First integrals & energy conservation

<u>Ex</u>: PM) $m\ddot{q} = -\nabla_q V(q) \underset{\text{test with } \dot{q}}{\Longrightarrow} \underbrace{m\ddot{q}\dot{q}}_{=\frac{d}{dt}\left(\frac{m}{2}|\dot{q}|^2\right)} = \underbrace{-\nabla_q V(q)\,\dot{q}}_{=-\frac{d}{dt}V(q)} \Rightarrow H = \frac{m}{2}|\dot{q}|^2 + V(q) = \text{const.}$

$\Rightarrow$ energy is conserved!

This energy conservation is a more general concept:

<u>Thm (energy conservation)</u>: If the Lagrangian L does not depend on time, then Newton's law implies energy conservation, i.e. $H(q,p) = \text{const.}$

proof: Test $(**)$ with $\dot{q} \Rightarrow 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\cdot\dot{q} - \frac{\partial L}{\partial q}\cdot\dot{q} = \dot{p}\cdot\dot{q} + \frac{\partial H}{\partial q}\dot{q} = \frac{\partial H}{\partial p}\cdot\dot{p} + \frac{\partial H}{\partial q}\dot{q} = \frac{dH}{dt}$ $\square$

This conservation law can be used to reduce the second order ode $(**)$ or the system of two odes $(***)$ to a single first order ode (in the case $N=1$ of one coordinate) by solving $H(q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q})) = H(q(0), p(0))$ for $\dot{q}$.

<u>Ex</u>: PMe) For $V(q) = -\frac{\gamma m M}{|q|}$ we obtain $H = \text{const.} \Rightarrow |\dot{q}| = \sqrt{\frac{2\gamma M}{|q|}}$, i.e. if we know $q_1(t), q_3(t)$ we obtain an ode for q_2

A conserved quantity G, i.e. one for which $\frac{dG}{dt} = 0$ under the dynamics $(**)$ is called a <u>first integral</u> and can always be used as above to reduce the ode. <u>Noether's Theorem</u> (Emmy Noether, 1915) states that any continuous symmetry of the Lagrangian implies the existence of a corresponding first integral. As seen above, a translational symmetry in time, i.e. $L(t+\Delta t, q, \dot{q}) = L(t, q, \dot{q})$ implies energy conservation.

<u>Thm (momentum conservation)</u>: If L is invariant under q_k, then p_k is conserved.

proof: $\dot{p}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)_k = \frac{\partial L}{\partial q_k} = 0$ $\square$

<u>Ex</u>: PMe) L is invariant under rotation, i.e. in polar coordinates,

$$L\left(t, \begin{pmatrix} r \\ \theta+\Delta\theta \end{pmatrix}, \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}\right) = L\left(t, \begin{pmatrix} r \\ \theta \end{pmatrix}, \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}\right) = \frac{m}{2}\left(r^2\dot{\theta}^2 + \dot{r}^2\right) + \frac{\gamma m M}{r}$$

$\Rightarrow$ angular momentum $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$ is conserved

Together with the conservation of $H = \frac{m}{2}(r^2\dot{\theta}^2 + \dot{r}^2) - \frac{\gamma m M}{r}$ we obtain

$\dot{r} = \sqrt{\frac{2H}{m} + \frac{2\gamma M}{r} - \left(\frac{p_\theta}{rm}\right)^2}$ or $\int \frac{dr}{\sqrt{\frac{2H}{m} + \frac{2\gamma M}{r} - \left(\frac{p_\theta}{rm}\right)^2}} = \int dt$, and then (after solving for r)

$\dot{\theta} = \frac{p_\theta}{m r(t)^2}$ or $\theta = \theta(0) + \int_0^t \frac{p_\theta}{m r^2}\, dt.$

A system is called <u>integrable</u>, if it can be reduced to a sequence of integrals as above. As a rule, an integrable system has as many first integrals as generalized coordinates q. (typically the energy and $N-1$ momenta). The 2-body pb for instance is integrable (energy, total linear momentum in x- and y-direction, total angular momentum), while the 3-body pb is not. Non-integrable systems are typically only predictable over short time intervals.

