<u>norms of subdeterminants</u>: $\|F\|_2 = \sqrt{\sum_{i,j} F_{ij}^2} = \sqrt{\operatorname{tr} C} = \sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}$

norm of stretch vector $\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix}$ => associated with length changes

$\|\operatorname{cof} F\|_2 = \sqrt{\underbrace{(\lambda_1\lambda_2)^2}_{} + (\lambda_2\lambda_3)^2 + (\lambda_1\lambda_3)^2} = \sqrt{II_C}$

area change in 1-2-plane => associated with area changes
(cofactor matrix is matrix of 2×2 subdeterminants;
by Cramer's rule $\operatorname{cof} F = \det F \, F^{-T}$)

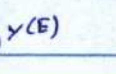
$|\det F| = \lambda_1\lambda_2\lambda_3 = \sqrt{\det C}$

describes volume change $(\operatorname{vol} y(E) = \int_E \det F \, dx)$

Stress

<u>body force</u>: $b(y,t) \in \mathbb{R}^3$ = force per unit volume exerted by external world

total force exerted on $E \subset \Omega$: $\int_{y(E,t)} b(y,t)\, dy$

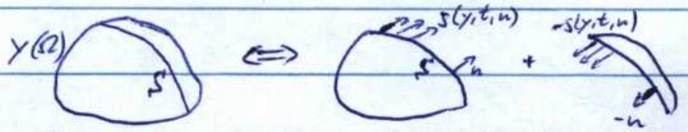


e.g. gravity: $b(y,t) = \overset{\nearrow}{g}\, \rho\!\overset{\nwarrow}{(y,t)} \begin{pmatrix}0\\0\\-1\end{pmatrix}$
gravitational constant density

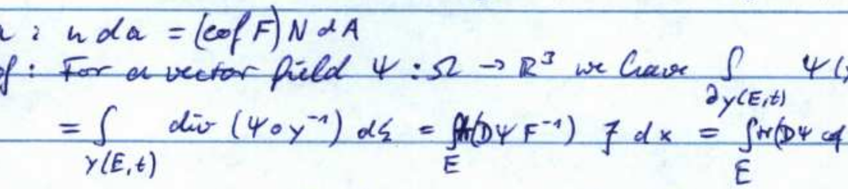
<u>surface force</u>: Let S be a two-dim. manifold with normal n and material on one side of S. A surface force $s(y,t) \in \mathbb{R}^3$, $y \in S$, is a force per unit area of S, acting on the material.

total force exerted on S : $\int_S s(y,t)\, dy$

<u>Cauchy hypothesis</u>: There is a vector field $s(y,t,n) \in \mathbb{R}^3$ such that for any smooth surface $S \subset y(\Omega,t)$ with normal n at y
$s(y,t,n)$ is the surface force exerted by the material on one side of S onto the material on the other side.



$s(y,t,n)$ is called <u>Cauchy stress vector</u>. The (first) <u>Piola-Kirchhoff stress vector</u> $S_R(x,t,N)$ is parallel to s, but measures the surface force per unit area in the reference configuration, acting across the deformed surface having normal N in the reference configuration.



Let da be a surface element on the deformed configuration and dA the corresponding surface element on the reference configuration, then $S_R(x,t,N)\,dA = s(y,t,n)\,da$.

lemma : $n\,da = (\operatorname{cof} F) N\, dA$

proof: For a vector field $\Psi : \Omega \to \mathbb{R}^3$ we have $\int_{\partial y(E,t)} \Psi(y^{-1}(\xi,t)) \cdot n\, da$

$$= \int_{y(E,t)} \operatorname{div}(\Psi \circ y^{-1})\, d\xi = \int_E \operatorname{tr}(D\Psi F^{-1})\, J\, dx = \int_E \operatorname{tr}(D\Psi \operatorname{cof} F^T)\, dx$$

$$\underset{\substack{\uparrow}}{=} \int_E \operatorname{div}(\operatorname{cof} F^T \Psi)\, dx = \int_{\partial E} \Psi \cdot [(\operatorname{cof} F)N]\, dA \qquad \square$$

Piola's identity: $\operatorname{div}(\operatorname{cof} F) = 0$

Hence, $n = \frac{(\operatorname{cof} F)N}{\|(\operatorname{cof} F)N\|}$, $da = \|(\operatorname{cof} F)N\|\, dA$, $s_R = \|(\operatorname{cof} F)N\|\, s$.

