Mechanics I
Homework Feb. 13^{th}; due Feb. 27^{th}

1. Derive the mass balance law in Eulerian coordinates,

$$\frac{\partial \rho(y,t)}{\partial t} + \operatorname{div}_y(\rho(y,t)v(y,t)) = 0\,,$$

for example by applying the transformation rule to $\int_E \rho_R(x)\,\mathrm{d}x$ (for an arbitrary $E \subset \mathbb{R}^3$) and then differentiating in time.

2. If the surface force vanishes on some part $y(\partial\Omega_1, t)$ of the boundary of $y(\Omega, t)$, show that for any $\hat{y} \in y(\partial\Omega_1, t)$ the Cauchy stress vector on any plane through $\hat{y}$ is tangent to the boundary. Show also that the strongest stress vector occurs on a plane perpendicular to $y(\partial\Omega_1, t)$.

3. A hyperelastic material has the stored energy function $W(F) = \frac{1}{2}|F|^2 + \frac{1}{\det F}$ for $|F|^2 = \operatorname{tr}(F^T F)$. Show that W is minimized for $F \in \mathrm{SO}(3)$. Compute the Piola–Kirchhoff and the Cauchy stress tensor for the deformation $y(x) = (x_1 + \gamma x_2, x_2, x_3)$.

4. Show that the second Piola–Kirchhoff stress tensor $S = F^{-1}T_R$ is symmetric.

5. The stored energy function of a St. Venant–Kirchhoff material is given by $W(F) = \frac{\lambda}{2}(\operatorname{tr}E)^2 + \mu \operatorname{tr}E^2$ for the Green–Lagrange strain $E = \frac{1}{2}(F^T F - I)$. Show that this material has a linear constitutive law in the sense that the second Piola–Kirchhoff stress tensor is a linear function of the strain, $S = \lambda(\operatorname{tr}E)I + 2\mu E$ (λ, μ are called the Lamé constants).

   What are the problems of this material law concerning material self-penetration and polyconvexity?

6. Given the form $W(F) = a|F|^p + b|\operatorname{cof}F|^q + \Gamma(\det F)$ of a stored energy function with $a, b > 0$, $p \geq 2$, $q \geq 0$, $\Gamma : \mathbb{R} \to \mathbb{R}$, find the conditions on the coefficients and on Γ such that for zero strain the energy and the stress are zero.

   Next find conditions such that for small deformations $y \approx \mathrm{id}$ we have $T \approx \frac{\lambda}{2}(\operatorname{tr}\epsilon)I + \mu\epsilon$ with $\epsilon = \frac{1}{2}[(\nabla y - I) + (\nabla y - I)^T]$ (hint: you need to linearize T around $\nabla y = I$).

7. Incompressible materials are those which require an infinite amount of energy to change their volume. Their stored energy function $W(F)$ only makes sense for $\det F = 1$. In particular, for any $p \in \mathbb{R}$ the energy $\tilde{W}(F) = W(F) + p(\det F - 1)$ models the same material. Derive that the Cauchy stress tensor for such materials is only determined up to a hydrostatic pressure pI.

8. Consider a unit cube of an incompressible Mooney material with stored energy function

$$(\lambda_1, \lambda_2, \lambda_3) \mapsto \alpha(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)\,.$$

Applying a normal dead load traction of magnitude τ, show that for some parameters there can be solutions of the form $y(x) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$ with the λ_i not all 1 (this is called the Rivlin-cube problem).

9. For an isotropic material with stored energy function

$$(\lambda_1, \lambda_2, \lambda_3) \mapsto \Phi(\lambda_1, \lambda_2, \lambda_3)$$

show that for a deformation $y(x) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$ the Piola–Kirchhoff stress tensor is given by $\mathrm{diag}(\partial_1\Phi, \partial_2\Phi, \partial_3\Phi)$.

10. Let $0 \le R_1 < R_2$ and $y(x) = r(|x|)\frac{x}{|x|}$ be a radial equilibrium solution on $[R_1, R_2]$ for an isotropic material. We shall assume that a stronger stretch implies a stronger strain (the "tension-extension inequality"; in the language of the previous question, $\partial_i^2\Phi > 0$). Show that if $r'(R_0) = r(R_0)/R_0$ for some $R_0 \in (R_1, R_2)$, then $r(|x|) = \lambda|x|$ for all x and some $\lambda > 0$. (E. g. you can derive an ode for r from the balance laws and then show that it is solved by the above; you may simply assume the solution to be unique.)