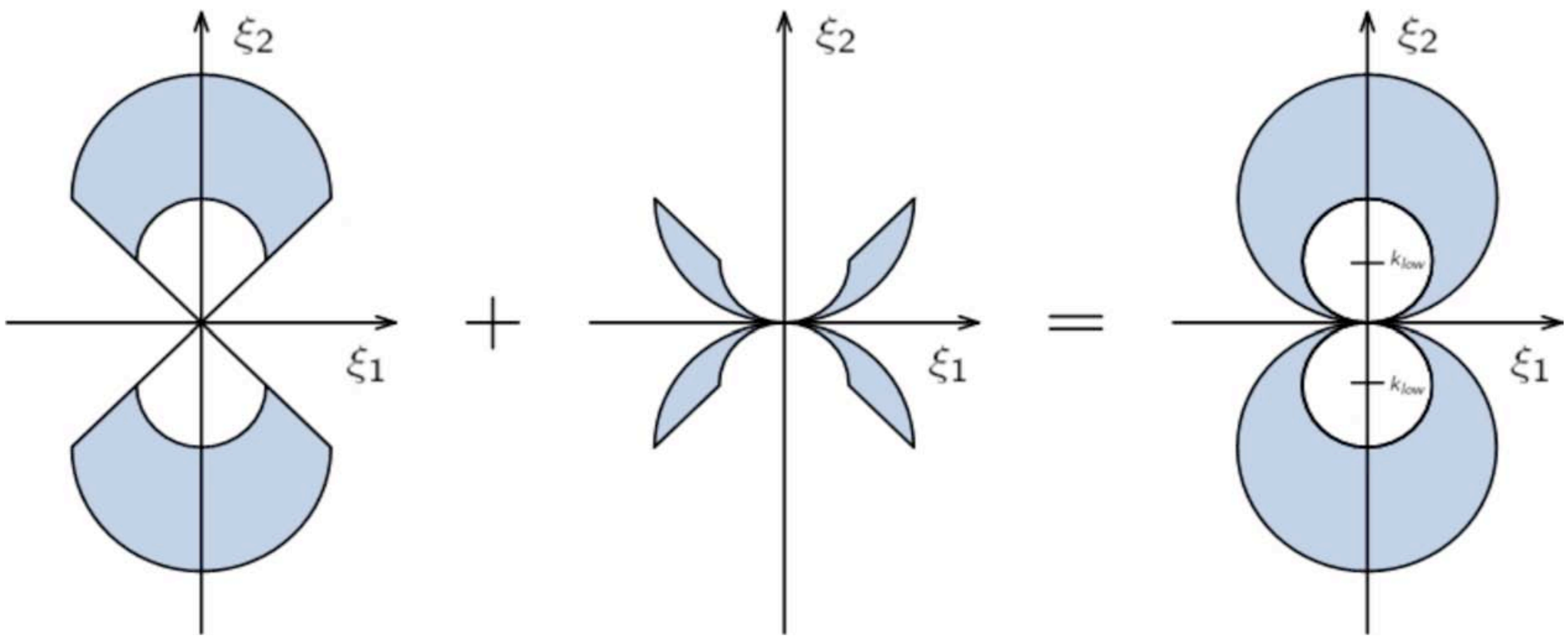


Inverse problem: Find f from $(R_y(f))(t) = u(y,t)$, y on the flight track, $t > 0$

SAR Image of Elbe River Valley (ESA ASAR)

