Dimension reduction of dynamic super-resolution

Alexander Schlüter, collab. Benedikt Wirth, Martin Holler

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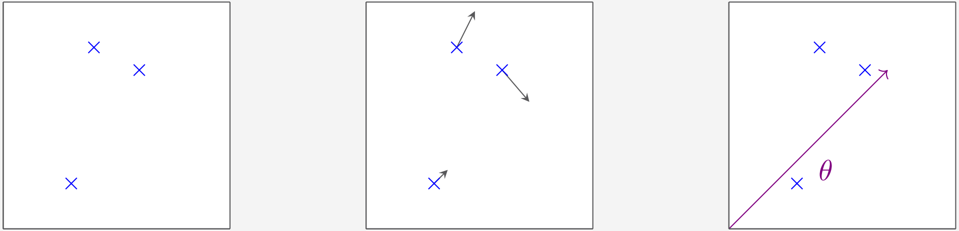
Outline



Static → Dynamic → Dimension reduction

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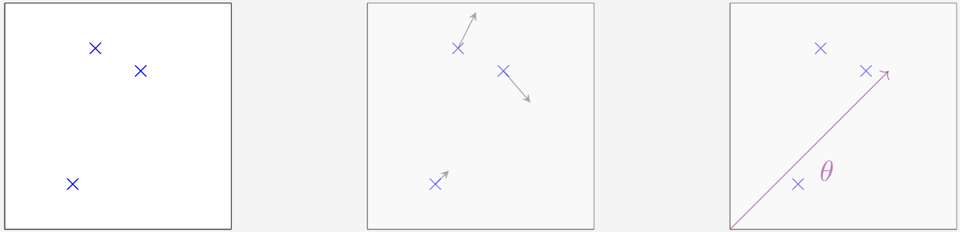
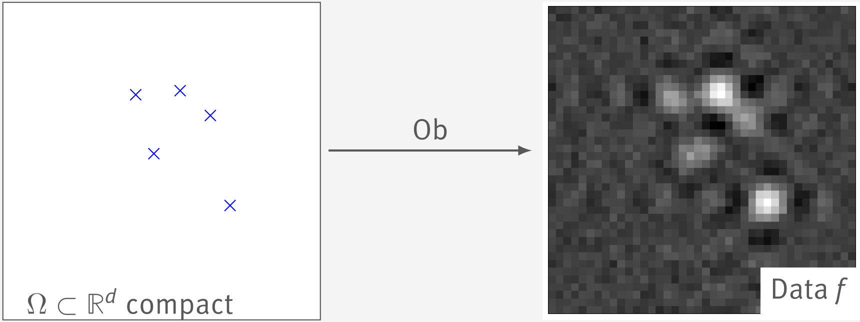


Static → Dynamic → Dimension reduction

Spike super-resolution



Collection of point sources (stars, fluorescent molecules, cells, ...)

Measurement filters out fine scale information, adds noise

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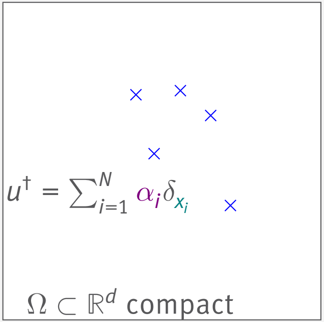
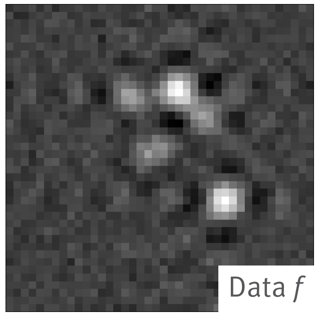


$\xrightarrow{\text{Ob}}$



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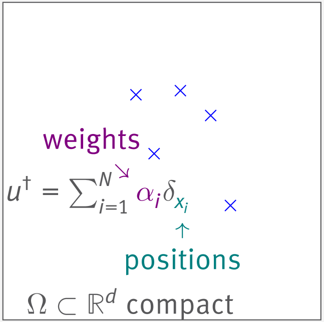

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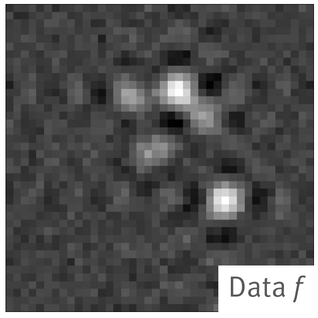

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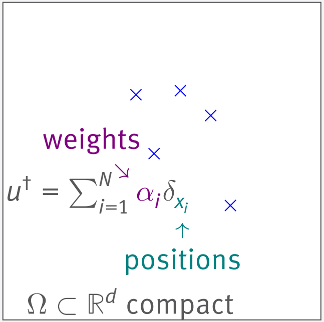
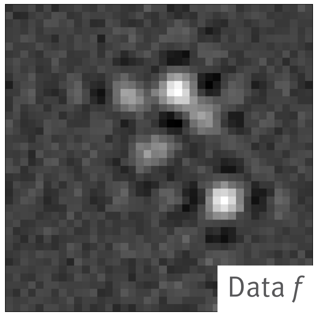


Ob

Reconstruction?



