When does equilibrium state satisfy a variational principle?

Gibbs free energy $\quad E[y] = \underbrace{\int_\Omega W(Dy)\,dx}_{\text{stored elastic energy}} + \underbrace{\int_\Omega \Psi(y)\,dx}_{\text{potential energy of body forces}} + \underbrace{\int_{\partial\Omega_2} \chi(y)\,dA(x)}_{\text{potential energy of surface forces}}$

A minimizer satisfies the Euler-Lagrange equation: for all smooth ϑ with $\vartheta=0$ on $\partial\Omega_1$,

$$0 = \partial_y E[y](\vartheta) = \int_\Omega W_{,A}(Dy) : D\vartheta + \nabla_y \Psi \cdot \vartheta\,dx + \int_{\partial\Omega_2} \nabla_y \chi \cdot \vartheta\,dA(x)$$

$$= \int_\Omega (-\operatorname{div} T_R(x) + \nabla_y \Psi)\cdot\vartheta\,dx + \int_{\partial\Omega_2 \cup \partial\Omega_3} (T_R(x)N(x) + \nabla_y\chi)\cdot\vartheta\,dA(x)$$

$\Rightarrow$ equil. state satisfies var. princ. iff
- body force is conservative, i.e. $f\,b = -\nabla_y \Psi(y(x))$
  (e.g. gravity $b = \rho_R/J \begin{pmatrix}0\\0\\-g\end{pmatrix} \Rightarrow \Psi(y(x)) = \rho_R(x)\,g\,y_3(x)$)
- surface load is conservative, $s = -\nabla_y \chi(y(x))$
  (e.g. dead load $s_R(x) \Rightarrow \chi = -s_R(x)\cdot y(x)$)

Ex: gravity & dead load: equil. state minimizes $E[y] = \int W(Dy)\,dx + g\int_\Omega \rho_R(x)\,y_3(x)\,dx - \int_{\partial\Omega_2} s_R(x)\cdot y(x)\,dA$

<u>"dead load"</u>: $T_R(x)N(x) = s_R(x)$; keeps direction and force per unit reference area

<u>"live load"</u>: $T(y)\,n(y) = s(y)$

<u>Existence theory for minimizers of E</u>

Assume $W(F) \ge c\,\|F\|_2^p$, $p>3$, $b = \frac{\hat b}{J}$ with $\hat b \in W^{-1,p^*}(\Omega)$, $s_R \in L^{p^*}(\partial\Omega_3)$, $\frac{1}{p}+\frac{1}{p^*}=1$, Ω Lipschitz

Define $E[y]=\infty$ if $y \notin W^{1,p}(\Omega)$ or $y \ne y_0$ on $\partial\Omega_1$.

<u>Direct method of calculus of variations</u>

0) E is not identically ∞ & is bdd from below.

Here: $E[y] \ge c\|\nabla y\|_{L^p}^p - \underbrace{\int_\Omega \hat b\cdot y\,dx}_{\le \|\hat b\|_{W^{-1,p^*}}\|y\|_{W^{1,p}}} - \underbrace{\int_{\partial\Omega_2} s_R\cdot y\,dA(x)}_{\le \|s_R\|_{L^{p^*}}\|y|_{\partial\Omega_2}\|_{L^p} \le \|s_R\|_{L^{p^*}}\|y\|_{W^{1,p}}} \overset{\text{Poincaré inequality}}{\ge} c\|y\|_{W^{1,p}}^p - (\|\hat b\|_{W^{-1,p^*}} + \|s_R\|_{L^{p^*}})\|y\|_{W^{1,p}}$ (+)

$\Rightarrow$ bounded below

$E[y_0] < \infty$

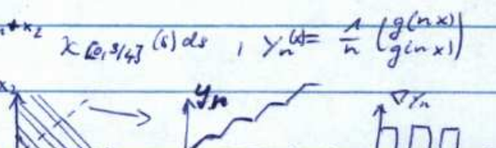
1) Consider "minimizing sequence" $y_1, y_2, \ldots$ with $E[y_n] \to \inf_y E[y]$

2) Show that a subsequence converges in some special space (to be appropriately chosen).
In detail: Derive compactness of $\{y_1, y_2, \ldots\}$ in that space from energy boundedness.
Here: The space will be $W^{1,p}(\Omega)$ with weak topology.
$E[y_n] < C \underset{(+)}{\Rightarrow} \|y_n\|_{W^{1,p}} < \tilde C \Rightarrow$ a subsequence converges weakly, $y_n \xrightarrow[W^{1,p}]{} y^* \in W^{1,p}(\Omega)$

