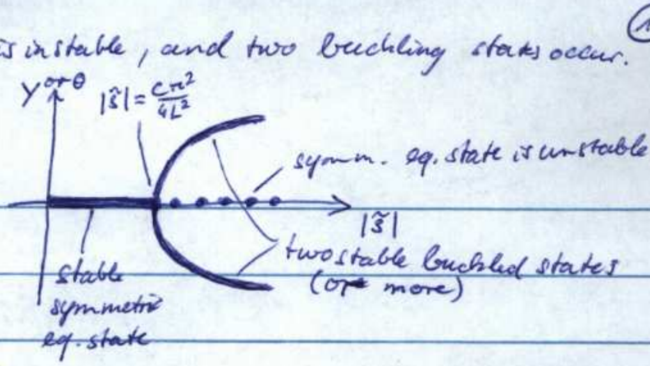
$\Rightarrow$ for $|\tilde{\sigma}| > \frac{C\pi^2}{4L^2}$ the symmetric eq. state is instable, and two buckling states occur. (11)

this is a pitchfork bifurcation,



formal calculation justifies this picture:

let $|\tilde{\sigma}|(\varepsilon) = \frac{C\pi^2}{4L^2} + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \dots$

$$\theta(x,\varepsilon) = \hat{\theta}(x) + \varepsilon\theta_1(x) + \varepsilon^2\theta_2(x) + \dots$$

insert into (*) & expand in powers of ε:

$$0 = C\left(\varepsilon\theta_1(x) + \varepsilon^2\theta_2(x) + \dots\right)'' + \left(\frac{C\pi^2}{4L^2} + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \dots\right)\sin\left(\varepsilon\theta_1(x) + \varepsilon^2\theta_2(x) + \dots\right)$$

$$= C\left(\varepsilon\theta_1'' + \varepsilon^2\theta_2'' + \dots\right) + \left(\frac{C\pi^2}{4L^2} + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \dots\right)\left(\varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3\left(\theta_3 - \frac{\theta_1^3}{6}\right) + \dots\right)$$

$\sin\gamma = \gamma - \frac{\gamma^3}{6} + \dots$

We get at order ε: $C\theta_1'' + \frac{C\pi^2}{4L^2}\theta_1 = 0$, $\theta_1(0) = \theta_1'(L) = 0 \Rightarrow \theta_1(x_1) = g\sin\left(\frac{\pi}{2L}x_1\right)$, $g \in \mathbb{R}$

at order ε^2: $C\theta_2'' + \frac{C\pi^2}{4L^2}\theta_2 = -\alpha_1\theta_1$, $\theta_2(0) = \theta_2'(L) = 0$

Solution exists, iff rhs is $\perp$ to nullspace of lhs, i.e. iff $\int_0^L \alpha_1 g\sin^2\left(\frac{\pi}{2L}x_1\right)dx_1 = 0$.

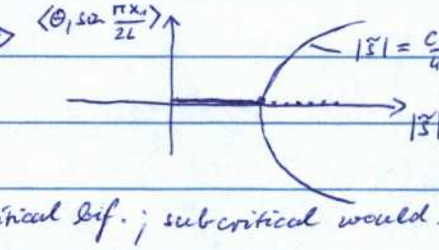
Assuming $g \neq 0$, we obtain $\alpha_1 = 0$, $\theta_2 = h\sin\left(\frac{\pi}{2L}x_1\right)$, $h \in \mathbb{R}$

at order ε^3: $C\theta_3'' + \frac{C\pi^2}{4L^2}\theta_3 = -\alpha_2\theta_1 - \underbrace{\alpha_1}_{0}\theta_2 + \frac{C\pi^2}{4L^2}\frac{\theta_1^3}{6}$, $\theta_3(0) = \theta_3'(L) = 0$

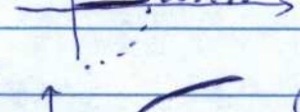
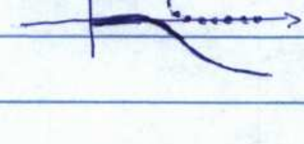
Solution exists iff $\underbrace{\int_0^L -\alpha_2 g\sin^2\left(\frac{\pi}{2L}x_1\right) + \frac{C\pi^2}{24L^2}g^3\sin^4\left(\frac{\pi}{2L}x_1\right)dx_1}_{-\alpha_2 g\frac{L}{2} + \frac{C\pi^2}{24L^2}g^3\frac{3L}{8}} = 0$

$\Rightarrow \alpha_2 = \frac{C\pi^2}{32L^2}g^2$

$\Rightarrow$



$|\tilde{\sigma}| = \frac{C\pi^2}{4L^2}\left(1 + \frac{g^2}{8}\varepsilon^2 + O(\varepsilon^3)\right)$, $\langle\theta, \sin\frac{\pi x_1}{2L}\rangle = \varepsilon\frac{gL}{2} + O(\varepsilon^2)$

- This is a supercritical bif.; subcritical would be  , i.e. $\alpha_2 < 0$
- If the rod is slightly asymmetric, diagram turns into  (saddle node bifurcation)
- rigorous treatment: Lyapunov–Schmidt reduction

Linearized Elasticity

Assume material to be stress-free in reference configuration, and write $y(x,t) = x + u(x,t)$. Linearized elasticity assumes the displacement to be small and simply ignores all higher order terms. Abbreviate $H = Du$, then

$$W(Dy) = W(I + H) = W(I) + \underbrace{W_{,A}(I)}_{=0} : H + \frac{1}{2}W_{,AA}(I)(H,H) + o(|H|^2)$$

$$T_R(Dy) = \underbrace{T_R(I)}_{=0} + T_{R,A}(I)\,H + o(|H|)$$

Elasticity tensor $C = T_{R,A}(I) = W_{,AA}(I) : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$, $(CH)_{ij} = \sum_{k,\ell} C_{ijk\ell}H_{k\ell}$ with $C_{ijk\ell} = \frac{\partial^2 W(I)}{\partial F_{ij}\,\partial F_{k\ell}}$.