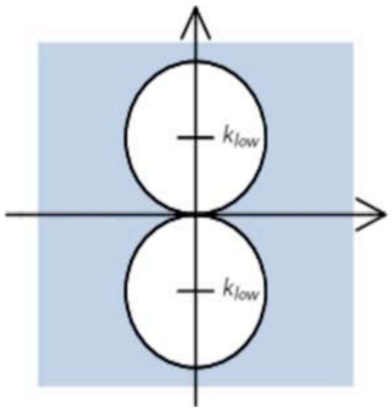


Fourier Analysis of Reflection/Transmission Imaging

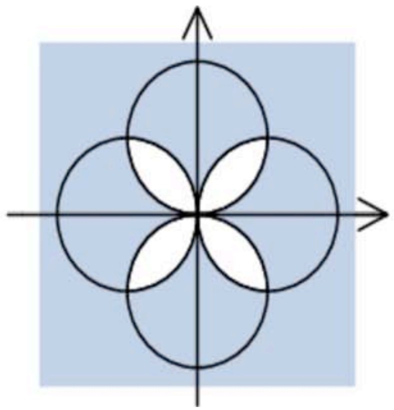


Reflection Transmission Combined

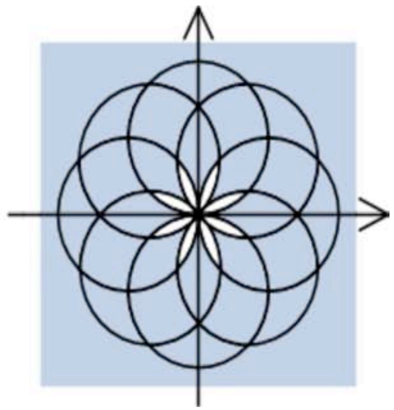
Fourier Coverage for several incoming waves