3) Show sequential lower semi-continuity of E in that space, i.e. $E[y^*] \le \liminf_{n\to\infty} E[y_n]$
This implies $E[y^*] = \inf_y E[y]$.
Here:
- $y_0 = y_n|_{\partial\Omega_1} \xrightarrow{L^p} y^*|_{\partial\Omega_1}$ (by trace thm)
- $\int_{\partial\Omega_2} s_R\cdot y_n\,dA \longrightarrow \int_{\partial\Omega_2} s_R\cdot y^*\,dA$ (by $y_n|_{\partial\Omega_2} \xrightarrow{L^p} y^*|_{\partial\Omega_2}$)
- $\int_\Omega \hat b\cdot y_n\,dx \longrightarrow \int_\Omega \hat b\cdot y^*\,dx$ (by weak conv.)
- $\int_\Omega W(Dy^*)\,dx \le \liminf_{n\to\infty}\int_\Omega W(Dy_n)\,dx$?? $\quad(\times)$

<u>Example where $(\times)$ does not hold</u> (2D): $g(x) = \int_0^{x_1+x_2} \chi_{[0,3/4]}(s)\,ds$, $y_n(x) = \frac{1}{n}\begin{pmatrix}g(nx)\\g(nx)\end{pmatrix}$

$\Rightarrow Dy_n(x) = \chi_{[0,3/4]}(n(x_1+x_2))\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$



$y_n(x) \longrightarrow y^*(x) = \frac{3}{4}\begin{pmatrix}x_1+x_2\\x_1+x_2\end{pmatrix}$, $Dy^* = \frac{3}{4}\begin{pmatrix}1&1\\1&1\end{pmatrix}$

For $W(F) = (\|F\|_2 - 2)^2\|F\|_2^2$, $\int_{[0,1]^2} W(Dy_n)\,dx = 0 > \int_{[0,1]} W(Dy^*)\,dx = (\frac{3}{2}-2)^2(\frac{3}{2})^2$

Definition: $W:\mathbb{R}^{3\times3}\to\mathbb{R}$ is called "polyconvex" if there is a convex $\hat W:\mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3}\times\mathbb{R}\to\mathbb{R}$ with $W(F)=\hat W(F,\operatorname{cof}F,\det F)\ \forall F\in\mathbb{R}^{3\times3}$.

Thm: If W is non-negative, lower semi-continuous, and polyconvex, then $\int_\Omega W(Dy)\,dx$ is seq. lower semi-cont. wrt. weak $W^{1,p}$-convergence $(p>3)$.

proof: Let $y_n \rightharpoonup y^*$ in $W^{1,p}(\Omega)$. Need to show $\int_\Omega W(Dy^*)\,dx \le \liminf_{n\to\infty}\int_\Omega W(Dy_n)\,dx$.
By taking a subsequence, liminf can be replaced by lim.
Since Dy_n is bdd in L^p, $\operatorname{cof} Dy_n$ is bdd in $L^{p/2}$ and $\det Dy_n$ in $L^{p/3}$.
$\Rightarrow$ By taking a subsequence, $w_n := (Dy_n, \operatorname{cof}Dy_n, \det Dy_n) \xrightharpoonup{L^p\times L^{p/2}\times L^{p/3}} (Dy^*,A,b) =: w^*$.
~~Upon taking another subsequence, conv~~ By Mazur's lemma there exists a sequence of convex combinations $\hat\omega_n := \sum_{j=n}^{N_n}\alpha_j^n w_j$, $N_n\in\mathbb{N}$, $\alpha_j^n\in[0,1]$, $\sum_{j=n}^{N_n}\alpha_j^n = 1$
with $\hat\omega_n \xrightarrow{L^p\times L^{p/2}\times L^{p/3}} w^*$ strongly. Upon taking a subsequence, convergence is even pointwise a.e., hence $A=\operatorname{cof}Dy^*$, $b=\det Dy^*$. Now

$$\int_\Omega W(Dy^*)\,dx = \int_\Omega \hat W(w^*)\,dx \underset{\substack{\uparrow\\ \hat W \text{ is lsc}\\ +\text{ pointwise convergence}}}{\le} \int_\Omega \liminf_{n\to\infty} \hat W(\hat\omega_n)\,dx \underset{\substack{\uparrow\\ \text{Fatou's lemma}}}{\le} \liminf_{n\to\infty}\int_\Omega \hat W(\hat\omega_n)\,dx$$

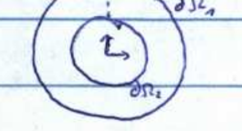
$$\underset{\substack{\uparrow\\ \hat W\text{ convex}}}{\le} \liminf_{n\to\infty}\int_\Omega \sum_{j=n}^{N_n}\alpha_j^n\,\hat W(w_n)\,dx = \liminf_{n\to\infty}\sum_{j=n}^{N_n}\alpha_j^n\int_\Omega W(Dy_n)\,dx = \liminf_{n\to\infty}\int_\Omega W(Dy_n)\,dx$$

□

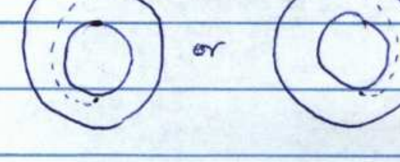
Rk: • With additional conditions one can relax $p>3$
• In fact, polyconvexity is too strong; one only needs "$W^{1,p}$-quasi-convexity", but this notion is difficult to use/understand.

Non-Uniqueness of minimizers/equilibrium states

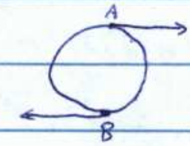
pure Dirichlet bc: (Ex. by Fritz John)



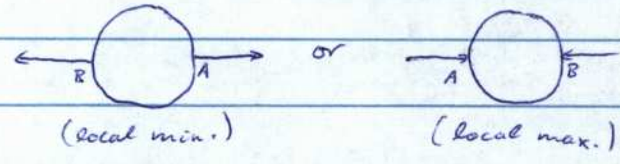
$y(x)=x$ on $\partial\Omega_1$
$y(x)=-x$ on $\partial\Omega_2$ $\rightsquigarrow$

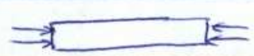


pure traction:



$\rightsquigarrow$





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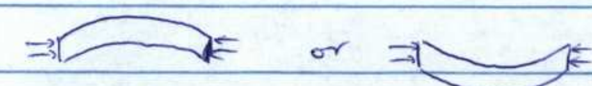
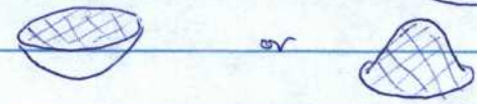
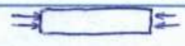


zero traction:

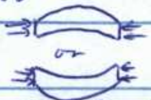


Buckling

Often, non-uniqueness is associated with "buckling", a bifurcation phenomenon: By slowly increasing the load, a stable eq. state suddenly becomes an unstable saddle point, and two new energy minimizers occur.

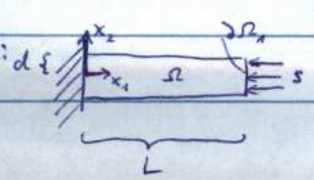


small load



high load

Exple:



$E[y] = \int_\Omega W(Dy)\,dx - \int_{\partial\Omega_N} s\cdot y\,dA$

$W(F) = \mu\,\mathrm{tr}\,E^2$ with $E=\frac{1}{2}(F^TF-I)$ (St. Venant-Kirchhoff material with Poisson ratio 0)

$W_{,A}(F) = \mu FE = \mu F(F^T F - I)$

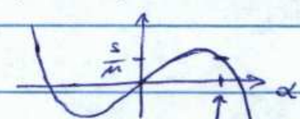
$W_{,AA}(F)(M,N) = -\mu M:N + \mu (MF^T):(NF^T) + \mu (F^T M):(F^T N) + \mu (F^T M):(N^T F)$ with $A:B = \operatorname{tr}(A^T B)$

E-L-eq.: $0 = \partial_y E[y](\vartheta) = \int_\Omega W_{,A}(Dy):D\vartheta \, dx - \int_{\partial\Omega_1} s\cdot\vartheta \, dA$

$\Rightarrow \operatorname{div} W_{,A}(Dy) = 0$ on Ω, $W_{,A}(Dy)N = -s$ on $\partial\Omega_1$

ansatz: $\hat y(x) = \begin{pmatrix} \alpha x_1 \\ x_2 \end{pmatrix} \Rightarrow D\hat y = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$

$-s = W_{,A}(D\hat y)N = \begin{pmatrix} \mu\alpha(\alpha^2-1) \\ 0 \end{pmatrix} \Rightarrow \hat y$ is solution for α satisfying $\alpha(1-\alpha^2) = \frac{s}{\mu}$



is $\hat y$ energy minimizer ?