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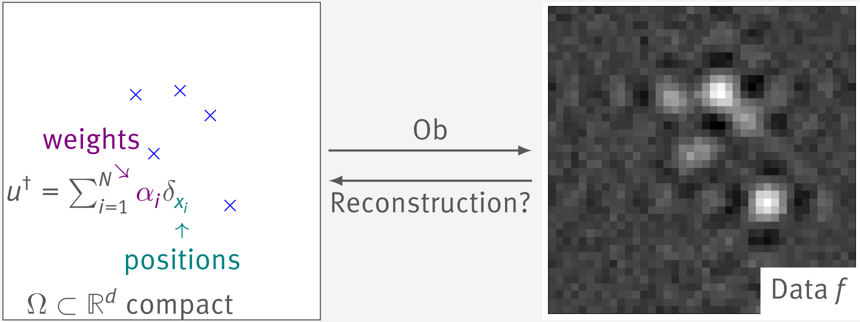

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Ideal frequency filter: $\Omega = [0,1]^d$, cutoff frequency $f_c \in \mathbb{N}$, Ob $= \mathcal{F}$ where

$$\mathcal{F}u = \left(\int_{[0,1]^d} e^{-2\pi i l \cdot x} \, \mathrm{d}u(x) \right)_{\substack{l \in \mathbb{Z}^d \\ \|l\|_\infty \leq f_c}}$$

Convex optimization

How can we reconstruct $u^\dagger = \sum_{i=1}^N \alpha_i \delta_{x_i}$ from data $f^\dagger = \mathrm{Ob} u^\dagger$?

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Consider the convex optimization problem

$$\min_{u \in \mathcal{M}(\Omega)} \|u\|_{\mathrm{TV}} \quad \text{s.t.} \quad \mathrm{Ob}u = f^\dagger \tag{ER}$$

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Consider the convex optimization problem

$$\underbrace{\min_{u \in \mathcal{M}(\Omega)}}_{\text{space of Radon measures}} \underbrace{\|u\|_{\mathrm{TV}}}_{\text{total variation norm of a measure}} \quad \text{s.t.} \quad \mathrm{Ob}u = f^\dagger \tag{ER}$$

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space of Radon measures

total variation norm of a measure

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- If $u = \sum_{i=1}^{N} \alpha_i \delta_{x_i}$, then $\|u\|_{\mathrm{TV}} = \sum_{i=1}^{N} |\alpha_i| = \|\alpha\|_1$
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- Continuous analog of ℓ^1-norm
- Convex, induces sparsity of solutions

Exact reconstruction

Goal: Prove that $u^\dagger$ is the unique solution to

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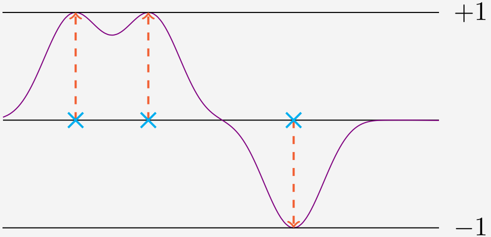
Candès, Fernandez-Granda: True if for every $\eta \in \mathbb{C}^N$ with $|\eta_i| = 1$, there exists a dual certificate

$$q(x) = \sum_{\|l\|_\infty \le f_c} c_l e^{2\pi i l\cdot x} \quad \text{such that} \quad \begin{cases} q(x_i) = \eta_i, & i = 1,\dots,N, \\ |q(x)| < 1, & x \in \Omega \setminus \{x_1,\dots,x_N\} . \end{cases}$$



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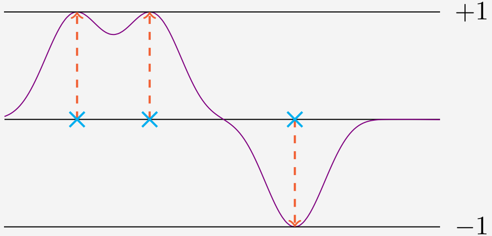
The low-frequency polynomial q interpolates between the signs in η.



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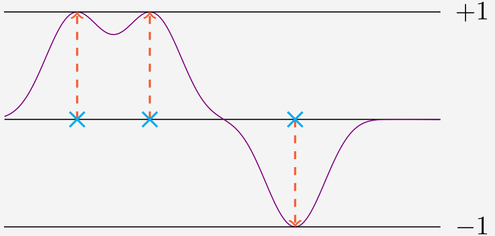
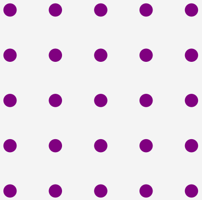
Thm.: Certicates exist as long as the positions $\{x_i\}$ are well separated.



Numerics: Grid-based vs. off-the-grid

Grid-based:



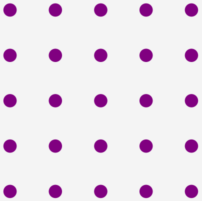
Grid points $\{p_1, \ldots, p_L\} \subset \Omega$

Numerics: Grid-based vs. off-the-grid

Grid-based: Solve

$$\min_{\xi \in \mathbb{C}^L} \|\xi\|_1 \quad \text{s.t.} \quad M\xi = f^\dagger$$

with measurement matrix
$M := (\mathrm{Ob}\delta_{p_1}, \dots, \mathrm{Ob}\delta_{p_L})$



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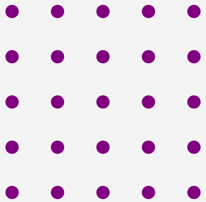

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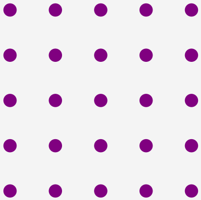