1 wave

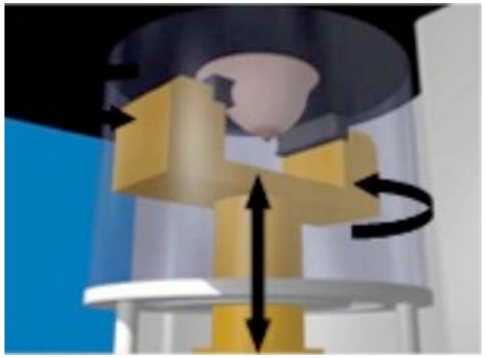


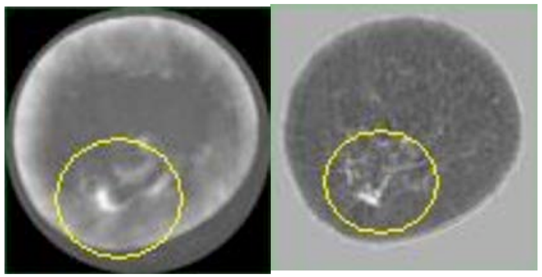
2 waves



4 waves

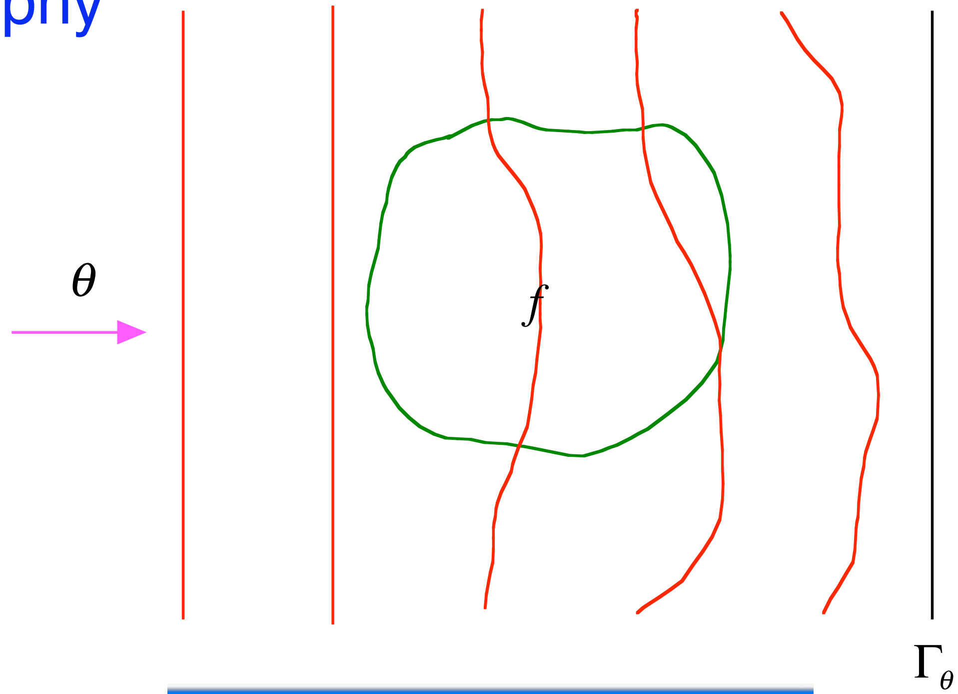
Ultrasound Tomography





Ultrasound Tomography

$$\Delta u(x) + k^2(1 + f(x))u(x) = 0,$$
$$u(x) = \exp(ikx\cdot\theta) + u_s(x).$$



Inverse problem: Find f from $u(x)$ for Γ_θ, $\theta \in S^1$

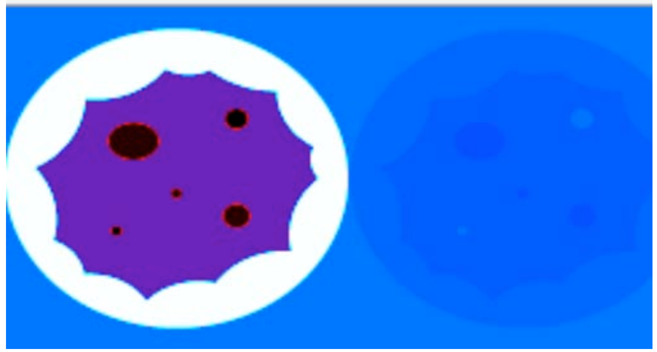
$$f(x) = \frac{c_0^2}{c^2} - 1 - \frac{i}{k}\frac{2\alpha c_0}{c},$$

$c = c(x)$ local speed of sound

c_0 speed of sound in ambient medium

$\alpha = \alpha(x)$ attenuation

$\mathrm{k} = \omega / \mathrm{c}_0$ wavenumber



$\Re f$ $\Im f$

Role of Parameter k

1. k controls spatial resolution. $\hat{f}$ is STABLY determined in the ball of radius $2k$ around origin. Spatial resolution $\pi / k = 0.75$ mm for 1MHz.

2. k large makes it difficult to solve the boundary value problem for the Helmholtz equation numerically.

Solve the Helmholtz equation by initial value techniques!

Initial Value Problem for the Helmholtz Equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + k^2 u = 0 \qquad u(x_1,0) = u_0(x_1),\ \frac{\partial u}{\partial x_2}(x_1,0) = u_1(x_1)$$

Fourier transform with respect to x_1:

$$\hat{u}(\xi_1,x_2) = (2\pi)^{-1/2} \int \exp(-i x_1 \xi_1) u(x_1,x_2)\,dx_1$$

Ordinary differential equation in x_2:

