Mechanics I
Homework Jan. 30^{th}; due Feb. 13^{th}

1. A body moves according to

$$y(x,t) = (x_1 + ktx_3, x_2 + ktx_3, x_3 - kt(x_1 + x_2))$$

for a constant $k > 0$. What is the (particle) velocity in Lagrangian coordinates? What the velocity at a point y in Eulerian coordinates? Show that a particle with reference position x moves along a straight line orthogonal to x.

2. Assume a deformation $y(x,t)$ to have the deformation gradient $F(x,t) = f(x)A(t)$, $f : \Omega \to \mathbb{R}^+$, $A : [0,\infty) \to \mathbb{R}^{3\times 3}$. Show that if the domain $\Omega \subset \mathbb{R}^3$ is connected, then f has to be constant.

3. Assume a deformation $y : \mathbb{R}^2 \to \mathbb{R}^2$ with deformation gradient to be given by

$$F(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \tfrac{b}{2\pi r} \left(\begin{smallmatrix} -\sin(\theta) & \cos(\theta) \\ 0 & 0 \end{smallmatrix}\right)$$

with $b > 0$ small and cylindrical coordinates $x = (r\cos\theta, r\sin\theta)$. Show that this is a valid deformation gradient on $\mathbb{R}^2 \setminus \{0\}$, but that there is a curl-concentration at $x = 0$ (you can apply Stoke's theorem to a small ball around $x = 0$ to compute the curl inside the ball). Such deformations do occur in physics; they are associated with material faults at $x = 0$, called dislocations.

4. Let $\Omega = \{x : 0 < x_3 < 1, 0 < \sqrt{x_1^2 + x_2^2} < 1\}$ be deformed by $y(x) = (r\cos(2\theta), r\sin(2\theta), x_3)$, where $x = (r\cos\theta, r\sin\theta, x_3)$. Show that y is smooth with $\det Dy > 0$ everywhere, but that y is not even *locally* invertible.

5. Show that the motion

$$y(x,t) = (x_1(x_1 + 2x_2), 2x_2(x_1 + x_2), (1+t)x_3)$$

is invertible on $\Omega = \{x_1 > 0, x_2 > 0, 1 < x_1 + x_2 < 2, 0 < x_3 < 1\}$, using the global inverse function theorem.

6. Given $F \in \mathbb{R}^{3\times 3}$ with $\det F > 0$ and the polar decomposition $F = RU$, show that R is the orthogonal projection of F onto SO(3), i. e.

$$|F - R| < |F - Q|$$

for all $Q \in \mathrm{SO}(3)$ with $Q \neq R$, where $|A| = \mathrm{tr}(A^T A)$.

7. Consider the shear deformation

$$y(x) = (x_1 + \gamma x_2, x_2, x_3)$$

for $\gamma \geq 0$. Represent the deformation by a composition of a rotation, a stretching/compression along the coordinate axes, and another rotation. Illustrate the composition by a sketch. (Hint: The polar decomposition of the deformation gradient reads $\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ \sin\psi & \frac{1+\sin^2\psi}{\cos\psi} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $\psi = \arctan\frac{\gamma}{2}$.)

8. A *rigid deformation* is a deformation $y(x)$ which preserve distances, i. e. $|y(x) - y(z)| = |x - z| \; \forall x, z \in \Omega$. Prove the equivalence of the following:

   - y is a rigid deformation.
   - $y(x) = a + Rx$ for a constant vector a and a constant rotation R.
   - The deformation gradient is a rotation everywhere.
   - The Lagrangian strain tensor is zero everywhere.