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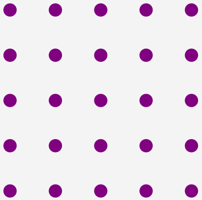

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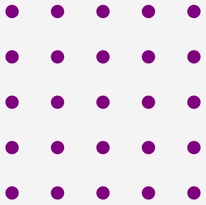

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Off-the-grid example: Alternating Descent Conditional Gradient method (ADCG), alternates between

× = True srces | Add source | Smooth opt. | Add source | Smooth opt.

supp = $\{\ \}$ | supp = $\{\bigcirc\}$ | supp = $\{\bigcirc\}$ | supp = $\{\bigcirc, \bigcirc\}$ | supp = $\{\bigcirc, \bigcirc\}$

[Boyd et al. 2015]

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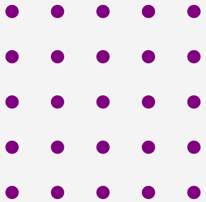

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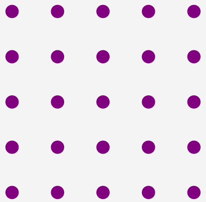

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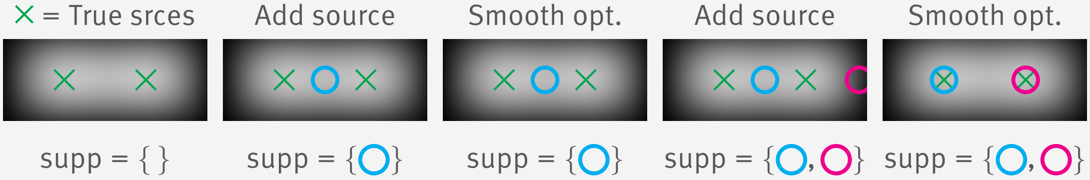
Off-the-grid example: Alternating Descent Conditional Gradient method (ADCG), alternates between

- adding new source points to solve global convex problem
- performing local, differentiable optimization on positions and weights



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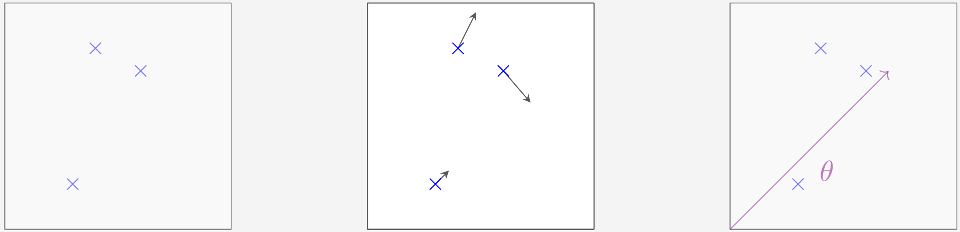
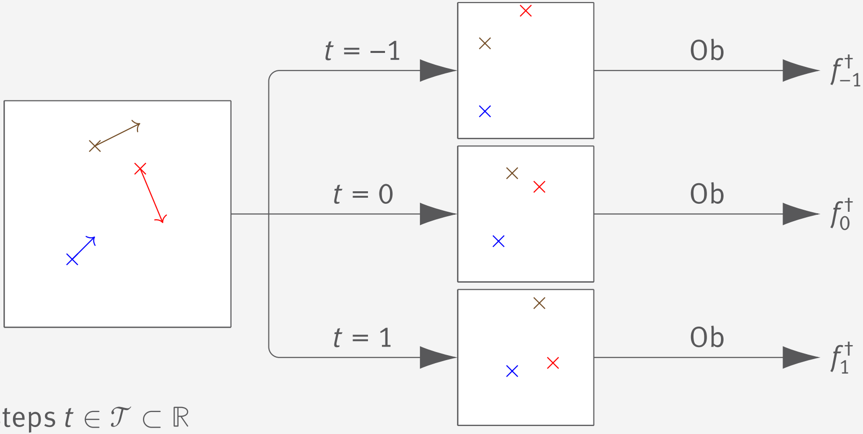


