Mechanics I
Homework Feb. 27$^{\text{th}}$; due Mar. 13$^{\text{th}}$

1. Use the direct method of the calculus of variations to show that the function
$$f : \mathbb{R}^{3\times 3} \to \mathbb{R}\,, \quad A \mapsto \begin{cases} -c & \text{if } A \text{ is singular} \\ \det A + |A|^5 & \text{else} \end{cases}$$
for some fixed $c > 0$ possesses a minimizer.

2. Use the direct method of the calculus of variations to show that the energy of linearized elasticity,
$$\int_\Omega (C\epsilon(u)) : \epsilon(u) - \hat{b} \cdot u \,\mathrm{d}x - \int_{\partial\Omega_2} s \cdot u \,\mathrm{d}a\,,$$
has a minimizer among all $u \in H^1(\Omega)$ with $u = u_0$ on $\partial\Omega_1$, assuming $(C\epsilon(u)) : \epsilon(u) \geq c|\epsilon(u)|^2$ for some $c > 0$, $\hat{b} \in H^{-1}(\Omega)$, $s \in L^2(\partial\Omega_2)$, $u_0 \in H^1(\Omega)$.

3. Let Ω be a rod that is uniformly dilated along its longitudinal axis by the factor $1 + \epsilon_1$ without restraining its sides (i. e. zero traction on its sides). The rod's behavior shall be described by linearized elasticity. Let σ_1 be the magnitude of cross-sectional tension, then Young's modulus E is defined as σ_1/ϵ_1. Let $1+\epsilon_2$ be the dilation factor in a direction orthogonal to the longitudinal axis, then Poisson's ratio ν is defined as ϵ_2/ϵ_1. Express E and ν in terms of the Lamé constants λ and μ, give bounds for E and ν so that the elasticity tensor is positive semi-definite (i. e. $(C\epsilon) : \epsilon \geq 0$), and give an expression for the stress $C\epsilon(u)$ in terms of E and ν.

4. Now let a piece of material Ω with volume V be subjected to a hydrostatic pressure p, and let $e = \Delta V/V$ be the resulting relative volume change. The bulk modulus K is defined as $-\lim_{p\to 0} p/e$. Assuming linearized elasticity, express K in terms of the Lamé constants λ and μ, give bounds for K and μ so that the elasticity tensor is positive semi-definite, and give an expression for the stress $C\epsilon(u)$ in terms of K and μ.

5. The constitutive law of Euler's elastica model of elastic rods with constant cross-section can be derived based on particular kinematic assumptions:

   Consider the simpler 2D case (3D works analogously), i. e. let a rod of thickness $2d$ and length L be described by $\Omega = (0, L) \times (-d, d)$. Let the material have a Young's modulus E and Poisson's ratio $\nu = 0$. When the rod is bent, there is a fiber $(0, L) \times \{x_2\}$ which is neither elongated nor compressed (the "stress-free" or "central" fiber). The other parallel fibers are compressed or elongated, depending on whether they are on the outside or the inside of the bending. All cross-sections orthogonal to the central fiber in the reference configuration stay undeformed and are merely rotated so as to stay orthogonal to the central fiber also in the deformed configuration.

Derive that the bending moment of the rod at a position x_1 is given by $m(x_1) = EI\kappa(x_1)$ for $\kappa(x_1)$ the bending curvature of the central fiber at x_1 and $I = \frac{2d^3}{3}$ the area moment of inertia.

- First show from the kinematic assumptions that the normal strain (and thus the normal stress) in the rod's cross-section is affine in x_2.
- Next derive from the balance of linear momentum that the central fiber is $(0, L) \times \{0\}$.
- Finally compute the angular momentum around $(x_1, 0)$.

(The derivation also works for non-zero Poisson's ratio, however, then the assumption of undeformed cross-sections is typically not satisfied.)

6. Derive the governing ode for the deformation of thin rods as in the lecture, only this time add body forces.