Balance laws

Any physical quantity $f(y(x,t),t)$ can be expressed as a function of x and t, $\tilde{f}(x,t) = f(y(x,t),t)$. For simplicity write $f = \tilde{f}$, i.e. $f(x,t) = f(y,t)$.

conservation of mass

Let $\rho(y,t)$ be the material density and $\rho_R(x)$ the density in the reference configuration, then for any $E \subset \Omega$ (measurable) we have

$$\int_E \rho_R(x)\,dx = \int_{y(E,t)} \rho(y,t)\,dy = \int_E \rho(x,t)\,J(x)\,dx\ ,$$

thus $\rho J = \rho_R$.

conservation of linear momentum

For all $E \subset \Omega$ we have $\frac{d}{dt}\int_E \rho_R \dot{y}\,dx = \int_{\partial E} s_R(x,t,N)\,dA + \int_E J b(x,t)\,dx$. (*)

("axion of force balance")

Cauchy stress thm: $(\ast) \iff$ (1) $s_R(x,t,N) = T_R(x,t)\cdot N$ for some $T_R(\cdot,t) : \Omega \to \mathbb{R}^{3\times 3}$
(the "Piola - Kirchhoff stress tensor")

(2) $\rho_R \ddot{y} = \operatorname{div} T_R + Jb$

proof: sufficiency obvious; necessity:

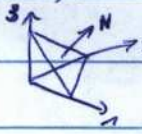
For any smooth bounded open set $G \subset \mathbb{R}^3$ and $\varepsilon > 0$ choose $E = x_0 + \varepsilon G$, then $\frac{d}{dt}\int_E \rho_R \dot{y}\,dx - \int_E J b(x,t)\,dx = \varepsilon^3 \int_G [\rho_R \ddot{y} - Jb](x_0 + \varepsilon\xi, t)\,d\xi = O(\varepsilon^3)$

and $\int_{\partial E} s_R(x,t,N)\,dA = \varepsilon^2 \int_{\partial G} s_R(x_0 + \varepsilon\xi, t, N(\xi))\,dA$.

Dividing (*) by ε^2 and letting $\varepsilon \to 0$ we obtain $\int_{\partial G} s_R(x_0,t,N(\xi))\,dA = 0$

(surface forces dominate body forces).

Taking G as the tetrahedron



one obtains

$$0 = s_R(x_0,t,N) + \sum_{i=1}^{3} N_i s_R(x_0,t,-e_i)$$

which implies (1).

Thus $\int_{\partial E} s_R\,dA = \int_E \operatorname{div} T_R\,dx$ and so $(\ast) \Rightarrow \int_E \rho_R \ddot{y} - \operatorname{div} T_R - Jb\,dx = 0 \Rightarrow (2)$

$\square$

Now $s_R\,dA$ ~~implies~~ $= s\,da$ implies $T_R N\,dA = T_R(\operatorname{cof} F)^{-1} n\,da = s\,da$, thus $s(y,t,n) = T(y,t)\,n$ for the Cauchy stress tensor $T(y,t) = [T_R(y(x,t))\operatorname{cof} F(y(x,t))^{-1}]$.

The transformation rule yields

$$\rho_R = J\rho,\ \operatorname{div}_x(\operatorname{cof} F) = 0 \Rightarrow \operatorname{div}_x T_R = J \operatorname{div}_y T$$

$$0 = \int_E \rho_R \ddot{y} - \operatorname{div}_x T_R - Jb\,dx = \int_{y(E,t)} J^{-1}(\rho_R \ddot{y} - \operatorname{div}_x T_R - Jb)\,dx \overset{\downarrow}{=} \int_{y(E,t)} \rho\ddot{y} - \operatorname{div}_y T - b\,dx$$

letting $x(y,t)$ denote the inverse of $y(x,t)$ and $v(y,t) = \dot{y}(x(y,t),t)$, then $\ddot{y}(x(y,t),t) = \frac{\partial v}{\partial t} + (v\cdot\nabla)v$, since $D_y v = D_x \dot{y}\, D_y x$ and $\frac{\partial v}{\partial t} = \ddot{y} + (D_x \dot{y})\dot{x} \underset{\uparrow}{=} \ddot{y} - (D_x \dot{y})(D_x y)^{-1} v$,

$$0 = \frac{d}{dt}(y \circ x) = \frac{d}{dt} y(x(y,t),t) = \dot{y} + (D_x y)\dot{x}$$

thus (2) in Eulerian coords. reads

$$\rho\left(\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right) = \operatorname{div}_y T + b\,.$$

Note: $\left(\frac{\partial}{\partial t} + v\cdot\nabla\right)$ is called the "material derivative".

Conservation of angular momentum

For all $E \subset \Omega$ we have $\frac{d}{dt}\int_E y \wedge \rho_R \dot{y}\, dx = \int_{\partial E} y \wedge s_R\, dA + \int_E y \wedge f b\, dx$ $(**)$

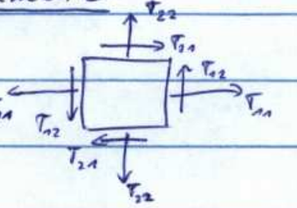
("axiom of momentum balance")

$$(**) \Leftrightarrow 0 = \int_E \underbrace{\dot{y} \wedge \rho_R \dot{y}}_{=0} + \underbrace{y \wedge \rho_R \ddot{y} - y \wedge f b}_{= y \wedge \operatorname{div} T_R \text{ by } (2)}\, dx \; \underbrace{- \int_{\partial E} y \wedge T_R N\, dA}_{= \int_E \operatorname{div}(\underset{\substack{\uparrow \\ \text{columnwise}}}{y \wedge T_R})\, dx = \int_E y \wedge \operatorname{div} T_R + \begin{pmatrix} T_{R3:} \cdot Dy_2 - T_{R2:} \cdot Dy_3 \\ T_{R1:} \cdot Dy_3 - T_{R3:} \cdot Dy_1 \\ T_{R2:} \cdot Dy_1 - T_{R1:} \cdot Dy_2 \end{pmatrix} dx}$$

$$\Leftrightarrow 0 = \int_E \begin{pmatrix} (T_R F^T)_{32} - (T_R F^T)_{23} \\ (T_R F^T)_{13} - (T_R F^T)_{31} \\ (T_R F^T)_{21} - (T_R F^T)_{12} \end{pmatrix} dx \quad \forall E \subset \Omega \Leftrightarrow T_R F^T \text{ symmetric}$$

$\Leftrightarrow$ <u>T is symmetric</u>

Intuition (2D):



if $T_{12} \neq T_{21}$, the angular momentum of an infinitesimal volume element is unbalanced.

Constitutive Laws (to make equations determinate)

So far, everything applies to all materials for which the Cauchy-stress-hypothesis holds. To specify a material, need to express the stress a terms of a motion.

In principle one could have $T_R = \hat{T}_R(x, y, D_x y, D_x^2 y, \dots, \dot{y}, D_x \dot{y}, D_x^2 \dot{y}, \dots, \ddot{y}, D_x \ddot{y}, \dots)$.

<u>Frame indifference</u>: An observer B, whose coordinate system ("frame of reference") moves relatively to another observer A, should still observe the same stresses.

<u>Thm</u>: Frame indifference $\Leftrightarrow$ $\hat{T}_R$ is indep. of $y, \dot{y}, \ddot{y}, \dots$ and

$$\hat{T}_R(x, QD_x y, QD_x^2 y, \dots, QD_x \dot{y} + S_1 D_x y, QD_x^2 \dot{y} + S_1 D_x^2 y, \dots, QD_x \ddot{y} + 2S_1 D_x \dot{y} + S_2 D_x y, \dots)$$
$$= Q \hat{T}_R(x, D_x y, D_x^2 y, \dots, D_x \dot{y}, D_x^2 \dot{y}, \dots, D_x \ddot{y}, \dots) \quad \forall Q \in SO(3),\ S_1, S_2, \dots \text{ skew.}$$

proof: Let coord. system B result from coord. system A by a translation $-c(t) \in \mathbb{R}^3$ and a rotation $Q^T(t) \in SO(3)$, i.e. $y_B(x,t) = Q(t) y_A(x,t) + c(t)$.

$\Rightarrow \dot{y}_B = Q\dot{y}_A + \dot{Q} y_A + \dot{c}$, $\ddot{y}_B = Q\ddot{y}_A + 2\dot{Q}\dot{y}_A + \ddot{Q} y_A + \ddot{c}$

Now $s_{R,B}(x,t,N) = Q(t) s_{R,A}(x,t,N) \Leftrightarrow T_{R,B}(x,t) = Q(t) T_{R,A}(x,t)$

$$\Leftrightarrow \hat{T}_R(x, y_B, D_x y_B, D_x^2 y_B, \dots, \dot{y}_B, D_x \dot{y}_B, D_x^2 \dot{y}_B, \dots, \ddot{y}_B, D_x \ddot{y}_B, \dots)$$
$$= \hat{T}_R(x, Qy_A + c, QD_x y_A, QD_x^2 y_A, \dots, Q\dot{y}_A + \dot{Q} y_A + \dot{c}, QD_x \dot{y}_A + \dot{Q} D_x y_A, \dots, Q\ddot{y}_A + 2\dot{Q}\dot{y}_A + \ddot{Q} y_A + \ddot{c}, QD_x \ddot{y}_A + 2\dot{Q} D_x \dot{y}_A + \ddot{Q} D_x y_A, \dots)$$
$$= Q \hat{T}_R(x, y_A, D_x y_A, D_x^2 y_A, \dots, \dot{y}_A, D_x \dot{y}_A, \dots, \ddot{y}_A, D_x \ddot{y}_A, \dots)$$

Choose: $Q = I$, $c = \text{const} \Rightarrow \hat{T}_R$ is indep. of y

$c = t \cdot \text{const} \Rightarrow$ —— " —— $\dot{y}$

$c = t^2 \cdot \text{const} \Rightarrow$ —— " —— $\ddot{y}$

⋮

Result now follows from $\dot{Q}, \ddot{Q}, \dots$ being skew-symmetric for $Q: \mathbb{R} \to SO(3)$. □

A material is
- <u>viscous</u> if $\hat{T}_R = \hat{T}_R(x, \overbrace{D_x \dot{y} (D_x y)^{-1}}^{D_y v})$ (or even dependence on higher time derivatives)
- <u>elastic</u> if $\hat{T}_R = \hat{T}_R(x, D_x y)$
- <u>hyperelastic</u> if in addition $\hat{T}_R(x, F) = D_F W(x, F)$ for "<u>stored energy fcn</u>" W
- <u>viscoelastic</u> if $\hat{T}_R = \hat{T}_R(x, D_x y, D_x \dot{y})$ (or even dependence on higher time derivatives)
- <u>multipolar</u> if $\hat{T}_R$ also depends on higher spatial derivatives