Static

Dynamic

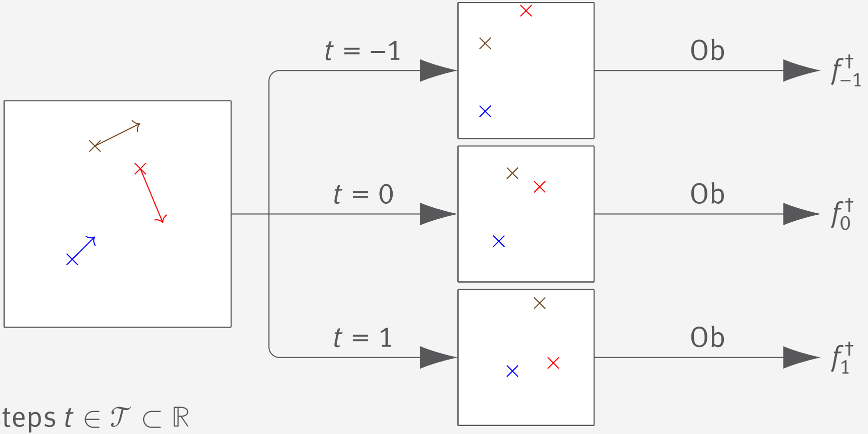
Dimension reduction

Dynamic super-resolution



- Time steps $t \in \mathcal{T} \subset \mathbb{R}$

Dynamic super-resolution



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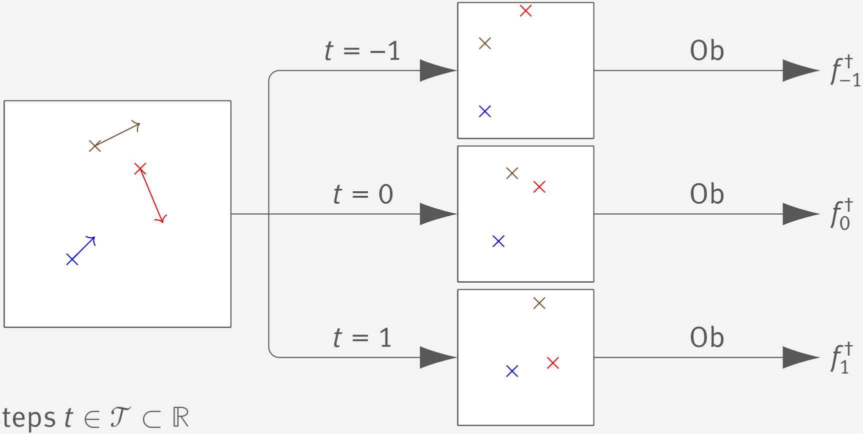

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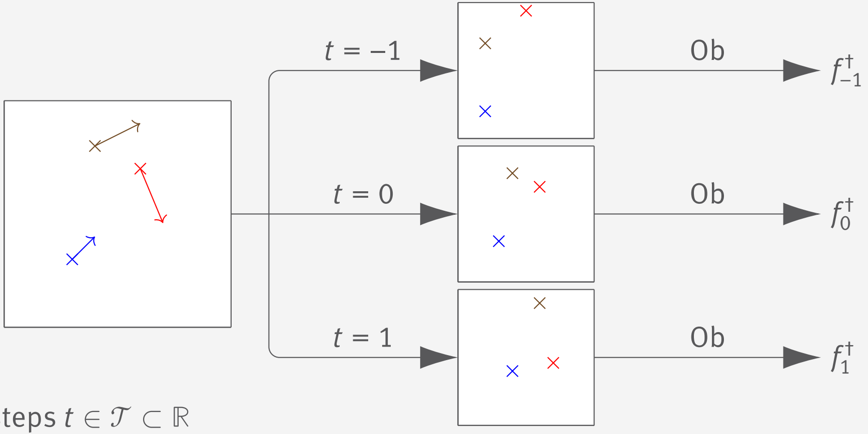

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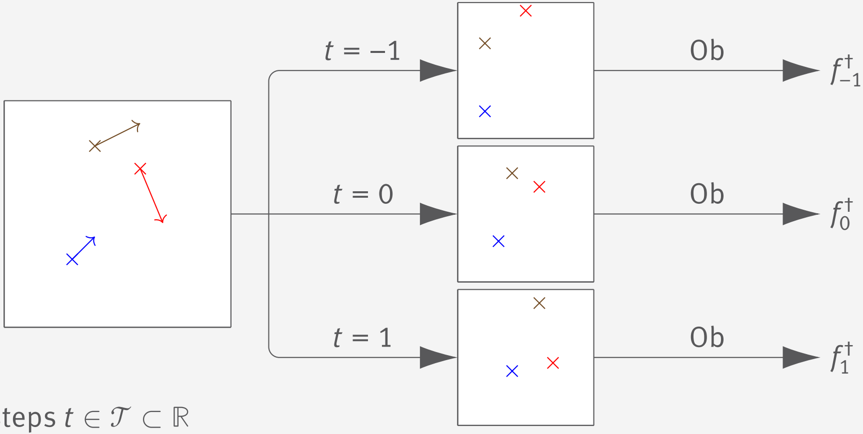
Goals:

Dynamic super-resolution



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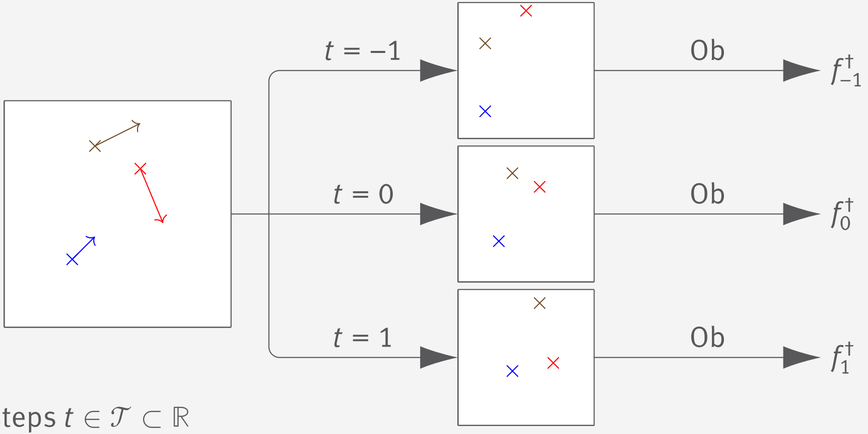