a) $C_{ijkl} = \dfrac{\partial^2 W(I)}{\partial F_{ij}\partial F_{kl}} \Rightarrow C_{ijkl} = C_{klij}$ "major symmetries"

b) frame indifference $\Rightarrow T_R(e^{Kt}) = 0\ \forall$ skew $K \in \mathbb{R}^{3\times 3}$, since $e^{Kt} \in SO(3)$ $\quad (e^{Kt}(e^{Kt})^T = e^{(K+K^T)t} = I)$, thus $\frac{d}{dt} T_R(e^{Kt})\big|_{t=0} = CK = 0 \Rightarrow C_{ijkl} = C_{ijlk}$

c) a) & b) $\Rightarrow C_{ijkl} = C_{jikl}$

"minor symmetries"

Rk: • Cauchy derived a theory in which additionally $C_{ijkl} = C_{ikjl}$ (based on pair interaction forces between atom centers), but this does not hold for many materials

• $\forall H \in \mathbb{R}^{3\times 3}$, $CH = C\left[\frac{H+H^T}{2} + \frac{H-H^T}{2}\right] = C\left(\frac{H+H^T}{2}\right) \Rightarrow CDu = C\varepsilon(u)$ for <u>linear strain tensor</u> $\varepsilon(u) = \frac{Du+Du^T}{2}$

• Counting yields: C can have 21 indep. components

• Can show: For isotropic materials, there are just 2 indep. components, in particular, $C\varepsilon = 2\mu\varepsilon + \lambda(\mathrm{tr}\,\varepsilon)I$ for <u>Lamé constants</u> μ, λ

• For description of isotropic materials each Lamé constant can be equivalently replaced by any of the following (also see homeworks)

- <u>Young's modulus</u> E : proportionality constant between longitudinal stress & strain for uniaxial tension
- <u>Poisson's ratio</u> ν : neg. ratio for uniaxial tension between strain orthogonal and parallel to axis
- <u>bulk modulus</u> K : $3\times$ proportionality constant between isotropic tension & strain (or proportion. const. betw. hydrostatic pressure & volume change)

• writing $\hat{b} = bf$, ignoring h.o.t., the momentum balance $\rho_R \ddot{y} = \mathrm{div}\, T_R(Dy) + \hat{b}$ becomes $\rho\ddot{u} = \mathrm{div}(CDu) + \hat{b}$

(notice that we no longer distinguish betw. reference coord. x and def. coord. y, since u is supposed small).

• rotations are replaced by their linearizations, i.e. skew-symmetric Du (see b) above), i.e. the stress $CDu = 0$ for Du skew.

$\Rightarrow$ if the physical problem involves large rotations, linearized elasticity will yield ~~displacements with large skew Du, which changes the size of the deformed objects~~

$\Rightarrow$ linearized elasticity not applicable

Since reference configuration is stress-free state, $W(I + \tau\varepsilon) \geq W(I)\ \forall \tau \in \mathbb{R}, \varepsilon \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, thus we need $(C\varepsilon):\varepsilon \geq 0$.

<u>Thm</u>: For an isotropic material, $(C\varepsilon):\varepsilon \geq 0 \iff \mu \geq 0$ and $3\lambda + 2\mu \geq 0$

proof: "$\Leftarrow$": $(C\varepsilon):\varepsilon = 2\mu|\varepsilon|^2 + \lambda(\mathrm{tr}\,\varepsilon)^2 = 2\mu\left|\varepsilon - \frac{1}{3}(\mathrm{tr}\,\varepsilon)I\right|^2 + \frac{3\lambda + 2\mu}{3}(\mathrm{tr}\,\varepsilon)^2$

"$\Rightarrow$": In above line, take ε with $\mathrm{tr}\,\varepsilon = 0$ or $\varepsilon = I$. $\square$

<u>Variational principle</u>: A solution of

$$\left.\begin{aligned} \mathrm{div}(C\varepsilon(u)) + \hat{b} &= 0 && \text{in } \Omega \\ u &= u_0 && \text{on } \partial\Omega_1 \\ (C\varepsilon)\, n &= s && \text{on } \partial\Omega_2 \end{aligned}\right\} (*)$$

is a critical point of $E[u] = \int_\Omega \frac{1}{2}(C\varepsilon(u)):\varepsilon(u) - \hat{b}\cdot u \,dx - \int_{\partial\Omega_2} s\cdot u \,da$, $u = u_0$ on $\partial\Omega_1$

(simply check Euler-Lagrange conditions)

<u>Lemma (Korn's 1st inequality)</u>: Let $\Omega \subset \mathbb{R}^3$ be bdd. with piecewise smooth bdy, then $\exists\, c = c(\Omega) > 0$ with

$$\int_\Omega u^2 + |\varepsilon(u)|^2 \,dx \geq c\int_\Omega u^2 + |\nabla u|^2 \,dx \quad \forall u \in H^1(\Omega).$$

<u>Lemma (Korn's 2nd inequality)</u>: Let additionally $\partial\Omega_2 \subset \partial\Omega$ have positive 2D measure, then $\exists\, c = c(\Omega, \partial\Omega_2) > 0$ with

$$\int_\Omega |\varepsilon(u)|^2 dx \geq c\int_\Omega u^2 + |\nabla u|^2 dx \quad \forall u \in H^1(\Omega) \text{ with } u|_{\partial\Omega_2} = 0.$$