GENERICALLY INFINITE ENTROPY IN A SIMPLE AF ALGEBRA

ABSTRACT. We construct a simple unital AF algebra for which the *-automorphisms that are not approximately inner all have infinite entropy and form a point-norm dense open set, where entropy is taken in any of the Voiculescu-Brown, lower Voiculescu-Brown, or contractive approximation senses.

1. INTRODUCTION

It is typically a hopeless task to try to understand the behaviour of all possible dynamical systems on a given space. However, one can frequently isolate properties that hold for a generic system, where generic is taken in the sense of Baire category. The study of generic dynamics has a long history stretching back to the pioneering work of Oxtoby and Ulam, who showed that for compact manifolds of dimension at least 2 a homeomorphism preserving a fixed probability measure of full support is generically ergodic for the uniform topology, and Halmos, who showed that an automorphism of a nonatomic Lebesgue space is generically weakly mixing for the weak topology. For references and surveys of the subsequent development of the subject as it involves measurable dynamics and measure-preserving topological dynamics and the relation between them, see [4, 2]. As a point of comparison for the present work, we mention in particular a result of Rokhlin which asserts that zero Kolmogorov-Sinai entropy is generic for automorphisms of a nonatomic Lebesgue space with respect to both the weak and uniform topologies.

In the purely topological setting, Akin, Hurley, and Kennedy recently carried out an extensive investigation of the generic behaviour, with respect to the uniform topology, of homeomorphisms on a large class of compact metric spaces [1]. In particular they demonstrated that a generic homeomorphism of a compact manifold of dimension at least 2, while almost equicontinuous, has a rather complicated geometric structure, including such features as a Cantor chain recurrent set, uncountably many nested sequences alternating between attractors and repellors, and a large collection of chain components which have a nontrivial subshift of finite type as a factor. Concerning topological entropy, which in its generalized senses is the focus of this note, Akin, Hurley, and Kennedy remark that, as a consequence of their results, an infinite value is generic in the above context. Glasner and Weiss showed meanwhile in [8] that infinite entropy is generic for homeomorphisms of the Hilbert cube, while zero entropy is generic for homeomorphisms of the Cantor set. The proof of the former result proceeds by locally perturbing an arbitrary homeomorphism around a fixed point to create a topological horseshoe, which is a stable entropy-producing structure which cannot exist in the zero-dimensional setting.

At the C^*-algebra level, the Cantor set provides an example of a unital commutative AF algebra, indeed the unique such with no isolated irreducible *-representations. Keeping within the zero-dimensional framework of unital AF algebras, what can we say about

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entropy of a generic *-automorphism at the opposite extreme when there are no nontrivial ideals?

In the case of the CAR algebra, it was shown in [10] that a generic *-automorphism has zero CA (contractive approximation) entropy. This is also the case for lower Voiculescu-Brown entropy, as is straightforward to deduce from the definition (see the end of this section and the beginning of Section 2) and the fact that every *-automorphism of the CAR algebra is approximately inner and hence approximable by an inductive limit *-automorphism. For Voiculescu-Brown entropy the situation is not as clear, although it is still true that zero values occur on a dense set of *-automorphisms.

In this note we construct a simple unital AF algebra which, from the viewpoint of dynamical entropy, exhibits behaviour that is starkly different from that of the Cantor set and the CAR algebra. This AF algebra possesses the property that the *-automorphisms that are not approximately inner all have infinite entropy and form a point-norm dense open set, with entropy taken in any of the above three senses. Exploiting the classification theory for AF algebras, the idea is to combine the algebraic control afforded by dimension group structure with the topological flexibility in having any metrizable Choquet simplex as a possible tracial state space. To establish lower bounds we employ a couple of results from the local theory of Banach spaces as formulated in [9, 10].

In contrast to the case of compact manifolds of dimension at least 2 [1] and the Hilbert cube [8], the stability of a horseshoe-type structure does not play a role here. The desired stability of lower entropy bounds will instead result from the rigidity of the dimension group. This rigidity permits us to lock in an infinite-entropy-producing geometric configuration on the tracial state space. On compact manifolds of dimension at least 2 and the Hilbert cube, only finite-entropy-producing topological horseshoe configurations are stable under perturbation, and so the conclusion for the set of infinite entropy homeomorphisms in this case is dense G_δ containment.

Before proceeding to the construction, we introduce some general notation and recall the definitions for the relevant notions of entropy.

Let X be a compact Hausdorff space. We denote by $C(X)$ the C^*-algebra of complex-valued functions on X, and by $C_{\mathbb{R}}(X)$ the real linear subspace of real-valued functions. We write M_X for the compact convex set of Borel probability measures on X and $\mathrm{Aff}(M_X)$ for the closed subspace of $C_{\mathbb{R}}(M_X)$ consisting of affine functions. Given a homeomorphism $T : X \to X$ we write α_T for the induced *-automorphism $f \mapsto f \circ T$ of $C(X)$. We denote by $\mathfrak{H}(X)$ the set of all homeomorphisms from X to itself, and equip this with the uniform topology. This topology can be described as that generated by the sets of the form

$$U_{T,\Omega,\varepsilon} = \left\{ S \in \mathfrak{H}(X) : \|\alpha_S(f) - \alpha_T(f)\| < \varepsilon \right\}$$

where $T \in \mathfrak{H}(X)$, Ω is a finite subset of $C(X)$, and $\varepsilon > 0$.

Let A be a C^*-algebra. We denote by $\mathrm{Aut}(A)$ the set of $*$-automorphisms of A equipped with the point-norm topology. When A is separable, $\mathrm{Aut}(A)$ is Polish and hence a Baire space. In the case that A is unital, we denote by T_A the set of tracial states on A equipped with the weak*-topology, and given an $\alpha \in \mathrm{Aut}(A)$ we write T_α for the induced affine homeomorphism $\tau \mapsto \tau \circ \alpha$ of T_A.

Let A be an exact C^*-algebra and $\pi : A \to \mathcal{B}(\mathcal{H})$ a faithful *-representation. Let Ω be a finite subset of A and $\delta > 0$. We denote by $\mathrm{CPA}(\pi, \Omega, \delta)$ the collection of triples (ϕ, ψ, B)

where B is a finite-dimensional C^*-algebra and $\phi : X \to B$ and $\psi : B \to Y$ are contractive completely positive linear maps such that

$$\|\psi \circ \phi(a) - \pi(a)\| < \delta$$

for all $a \in \Omega$. The completely positive rank is defined by

$$\operatorname{rcp}(\Omega, \delta) = \inf\{\operatorname{rank} B : (\phi, \psi, B) \in \operatorname{CPA}(\iota, \Omega, \delta)\}$$

and is independent of π by Arveson's extension theorem. For a *-automorphism α of A we define the *Voiculescu-Brown entropy* by

$$\operatorname{ht}(\alpha) = \sup_{\Omega} \sup_{\delta > 0} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{rcp}(\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta)$$

with Ω ranging over all finite subsets of A. The *lower Voiculescu-Brown entropy* $\operatorname{lht}(\alpha)$ is a variation on the above definition in which we take the limit infimum instead of the limit supremum.

For an isometric automorphism α of a Banach space V, the *contractive approximation entropy* or *CA entropy* $\operatorname{hc}(\alpha)$ is defined in formally the same way as Voiculescu-Brown entropy, only now using the contractive rank $\operatorname{rc}(\Omega, \delta)$. See [10] for the precise definition and more information.

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2. The construction

In [10] it was shown that, for a separable Banach space V, the isometric automorphisms of V with zero CA entropy form a G_δ subset of the set of all isometric automorphisms of V with the point-norm topology. A similar statement holds for lower Voiculescu-Brown entropy. Indeed if A is a separable exact C^*-algebra and $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ is an increasing collection of finite subsets of A whose union is dense in A, then the set of *-automorphisms of A with zero lower Voiculescu-Brown entropy is equal to

$$\bigcap_{i=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \big\{\alpha \in \operatorname{Aut}(A) : \operatorname{rcp}(\Omega_i \cup \alpha\Omega_i \cup \cdots \cup \alpha^m \Omega_i, 1/k) < e^{m/k} \text{ for some } m \geq n\big\},$$