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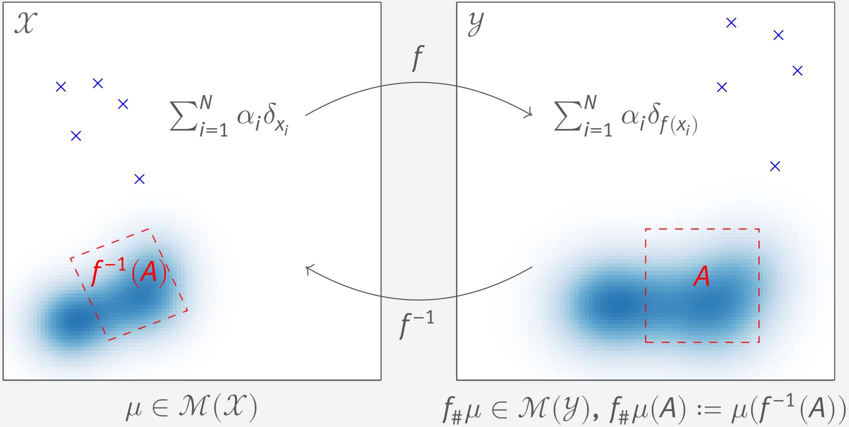
Goals:

- Improve reconstruction by combining information from multiple measurements
- Reconstruction of velocities

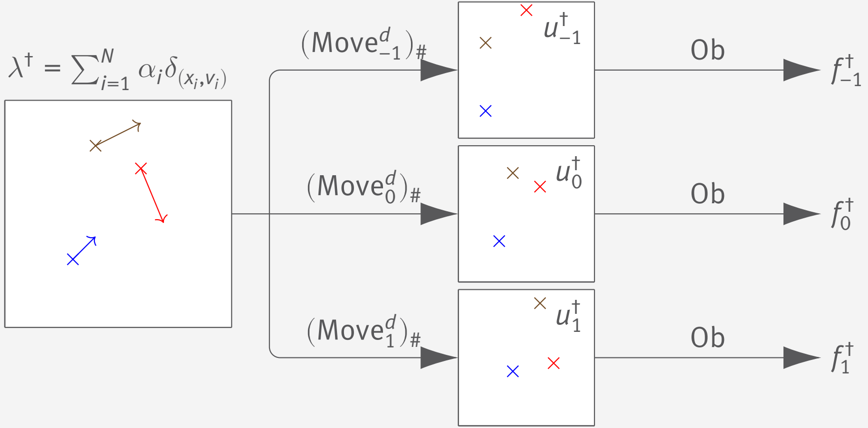
Pushforward



The pushforward transports mass along f.

Phase space lifting

Idea (Alberti et al.): Change to a representation in phase space.

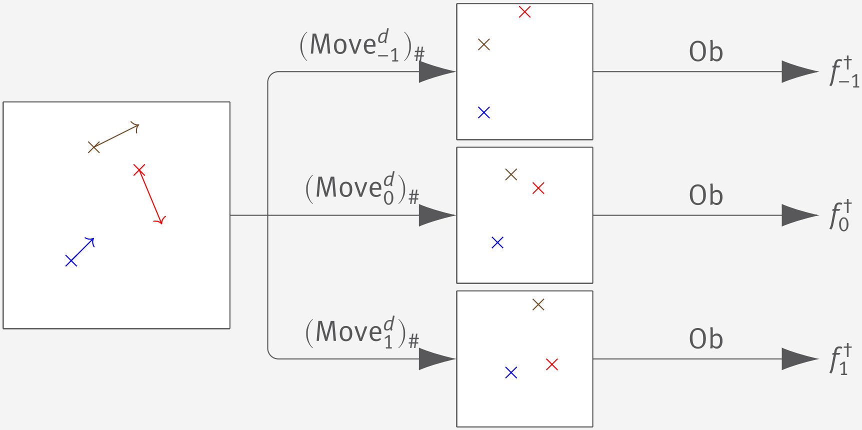


$$\begin{aligned}\mathrm{Move}^d_t \colon \mathbb{R}^d \times \mathbb{R}^d &\to \mathbb{R}^d\\ (x,v) &\mapsto x + tv\end{aligned}$$

$$\begin{aligned}u^\dagger_t &:= (\mathrm{Move}^d_t)_\# \lambda^\dagger\\ &= \sum_{i=1}^N \alpha_i \delta_{x_i + tv_i}\end{aligned}$$

$$f^\dagger_t := \mathrm{Ob}\, u^\dagger_t$$

Phase space lifting



Full-dimensional model by Alberti et al.:

$$\min_{\lambda\in\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)} \|\lambda\|_{\mathrm{TV}} \quad \text{subject to} \quad \mathrm{Ob}(\mathrm{Move}^d_t)_\#\lambda = f_t^\dagger \ \forall t\in\mathcal{T} \tag{ERdyn}$$

Dynamic vs. static

- Dynamical reconstruction:

$$\min_{\lambda \in \mathcal{M}(\Omega_{\text{dyn}})} \|\lambda\|_{\text{TV}} \quad \text{s.t.} \quad \text{Ob}(\text{Move}_t^d)_\# \lambda = f_t^\dagger \ \forall t \in \mathcal{T} \tag{ERdyn}$$

- Static reconstruction for $t \in \mathcal{T}$:

$$\min_{u \in \mathcal{M}([0,1]^d)} \|u\|_{\text{TV}} \quad \text{s.t.} \quad \text{Ob}u = f_t^\dagger \tag{ERt}$$

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- **Assume that dual certificates exist for the static problems** (ERt) **for some time steps** t (e.g. the particles are well-separated at those times)

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Question

What can we infer about solutions to the dynamical reconstruction problem?