Virial Theorem

This time test $(**)$ with q and integrate over $[0,\tau]$; $0 = q\cdot\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - q\cdot\frac{\partial L}{\partial q} = \frac{d}{dt}\left(q\cdot\frac{\partial L}{\partial \dot{q}}\right) - \dot{q}\cdot\frac{\partial L}{\partial \dot{q}} - q\cdot\frac{\partial L}{\partial q}$

$\Rightarrow 0 = q\cdot\frac{\partial L}{\partial \dot{q}}\Big|_0^\tau - \int_0^\tau \dot{q}\cdot\frac{\partial L}{\partial \dot{q}} + q\cdot\frac{\partial L}{\partial q}\, dt$ (alternatively, one can write $0 = p\cdot q\big|_0^\tau - \int_0^\tau (p\cdot q)^{\cdot}\, dt = p\cdot q\big|_0^\tau - \int_0^\tau p\cdot\dot{q} + \dot{p}\cdot q\, dt$)

If p and q are bounded, then dividing by τ and letting $\tau \to \infty$, we obtain

$$\overline{p\cdot\dot{q}} = \overline{\dot{p}\cdot q}$$

with $\overline{(\cdot)}$ being the time average.

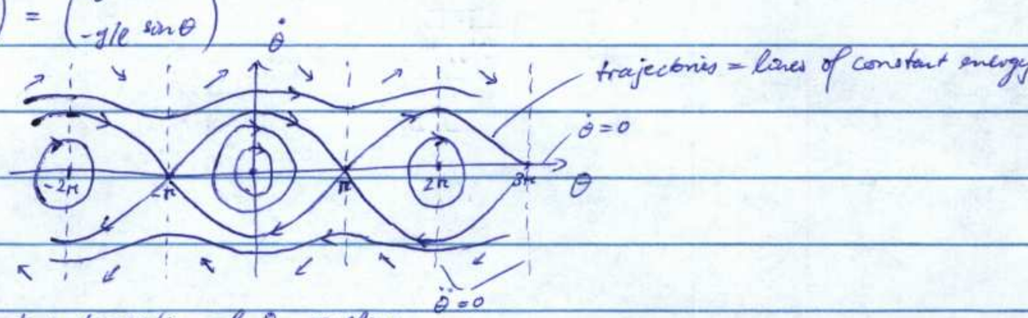
<u>Ex</u>: P) $\overline{m l^2 \dot{\theta}^2} = -mgl\,\overline{\theta \sin\theta}$. For small θ, $\frac{\theta\sin\theta}{2} \approx 1-\cos\theta$ so that the above becomes $2\overline{T} = mgl + 2\overline{V}$, so we have <u>equipartition</u> of energy between kinetic and potential energy.

The virial theorem always applies if from energy conservation one can imply boundedness of p and q.

Phase space

- Phase space is the space of all possible states $\binom{\dot q}{q} \in \mathbb{R}^{2N}$
- Every dynamical system traces out a trajectory in phase space.
- Energy conservation $\Rightarrow$ trajectories coincide with level lines of energy H.

Ex: P) $\frac{d}{dt}\begin{pmatrix}\theta \\ \dot\theta\end{pmatrix} = \begin{pmatrix}\dot\theta \\ -g/\ell \sin\theta\end{pmatrix}$



- steady states at intersection of 0-isoclines
- time parametrization cannot be read off (e.g. trajectory connecting two steady states takes infinitely long)

Lagrangian perspective (recall: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}$ $(**)$)