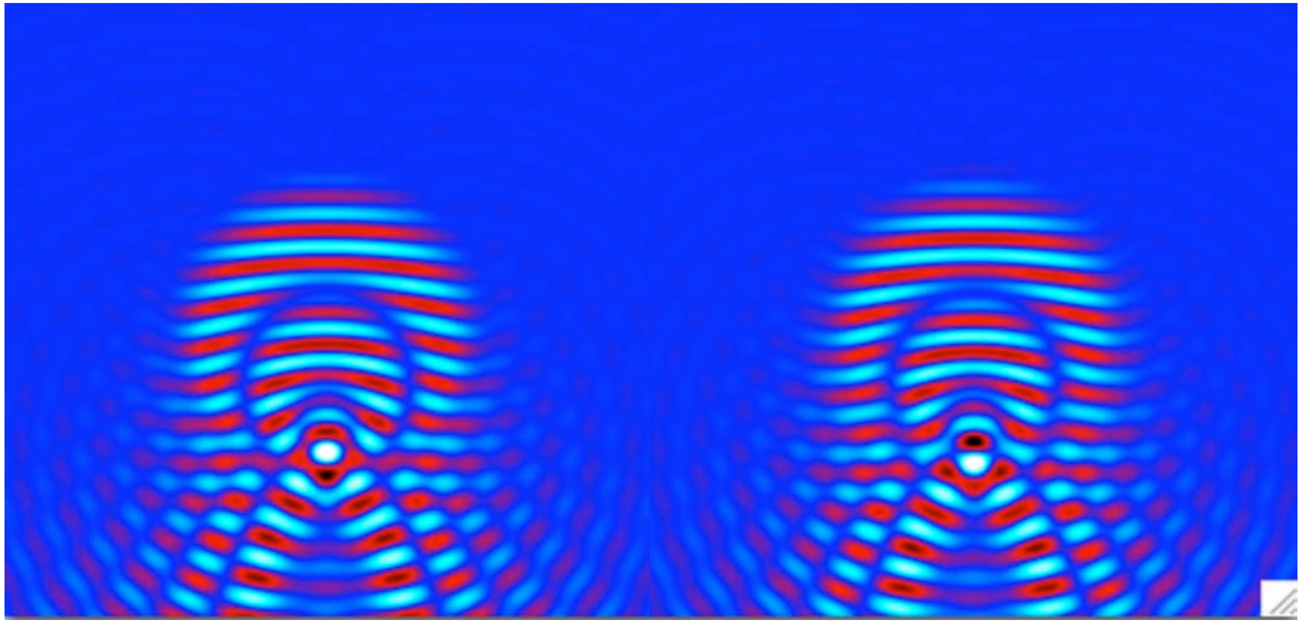
$$\frac{d^2 \hat{u}(\xi_1,x_2)}{dx_2^2} + (k^2 - \xi_1^2)\hat{u}(\xi_1,x_2) = 0$$

Solution:

$$\hat{u}(\xi_1,x_2) = \hat{u}_0(\xi_1)\cos(\kappa(\xi_1)x_2) + \frac{\hat{u}_1(\xi_1)}{\kappa(\xi_1)}\sin(\kappa(\xi_1)x_2),\ \kappa(\xi_1) = \sqrt{k^2 - \xi_1^2}$$

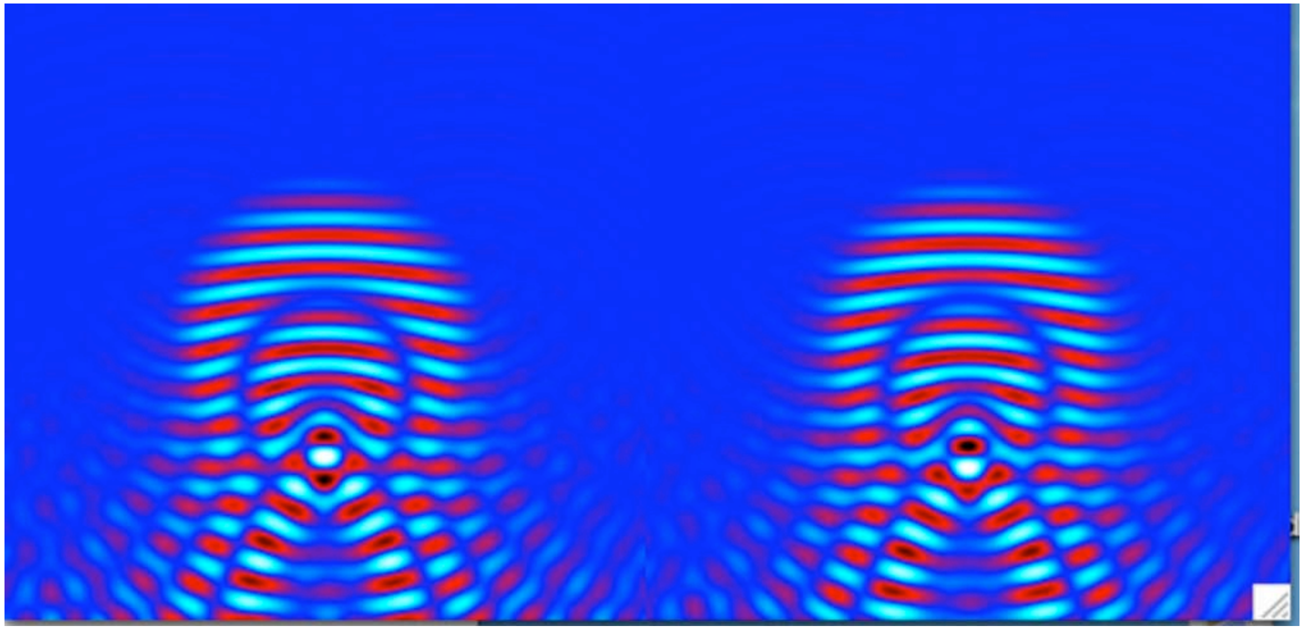
Stable as long as $\xi_1^2 \le k^2$

Exact (finite difference time domain, followed by Fourier transform



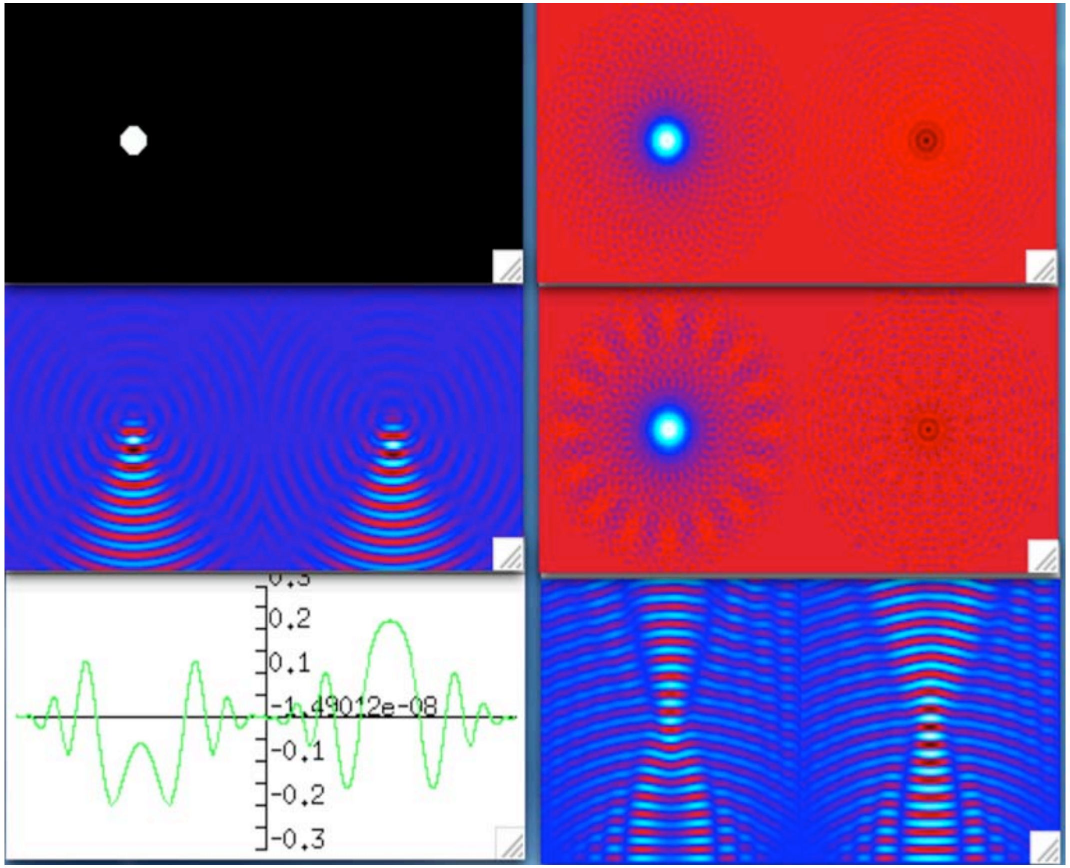
LUNEBERG LENSE

Initial value technique



Kaczmarz‘ Method for Ultrasound Tomography

$\Re$ $\Im$ $\Re$ $\Im$



object

32 superimposed backpropagated fields

scattered field