Dynamical reconstruction

Theorem (Alberti, Ammari, Romero, Wintz 2019)

Let $\{(x_i, v_i)\}_{i=1}^N \subset \Omega$ *be a configuration of N particles,* $\alpha \in \mathbb{C}^N$ *a vector of weights and* $\mathcal{T}' \subset \mathcal{T}$ *a subset of time steps with* $|\mathcal{T}'| \geq 3$.

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Consider the target measure

$$\lambda^\dagger = \sum_{i=1}^N \alpha_i \delta_{(x_i, v_i)} \in \mathcal{M}(\Omega_{\mathrm{dyn}}) .$$

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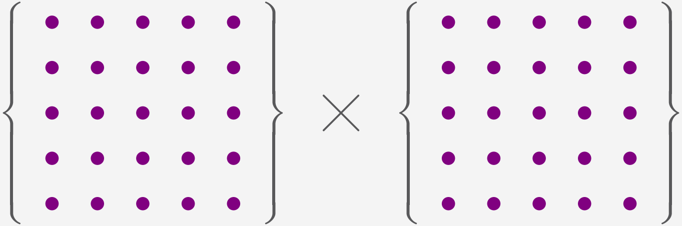
*Then $\lambda^\dagger$ is the **unique solution** to*

$$\min_{\lambda \in \mathcal{M}(\Omega_{\mathrm{dyn}})} \|\lambda\|_{\mathrm{TV}} \quad s.t. \quad \mathrm{Ob}(\mathrm{Move}_t^d)_\# \lambda = f_t^\dagger := \mathrm{Ob}(\mathrm{Move}_t^d)_\# \lambda^\dagger \; \forall t \in \mathcal{T} \tag{ERdyn}$$

Numerics



- Phase space measures live on subset of $\mathbb{R}^{2d}$, twice the dimension of the original static problem

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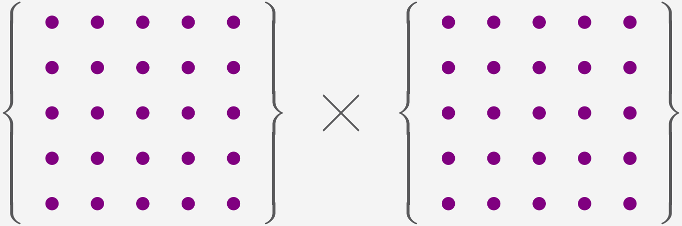

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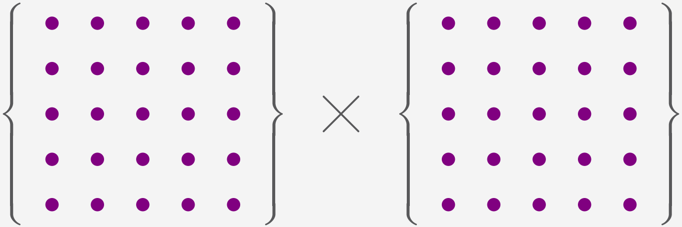

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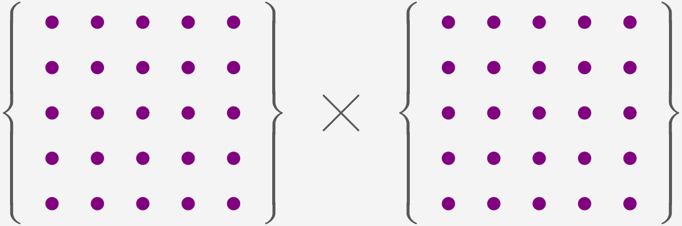

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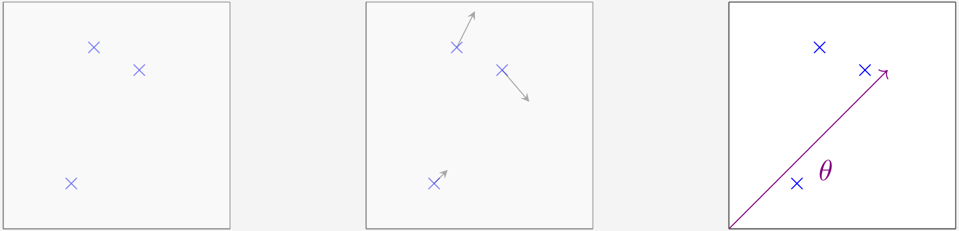
$\Longrightarrow$ **Need for dimension reduction!**

Outline



Static → Dynamic → **Dimension reduction**

Dimension reduction

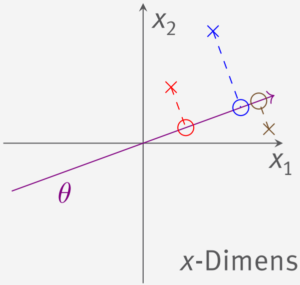
Classical Radon transform

$$f \mapsto (\mathcal{R}_\theta f)_{\theta \in \mathbb{S}^{d-1}}, \quad (\mathcal{R}_\theta f)(s) = \int_{H_\theta} f(s\theta + y)\, \mathrm{d}S(y)$$

Radon transform for measures

$$\mathcal{M}(\mathbb{R}^d) \ni u \mapsto (\mathrm{Rd}_\theta u)_{\theta \in \mathbb{S}^{d-1}}$$

$$\mathrm{Rd}_\theta u := [x \mapsto \theta \cdot x]_\# u \in \mathcal{M}(\mathbb{R})$$



x-Dimensions

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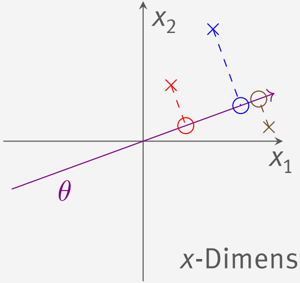
Joint Radon transform

$$\mathcal{M}(\mathbb{R}^{2d}) \ni \lambda \mapsto (\mathrm{Rj}_\theta \lambda)_{\theta \in \mathbb{S}^{d-1}}$$
$$\mathrm{Rj}_\theta \lambda := [(x, v) \mapsto (\theta \cdot x, \theta \cdot v)]_\# \lambda \in \mathcal{M}(\mathbb{R}^2)$$



