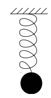
Mechanics I
Homework Apr. 10^{th}; due Apr. 24^{th}

1. Consider the mechanical system of a mass m, suspended from a spring and subject to the earth's gravitational field.



The force of the spring is proportional to its elongation with proportionality factor D. Letting the coordinate q denote the height of the mass, write down Newton's law of motion, the kinetic energy $T(\dot{q})$, the potential $V(q)$, the Lagrangian $L(t, q, \dot{q})$, the Hamiltonian $H(t, q, p)$, and the momentum p (as a function of $\dot{q}$). Compute the solution $q(t)$ for given initial data $q(0)$ and $\dot{q}(0)$, and make a sketch of the phase plane and the trajectories in it.

2. In 2D, consider the motion of a point mass (mass 1) in a central potential field U with center 0, i. e. for $q \in \mathbb{R}^2$ the position of the mass, Newton's law of motion reads

$$\ddot{q} = -\nabla_q U,$$

where U only depends on the distance r of the mass to the origin. Via the following steps, the system can be analyzed more easily.

- Write down the kinetic and the potential energy as well as the Lagrangian in polar coordinates (r, θ).
- Show that the angular momentum $p_\theta = \dot{q} \times q$ is conserved. To this end, it is easiest to show that p_θ is the momentum corresponding to the generalized coordinate θ and that this momentum is conserved due to a symmetry of the Lagrangian.
- Prove the following Theorem.
Theorem. For the motion of the mass point in the central potential field U with initial angular momentum p_θ, the distance r to the origin varies in the same way as the position r of a mass point in 1D with kinetic energy $\frac{1}{2}\dot{r}^2$ and the effective potential energy $V(r) = U(r) + \frac{p_\theta^2}{2r^2}$.
(Hint: E. g. write down Newton's law for the 1D and for the 2D problem, using polar coordinates and the conservation of angular momentum in 2D.)
- Show that the total energy H in the above 1D problem is conserved, and use this to derive

$$t = \int_{r(0)}^{r(t)} \frac{\mathrm{d}r}{\sqrt{2(H - V(r))}} \quad \text{as well as} \quad \theta(t) = \int_{r(0)}^{r(t)} \frac{p_\theta \mathrm{d}r}{r^2\sqrt{2(H - V(r))}}.$$

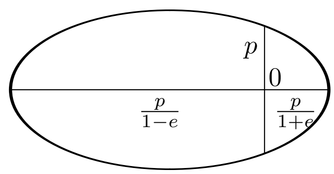
(Hint: First find a formula for $\dot{r}$ and then exploit $\mathrm{d}\theta/\mathrm{d}r = \dot{\theta}/\dot{r}$.)

3. Use the results from the previous question (for the potential $U(r) = -k/r$ with $k = \gamma Mm$) to derive Kepler's first two laws of planetary motion:

   - Derive the first law,

     A planet moves along an ellipse with the sun at one of its focal points.

     by first deriving $\theta(t) = \arccos \frac{\frac{p_\theta}{r(t)} - \frac{k}{p_\theta}}{\sqrt{2H + \frac{k^2}{p_\theta^2}}}$ and then $r(t) = \frac{p}{1+e\cos\theta(t)}$ for ellipse parameter $p = \frac{p_\theta^2}{k}$ and eccentricity $e = \sqrt{1 + \frac{2Hp_\theta^2}{k^2}}$. This is the equation of an ellipse in polar coordinates with center at a focal point.



   - Derive the second law,

     In planet motion, the radius vector sweeps out equal areas in equal time intervals.

     from the conservation of angular momentum.

4. Write down Newton's law of motion for the three-body system sun, earth, moon. Solve the ode numerically, looking up the correct masses of the three bodies, the gravitational constant γ, as well as a reasonable initial condition. The numerical solution of ODEs in Matlab is simple, you can follow e. g. Example 1 on
   http://www.mathworks.com/help/matlab/ref/ode23.html.
   Note: Different from the above central potential field problem, the three-body system is very hard to predict over long time intervals.