Action principle

The trajectory of a system with Lagrangian L from a generalized coordinate $q(0)$ to a coordinate $q(\tau)$ is a maximum or minimum or saddle point of the action

$$S[q(t)] = \int_0^\tau L(t, q(t), \dot q(t))\, dt .$$

Indeed, the Gâteaux derivative of S in some direction φ with $\varphi(0) = \varphi(\tau) = 0$ is

$$\partial_q S[q](\varphi) = \int_0^\tau \frac{\partial L}{\partial q}\cdot\varphi + \frac{\partial L}{\partial \dot q}\cdot\dot\varphi\, dt = \frac{\partial L}{\partial \dot q}\cdot\varphi\Big|_0^\tau + \int_0^\tau \varphi\cdot\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right) dt = 0$$

Note: Here, instead of specifying $q(0)$ and $\dot q(0)$ as usual, we specify $q(0)$ and $q(\tau)$.

Coordinate invariance

Ex: MP2) in Euclidean coordinates: $m\ddot q = -\frac{\gamma m M}{|q|^2}\frac{q}{|q|}$ is $(**)$ for $L(t,q,\dot q) = \frac{m}{2}|\dot q|^2 + \frac{\gamma m M}{|q|}$

in polar coordinates (2D): $m\left[\ddot r\binom{\cos\theta}{\sin\theta} + 2\dot r\binom{-\sin\theta}{\cos\theta}\dot\theta + r\ddot\theta\binom{-\sin\theta}{\cos\theta} - r\dot\theta^2\binom{\cos\theta}{\sin\theta}\right] = -\frac{\gamma m M}{r^2}\binom{\cos\theta}{\sin\theta}$

$$\underset{\substack{\text{1st line}\cdot\cos\theta + \text{2nd line}\cdot\sin\theta \\ \text{1st line}\cdot(-r\sin\theta) + \text{2nd line}\cdot(r\cos\theta)}}{\Longleftrightarrow} \quad \begin{aligned} m\ddot r - mr\dot\theta^2 &= -\gamma m M/r^2 \\ m(r^2\ddot\theta + 2r\dot r\dot\theta) &= 0 \end{aligned}$$

, which is $(**)$ for $L\left(t, \binom{r}{\theta}, \binom{\dot r}{\dot\theta}\right) = \frac{m}{2}(r^2\dot\theta^2 + \dot r^2) + \frac{\gamma m M}{r}$

In general: Our systems satisfy $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}$ for some $L(t,q,\dot q)$. If we introduce new generalized coordinates Q via a smooth bijection $q = f(Q)$, then the system also satisfies $\frac{d}{dt}\left(\frac{\partial \tilde L}{\partial \dot Q}\right) = \frac{\partial \tilde L}{\partial Q}$ for $\tilde L(t, Q, \dot Q) = L(t, f(Q), f'(Q)\dot Q)$, since

$$\frac{d}{dt}\left(\frac{\partial \tilde L}{\partial \dot Q}\right) - \frac{\partial \tilde L}{\partial Q} = \frac{d}{dt}\left(f'(Q)\frac{\partial L}{\partial \dot q}\right) - f'(Q)\frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot q}f''(Q)\dot Q = f'(Q)\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q}\right) = 0.$$

$\Rightarrow$ no matter in which coordinates we state the Lagrangian, $(**)$ always holds!

Systems with constraints

The action principle easily generalizes to constrained systems: If a system satisfies a kinematic constraint, then the system path $q(t)$ from one state $q(0)$ to another $q(\tau)$ minimizes the action among all paths that satisfy the constraint. This seems to be a physical fact, at least no counterexamples have been found.

Ex: P) polar coord: $q = \binom{r}{\theta} \Rightarrow L(q,\dot q) = T(\dot q) - V(q) = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) + gmr\cos\theta$

constraint: $r = \ell$

Of course, in this simple example one could directly substitute $r = \ell$ and then via $(**)$ arrive at $\ddot\theta + \frac{g}{\ell}\sin\theta = 0$. Alternatively (and more general) one can use the method of Lagrange multipliers: