<u>Consider a hyperelastic stored energy fcn</u> W; recall $\hat{T}_R(x, QF) = Q\hat{T}_R(x,F)$ $\forall Q \in SO(3)$.

<u>Thm</u>: $\hat{T}_R(x, QF) = Q\,\hat{T}_R(x,F) \iff W(x,QF) = W(x,F) \quad \forall Q \in SO(3)$.

proof: $\Leftarrow$: $\hat{T}_R(F) = \frac{d}{dF} W(F) = \frac{d}{dF} W(QF) = Q^T \underset{\substack{\uparrow\\ \text{derivative wrt matrix argument}}}{W_{,A}}(QF) = Q^T \hat{T}_R(QF)$

$\Rightarrow$: $\frac{d}{dF}\big(W(QF) - W(F)\big) = 0 \implies W(QF) - W(F) = W(Q) - W(I)$

by induction, $\frac{W(Q^n) - W(I)}{n} = W(Q) - W(I)$ (use $W(Q^n) - W(I) = W(Q^{n-1}Q) - W(Q) + W(Q) - W(I)$)

letting $n \to \infty$, lhs $\to 0$. $\square$

<u>Thm</u>: $W(x, QF) = W(x,F)$ $\forall Q \in SO(3) \implies T$ is symmetric

proof: Let K be skew, then $0 = \frac{d}{dt} W(e^{Kt}F)|_{t=0} = W_{,A}(e^{Kt}F) : Ke^{Kt}F|_{t=0} = \hat{T}_R(F) : KF = \mathrm{tr}(T_R F^T K^T)$
$= J\,\mathrm{tr}(TK^T)$. $\square$

An elastic material is
- <u>homogeneous</u> if $\hat{T}_R = \hat{T}_R(F)$
- <u>isotropic</u> if $\hat{T}_R(FQ) = \hat{T}_R(F)Q$ $\big(W(FQ) = W(F)\big)$ $\forall Q \in SO(3)$

By polar decomposition, $W(F) = W(R\sqrt{F^TF}) = W(\sqrt{F^TF})$.

<u>Thm (reduced form of stored energy)</u>: The following are equivalent:

(a) W is isotropic

(b) $W(F) = h(I_B, II_B, III_B)$ for some h

(c) $W(F) = \Phi(\lambda_1, \lambda_2, \lambda_3)$ for some symmetric Φ

(d) $T(F) = \alpha_0 I + \alpha_1 B + \alpha_2 B^2$ with the α_i fcns of I_B, II_B, III_B

<u>"Rivlin-Ericksen representation"</u>

proof: (a) $\iff W(F) = W(Q\Lambda R) = W(\Lambda) = W\left(\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}\right) = W\left(\begin{pmatrix}\lambda_2 & & \\ & \lambda_3 & \\ & & \lambda_1\end{pmatrix}\right) = \dots$ all permutations $\iff$ (c).

(c) $\Rightarrow$ (b) by $I_B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $II_B = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2$, $III_B = (\lambda_1\lambda_2\lambda_3)^2$

(b) $\Rightarrow$ (a) since invariants of FF^T are same as of $FQ(FQ)^T$

(d) $\Rightarrow T(FR) = T(F) \implies T(FR)\,\mathrm{cof}(FR) = T(F)\,\mathrm{cof}(F)\,\mathrm{cof}R \implies T_R(FR) = T_R(F)R \implies$ (a)

(b) $\Rightarrow$ (d) (for smooth h): $T(F) = T_R(F)\,\mathrm{cof}F^{-1} = \frac{W_{,A}(F)}{J}F^T = \frac{1}{J}\Big(\partial_1 h \underbrace{\frac{\partial I_B}{\partial F}}_{2F} + \partial_2 h \underbrace{\frac{\partial II_B}{\partial F}}_{2\|F\|^2F - 2FF^TF} + \partial_3 h \underbrace{\frac{\partial III_B}{\partial F}}_{=2J\,\mathrm{cof}F}\Big)F^T$
$= 2\partial_3 h\sqrt{III_B}\, I + \frac{2}{\sqrt{III_B}}(\partial_1 h + I_B\partial_2 h)B - \frac{2}{\sqrt{III_B}}\partial_2 h\, B^2$. $\square$

Examples: const. laws for rubbers (incompressible)

Let $I_1(z) = \begin{cases} 0 & \text{if } z = 1 \\ \infty & \text{if } z \neq 1 \end{cases}$

<u>Neo-Hookian material</u>: $W = \alpha(I_B - 3) + I_1(III_B)$, $\alpha > 0$

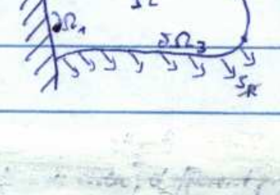
<u>Mooney-Rivlin material</u>: $W = \alpha(I_B - 3) + \beta(II_B - 3) + I_1(III_B)$, $\alpha, \beta > 0$

<u>Ogden material</u>: $W = \sum_{i=1}^{N} \alpha_i(\lambda_1^{p_i} + \lambda_2^{p_i} + \lambda_3^{p_i} - 3) + \sum_{i=1}^{M} \beta_i\big((\lambda_1\lambda_2)^{q_i} + (\lambda_2\lambda_3)^{q_i} + (\lambda_1\lambda_3)^{q_i} - 3\big) + I_1(III_B)$

for slight compressibility add $h(III_B)$ instead of $I_1(III_B)$

<u>Static equilibria as energy minimizers</u>

Consider the static elastic problem



$$\begin{aligned}
\mathrm{div}_x T_R(x) &= -f(x)\,b(x) && \text{in } \Omega \\
y(x) &= y_0(x) && \text{on } \partial\Omega_1 \\
T_R(x)N(x) &= 0 && \text{on } \partial\Omega_2 \\
T_R(x)N(x) &= s && \text{on } \partial\Omega_3 \\
T_R(x) &= W_{,F}(D_x y)
\end{aligned}$$

When does equilibrium state satisfy a variational principle? (8)

Gibbs free energy $E[y] = \underbrace{\int_\Omega W(Dy)\,dx}_{\text{stored elastic energy}} + \underbrace{\int_\Omega \Psi(y)\,dx}_{\substack{\text{potential energy}\\ \text{of body forces}}} + \underbrace{\int_{\partial\Omega_2} \chi(y)\,dA(x)}_{\substack{\text{potential energy}\\ \text{of surface forces}}}$

A minimizer satisfies the Euler-Lagrange equation: for all smooth ϑ with $\vartheta = 0$ on $\partial\Omega_1$,

$$0 = \partial_y E[y](\vartheta) = \int_\Omega W_{,A}(Dy) : D\vartheta + \nabla_y \Psi \cdot \vartheta\,dx + \int_{\partial\Omega_2} \nabla_y \chi \cdot \vartheta\,dA(x)$$

$$= \int_\Omega (-\operatorname{div} T_R(x) + \nabla_y \Psi) \cdot \vartheta\,dx + \int_{\partial\Omega_2 \cup \partial\Omega_3} (T_R(x)N(x) + \nabla_y \chi) \cdot \vartheta\,dA(x)$$

$\Rightarrow$ equil. state satisfies var. princ. iff
- body force is conservative, i.e. $J b = -\nabla_y \Psi(y(x))$
  (e.g. gravity $b = \rho_R / J \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \Rightarrow \Psi(y(x)) = \rho_R(x) g\, y_3(x)$)
- surface load is conservative, $s = -\nabla_y \chi(y(x))$
  (e.g. dead load $s_R(x) \Rightarrow \chi = -s_R(x) \cdot y(x)$)

Ex: gravity & dead load: equil. state minimizes $E[y] = \int_\Omega W(Dy)\,dx + g \int_\Omega \rho_R(x) y_3(x)\,dx - \int_{\partial\Omega_2} s_R(x) \cdot y(x)\,dA$

<u>"dead load"</u>: $T_R(x)N(x) = s_R(x)$; keeps direction and force per unit reference area

<u>"live load"</u>: $T(y)n(y) = s(y)$

<u>Existence theory for minimizers of E</u>

Assume $W(F) \geq c\|F\|_2^p$, $p > 3$, $b = \frac{\hat{b}}{J}$ with $\hat{b} \in W^{-1,p^*}(\Omega)$, $s_R \in L^{p^*}(\partial\Omega_3)$, $\frac{1}{p} + \frac{1}{p^*} = 1$, Ω Lipschitz

Define $E[y] = \infty$ if $y \notin W^{1,p}(\Omega)$ or $y \neq y_0$ on $\partial\Omega_1$.

<u>Direct method of calculus of variations</u>

0) E is not identically ∞ & is bdd from below.

Here: $E[y] \geq c\|\nabla y\|_{L^p}^p - \underbrace{\int_\Omega \hat{b} \cdot y\,dx}_{\leq \|\hat{b}\|_{W^{-1,p^*}} \|y\|_{W^{1,p}}} - \underbrace{\int_{\partial\Omega_2} s_R \cdot y\,dA(x)}_{\leq \|s_R\|_{L^{p^*}} \|y|_{\partial\Omega_2}\|_{L^p} \leq \|s_R\|_{L^{p^*}} \|y\|_{W^{1,p}}} \overset{\text{Poincaré inequality}}{\geq} c\|y\|_{W^{1,p}}^p - (\|\hat{b}\|_{W^{-1,p^*}} + \|s_R\|_{L^{p^*}})\|y\|_{W^{1,p}}$ (+)

$\Rightarrow$ bounded below

$E[y_0] < \infty$

1) Consider "minimizing sequence" $y_1, y_2, \ldots$ with $E[y_n] \to \inf_y E[y]$

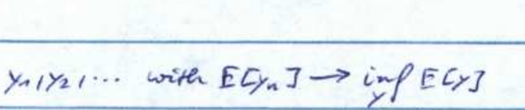
2) Show that a subsequence converges in some special space (to be appropriately chosen).
In detail: Derive compactness of $\{y_1, y_2, \ldots\}$ in that space from energy boundedness.
Here: The space will be $W^{1,p}(\Omega)$ with weak topology.

$E[y_n] < C \underset{(+)}{\Rightarrow} \|y_n\|_{W^{1,p}} < \tilde{C} \Rightarrow$ a subsequence converges weakly, $y_n \xrightarrow[W^{1,p}]{} y^* \in W^{1,p}(\Omega)$

3) Show sequential lower semi-continuity of E in that space, i.e. $E[y^*] \leq \liminf_{n\to\infty} E[y_n]$
This implies $E[y^*] = \inf_y E[y]$.
Here:
- $y_0 = y_n|_{\partial\Omega_1} \xrightarrow{L^p} y^*|_{\partial\Omega_1}$ (by trace thm)
- $\int_{\partial\Omega_2} s_R \cdot y_n\,dA \to \int_{\partial\Omega_2} s_R \cdot y^*\,dA$ (by $y_n|_{\partial\Omega_2} \xrightarrow{L^p} y^*|_{\partial\Omega_2}$)
- $\int_\Omega \hat{b} \cdot y_n\,dx \to \int_\Omega \hat{b} \cdot y^*\,dx$ (by weak conv.)
- $\int_\Omega W(Dy^*)\,dx \leq \liminf_{n\to\infty} \int_\Omega W(Dy_n)\,dx$?? (x)

<u>Example where (x) does not hold (2D)</u>: $g(x) = \int_0^{x_1+x_2} \chi_{[0,3/4]}(s)\,ds$, $y_n(x) = \frac{1}{n}\begin{pmatrix} g(nx) \\ g(nx) \end{pmatrix}$

$\Rightarrow \nabla y_n(x) = \chi_{[0,3/4]}(n(x_1+x_2)) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$



$y_n(x) \to y^*(x) = \frac{3}{4}\begin{pmatrix} x_1+x_2 \\ x_1+x_2 \end{pmatrix}$, $Dy^* = \frac{3}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

For $W(F) = (\|F\|_2 - 2)^2 \|F\|_2^2$, $\int_{[0,1]^2} W(Dy_n)\,dx = 0 > \int_{[0,1]} W(Dy^*)\,dx = (\frac{3}{2} - 2)^2 (\frac{3}{2})^2$