2nd order analysis: $\partial_y^2 E[\hat y](\vartheta,\Theta) = \int_\Omega W_{,AA}(D\hat y)(D\vartheta, D\Theta)\, dx$

$= \mu \int_\Omega -D\vartheta:D\Theta + (D\vartheta D\hat y^T):(D\Theta D\hat y^T) + (D\hat y^T D\vartheta):(D\hat y^T D\Theta) + (D\hat y^T D\vartheta):(D\Theta^T D\hat y)\, dx$

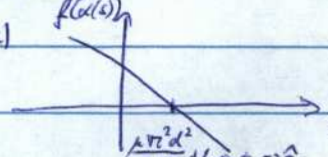
If quadratic form $-\partial_y^2 E[\hat y](\Theta,\Theta) < 0$ for some Θ (i.e. the associated Lax-Milgram operator has a negative eigenvalue for an eigenvector Θ), then $\hat y$ is a saddle!

Guessing critical eigenvector Θ:

$\Theta(x) = \varepsilon \begin{pmatrix} \frac{\pi x_2}{2L} \sin\left(\frac{\pi x_1}{2L}\right) \\ \cos\left(\frac{\pi x_1}{2L}\right) - 1 \end{pmatrix} \quad (+O(\varepsilon^2))$

$D\Theta = \varepsilon \begin{pmatrix} \left(\frac{\pi}{2L}\right)^2 x_2 \cos\left(\frac{\pi x_1}{2L}\right) & \frac{\pi}{2L}\sin\left(\frac{\pi x_1}{2L}\right) \\ -\frac{\pi}{2L}\sin\left(\frac{\pi x_1}{2L}\right) & 0 \end{pmatrix}$

$\Rightarrow \partial_y^2 E[\hat y](\Theta,\Theta) = \mu\varepsilon^2 \left(\frac{\pi}{2L}\right)^2 \left[(2\alpha^2+\alpha-1)\frac{\pi^2 d^3}{12L} + 2\alpha(\alpha-1)Ld\right] =: f(\alpha)$



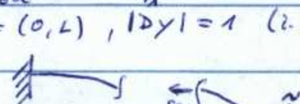
for $s > \hat s$, symmetric eq. state becomes instable!

<u>Analysis is simpler for 1D example</u>

1D inextensible rod

kinematics: $\Omega = (0,L)$, $|Dy| = 1$ (i.e. $Dy(x) = (\cos(\theta(x)), \sin\theta(x), 0)^T$)

kinetics/statics:



at each point x inside the rod there is a force $s(x)$ and a bending moment $m(x)$ (analogue to Cauchy stress hypothesis)

balance of lin. mom.: $s(x) = -\tilde s = (-|\tilde s|, 0, 0)$

balance of ang. mom.: $m(x) = \tilde s \times (y(L) - y(x))$ or equiv. $Dm(x) = -\tilde s \times Dy(x)$

constitutive law: $m(x) = (0,0,C\theta'(x))$ (linear law is good approximation for thin rods and small deformations, $C = \frac{\mu}{3}d^3$ can be derived)

$\Rightarrow [C\theta']' = -|\tilde s|\sin\theta \qquad$ with $\theta(0) = 0$, $\theta'(L) = 0 \qquad (*)$

This is Euler's elastica; there is also an associated variational principle:

$$E[\theta] = \int_\Omega \underbrace{\frac{C}{2}\theta'^2}_{\text{bending energy}} + \underbrace{|\tilde s|\cos\theta}_{\text{potential energy of load}} \, dx$$

Symmetric equil. state $\hat\theta \equiv 0$ is critical pt. of energy – is it a minimizer?

$\partial_\theta^2 E[\hat\theta](\vartheta) = \int_\Omega C\vartheta'^2 - |\tilde s|\cos(\hat\theta)\vartheta^2 \, dx = \int_\Omega C\vartheta'^2 - |\tilde s|\vartheta^2 \, dx = \int_0^L \vartheta \underbrace{(-C\vartheta'' - |\tilde s|\vartheta)}_{=: \mathcal{L}\vartheta} dx$

This is < 0 for some ϑ (and hence $\hat\theta$ unstable) iff smallest eigenvalue of $\mathcal{L}$ is < 0

smallest eigenvalue $\frac{C\pi^2}{4L^2} - |\tilde s|$, eigenvector $\vartheta = \sin\left(\frac{\pi}{2}\frac{x_1}{L}\right)$