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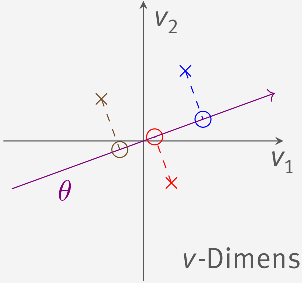

v-Dimensions

New variable:

$$\gamma_\theta^\dagger := \mathrm{Rj}_\theta \lambda^\dagger = \mathrm{Rj}_\theta \sum_{i=1}^N \alpha_i \delta_{(x_i, v_i)} = \sum_{i=1}^N \alpha_i \delta_{(\theta \cdot x_i, \theta \cdot v_i)}$$

Derivation of dim-reduced problem

Try to formulate a minimization problem for the new variable (γ_θ):

$$\min_{\lambda \in \mathcal{M}(\Omega_{\mathrm{dyn}})} \|\lambda\|_{\mathrm{TV}} \quad \text{s.t.}$$

$$\mathrm{Ob}(\mathrm{Move}_t^d)_\# \lambda = f_t^\dagger \; \forall t \in \mathcal{T}$$

Steps:

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The dimension-reduced problem can now be formulated:

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- What can be said about exact reconstruction?
- Simplification: Consider only nonnegative target measures, i.e. $\alpha_i\in[0,\infty)$, and restrict minimization to nonnegative real-valued measures

Exact reconstruction (dim-reduced)

Theorem

Let $\lambda^\dagger \in \mathcal{M}(\Omega_{\text{dyn}})$ *be a nonnegative discrete phase space measure. Assume that there is a subset* $\mathcal{T}' \subset \mathcal{T}$ *of time steps, such that*

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If (γ, u) is any solution of (ERdyn$^-$), *we have*

1. $u_t = u_t^\dagger \quad \forall t \in \Sigma$
2. $\gamma_\theta = \mathrm{Rj}_\theta \lambda^\dagger$ *for almost all $\theta \in \mathbb{S}^{d-1}$.*

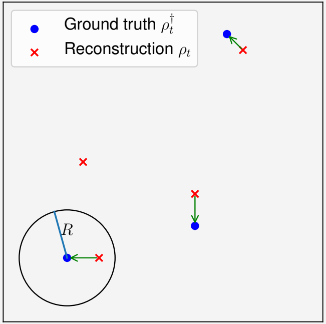
Analysis 2: Noisy data

Data f_t^δ is **distorted by noise** with intensity δ

$$\min_{\substack{\text{snapshots } u_t\\ \text{projections } \gamma_\theta}} \|\gamma_\theta\|_{\mathrm{TV}} + \frac{1}{2\alpha}\sum_{\text{times } t} \|\mathrm{Ob}(u_t) - f_t^\delta\|^2 \quad \text{subject to} \quad u_t, \gamma_\theta \text{ are } \textbf{consistent} \qquad (P_\delta)$$

Quantify error in unbalanced optimal transport cost:

$$\mathrm{OTCost}_R(\nu_1,\nu_2) = \inf\left\{W_2^2(\nu,\nu_2) + \tfrac{1}{2}R^2\|\nu_1 - \nu\|_{\mathrm{TV}} \,\middle|\, \nu \in \mathcal{M}_+(\mathbb{R}^n)\right\}$$



Theorem

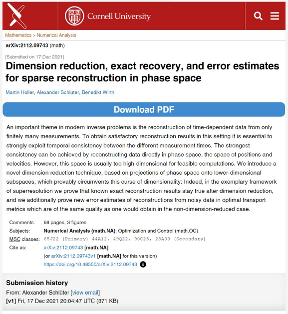
Choose $\alpha = \sqrt{\delta}$. There exist constants $R, C > 0$ such that, for all times t and directions θ "away" from overlaps, ghost particles:

1. $\mathrm{OTCost}_R(u_t, u_t^\dagger) \le C\sqrt{\delta}$
2. $\mathrm{OTCost}_R(\gamma_\theta, \gamma_\theta^\dagger) \le C\sqrt{\delta}$

Preprint

Model well-posedness and equivalence

Stability and convergence rates in unbalanced Wasserstein divergence


Mathematics > Numerical Analysis

arXiv:2112.09743 (math)

[Submitted on 17 Dec 2021]

Dimension reduction, exact recovery, and error estimates for sparse reconstruction in phase space

Martin Holler, Alexander Schlüter, Benedikt Wirth

Download PDF

An important theme in modern inverse problems is the reconstruction of time-dependent data from only finitely many measurements. To obtain satisfactory reconstruction results in this setting it is essential to strongly exploit temporal consistency between the different measurement times. The strongest consistency can be achieved by reconstructing data directly in phase space, the space of positions and velocities. However, this space is usually too high-dimensional for feasible computations. We introduce a novel dimension reduction technique, based on projections of phase space onto lower-dimensional subspaces, which provably circumvents this curse of dimensionality: Indeed, in the exemplary framework of superresolution we prove that known exact reconstruction results stay true after dimension reduction, and we additionally prove new error estimates of reconstructions from noisy data in optimal transport metrics which are of the same quality as one would obtain in the non-dimension-reduced case.