16 superimposed backpropagated fields

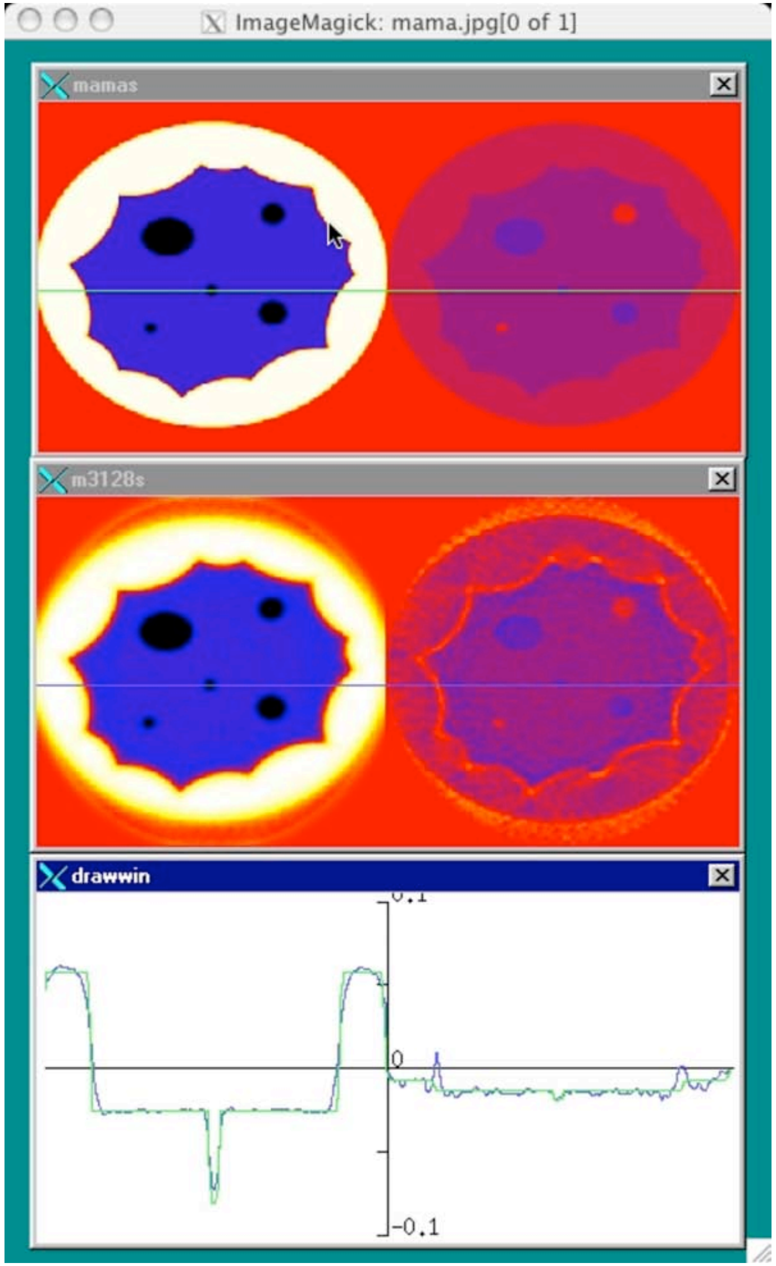
data

time reversed backpropagated field

Original

Reconstruction

Cross section



Concluding Remarks

Behind each imaging technology a differential equation is lurking

Image quality depends on the type of the differential equation

Kaczmarz‘ method intuitive paradigm for reconstruction algorithms