Mechanics I
Homework Apr. 24$^{\text{th}}$; due May 8$^{\text{th}}$

1. Show that the phase flow g^t of a Hamiltonian system is a group.

2. Consider a pendulum with mass $m = 1$, length $l = 4$, and gravitational acceleration $g = 1$. Sketch the $(\theta, \dot{\theta})$-phase plane as well as the image of the disk $\theta^2 + (\dot{\theta} - \frac{1}{2})^2 < \frac{1}{4}$ under the phase flow g^t for the time $t = \pi$. What is the relation between the areas of the disk and its image? How large is the area of the image in the (θ, p)-phase plane, where p is the momentum associated with θ?

3. For a linear system $\dot{x} = A(t)x$, $x(t) \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n\times n}$, let $x_1(t), \dots, x_n(t)$ denote n linearly independent solutions and introduce the Wronskian $W(t) = \det(x_1(t)|\dots|x_n(t))$. Prove Liouville's formula
$$W(t) = W(0)e^{\int_0^t \mathrm{tr}A(\tau)\,\mathrm{d}\tau}.$$
(Hint: The proof can proceed similarly to the proof of Liouville's theorem.)

4. Show that in a Hamiltonian system it is impossible to have asymptotically stable equilibrium points or asymptotically stable limit cycles in the phase space (using Liouville's theorem).

5. Consider the unit circle S^1 and let g be rotation by an angle α. Using Poincaré's recurrence theorem, show that if $\frac{\alpha}{2\pi}$ is irrational, the set of points $g^k x$, $k \in \mathbb{N}$, is dense on S^1 for any $x \in S^1$.

6. Solve Burgers' equation (which occurs e.g g. in modeling gas dynamics and traffic flow),
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,,$$
for initial data $u_0(x) = \max(0, 1 - |x|)$ at time $t = 0$, using the method of characteristics. When does the solution break down?

7. Fermat's principle states that a light ray travelling from one point to another always takes that path which needs the least time. The speed of light in vacuum is given by c, the speed in a medium by $v = c/n$ for the refractive index n of the medium. Of course, n may depend on the spatial location x.

   - Write down a Lagrangian $L(t, x, \dot{x})$ and the corresponding action functional for the motion of a light ray.
   - Assume, the light ray is never orthogonal to the x_3-axis and travels in the direction of positive x_3-values. Do a coordinate transform in which the time t is replaced by the x_3-coordinate (this corresponds just to reparameterizing the path) and write down the Lagrangian and action for this new time variable.

- Assume the half-space $\{x \in \mathbb{R}^3 : x_3 < 0\}$ to be occupied by a material with refractive index n_1 and the complement by a material with index n_2. Consider the light ray from $(0, 0, -1)$ to $(2, 0, 1)$. Derive an equation which describes the position $(x_1, x_2, 0)$ of the light ray at the material interface.
- Derive Snell's law of refraction: At a planar interface between two media with refractive indices n_1 and n_2, respectively, $\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}$ for the angles θ_1 and θ_2 between the light ray and the normal to the interface in medium 1 and 2, respectively.
- Give an example where the path of a light ray (i. e. the extremal of the action) is not unique. (Hint: Think of lenses. This phenomenon is also used in astronomy: For instance, around a black hole the space-time-metric is distorted so that the black hole acts like a lense and one can thus see stars behind the black hole multiple times.)