Comments:	68 pages, 3 figures
Subjects:	**Numerical Analysis (math.NA)**; Optimization and Control (math.OC)
MSC classes:	65J22 (Primary) 44A12, 49Q22, 90C25, 28A33 (Secondary)
Cite as:	arXiv:2112.09743 **[math.NA]**
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	https://doi.org/10.48550/arXiv.2112.09743

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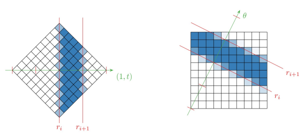
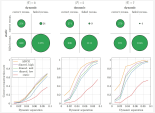
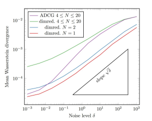
[v1] Fri, 17 Dec 2021 20:04:47 UTC (371 KB)

Numerical implementation & simulations







Exact reconstruction for noise-free data

Thank you for your attention!

References

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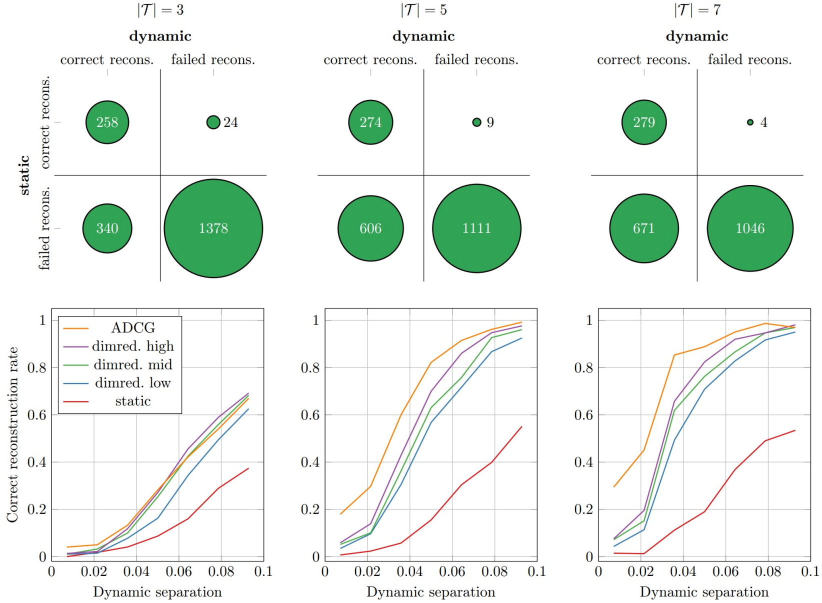
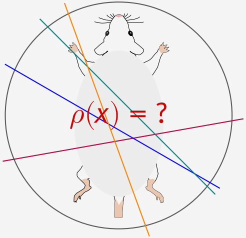
Simulations: noise-free



Application: Cell tracking in PET

- Project “TraCAR” with people from medicine, physics, biotechnology company
- Cancer immunotherapy using modified immune cells called **CAR T-cells**
- Development of new therapies require better insights into **cell movement near the tumor**
- Very little activity per cell -> need to be very **data efficient**, reconstruction directly from listmode PET without binning
- **Dynamics** of interest to estimate cell activity, tumor penetration

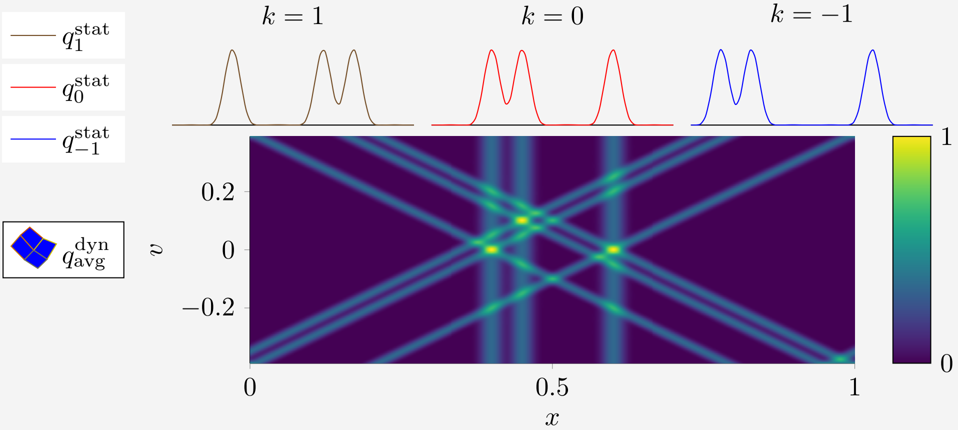


Time	Line of Response
t1	line1
t2	line2
t3	line3
t4	line4
⋮	⋮

Static average certificate

Idea (Alberti et al.): Build dynamic certificate by averaging static certificates

$$q^{\mathrm{dyn}}_{\mathrm{avg}}(x, v) := \frac{1}{3} \sum_{k=-1}^{1} q^{\mathrm{stat}}_{k}(x + k\Delta t\, v)$$



Ghost particles

$$q_{\text{avg}}^{\text{dyn}}(x, v) := \frac{1}{3} \sum_{k=-1}^{1} q_k^{\text{stat}}(x + k\Delta t\, v)$$