which is a G_δ subset of $\operatorname{Aut}(A)$. Since *-automorphisms of finite-dimensional C^*-algebras have zero lower Voiculescu-Brown entropy, it follows that, for a unital AF algebra A, zero lower Voiculescu-Brown entropy is generic in $\overline{\operatorname{Inn}(A)}$, and hence generic in $\operatorname{Aut}(A)$ when every *-automorphism is approximately inner, as is the case for the CAR algebra, for example. Thus to obtain generically nonzero lower Voiculescu-Brown entropy in $\operatorname{Aut}(A)$ it is necessary that the identity *-automorphism be approximable by *-automorphisms which are not approximately inner, as our construction will illustrate. These comments also apply to CA entropy for similar reasons. For Voiculescu-Brown entropy, we don't know in general when the subset of *-automorphisms with a zero value form a G_δ set, in particular for the case of the CAR algebra. However, zero Voiculescu-Brown entropy occurs for a dense set of *-automorphisms of the CAR algebra, since every *-automorphism is approximable by an inner *-automorphism given by a unitary with finite spectrum. We also remark that for unital C^*-algebras which are tensorially stable with respect to the Jiang-Su algebra $\mathcal{Z}$, which includes all unital AF algebras, infinite entropy occurs on a dense set of *-automorphisms, with entropy taken in any of the three senses (see Section 8 of [10]).

Proceeding now to the construction, let $d \in \{2, 3, \dots\}$. We denote by X_d the quotient of the disjoint union of $[0, 1]^d$ and $[0, 1]$ obtained by identifying the two points $(0, \dots, 0) \in [0, 1]^d$ and $0 \in [0, 1]$. For convenience we will regard $[0, 1]^d$ and $[0, 1]$ as subsets of X_d.

Let T_d be a homeomorphism of X_d which on $[0, 1]$ acts as $x \mapsto x^2$ and on $[0, 1]^d$ admits a sequence $\{U_q\}_{q\in\mathbb{N}}$ of pairwise disjoint invariant open sets such that for each q the restriction of T_d to some closed invariant subset of U_q has the shift on 2^q symbols as a factor (which can be ensured for example by constructing a Smale horseshoe with crossing number 2^q).

Let $\{t_j\}_{j\in\mathbb{N}}$ be a strictly increasing sequence in $[1/4, 1/2)$ with $t_1 = 1/4$. Note that the intervals $T^k(t_j, t_{j+1})$ for $j, k \in \mathbb{N}$ are pairwise disjoint. Take a sequence $\{g_j\}_{j\in\mathbb{N}}$ in $C_{\mathbb{R}}(X_d)$ such that the rational linear span of $\{g_j\}_{j\in\mathbb{N}} \cup \{1\}$ is norm dense in $C_{\mathbb{R}}(X_d)$. By an approximation argument involving scalar translations and perturbations of the g_j's, we may assume that for each $j \in \mathbb{N}$ there is an open interval $I_j = (b_j, b_{j+1}) \subseteq (t_j, t_{j+1})$ such that g_j is zero on I_j and is a nonconstant polynomial function on each of the intervals $[0, b_j]$ and $[b_{j+1}, 1]$. We may furthermore assume that the nonzero coefficients of these polynomials, when collected together over all $j \in \mathbb{N}$, are all distinct and form a rationally independent set of irrational numbers, which we will denote by Θ. Note in particular that this implies that the numbers $g_j(1)$ for $j \in \mathbb{N}$ are distinct and form a rationally independent set.

Denote by G_d the additive subgroup of $C_{\mathbb{R}}(X_d)$ generated by all rational multiples of the elements in

$$\{g_i \circ T_d^m : i \in \mathbb{N} \text{ and } m \in \mathbb{Z}\} \cup \{1\}.$$

Taking all rational multiples guarantees that the subset

$$\{g \in G_d : g(x) \in (0, 1) \text{ for all } x \in X_d\} \cup \{1\}$$

of G_d is a scale when viewing G_d as a dimension group with the strict ordering. What we will actually need is the corresponding fact for a dimension group built using these G_d's, as described below.

Proposition 2.1. Let $S : X_d \to X_d$ be a homeomorphism which induces a group automorphism of G_d. Then $S = T_d^k$ for some $k \in \mathbb{Z}$.

Proof. Let $j, l \in \mathbb{N}$ and consider the function $g_j \circ T_d^l \circ S$. This can be expressed as

$$(*) \qquad a_0 + \sum_{i=1}^{s} \sum_{m=-r}^{r} a_{i,m}(g_i \circ T_d^m)$$

for some $s, r \in \mathbb{N}$, where a_0 and the $a_{i,m}$ are rational numbers. Since $T_d^l \circ S$ must map $[0, 1]$ homeomorphically to itself, the function $g_j \circ T_d^l \circ S$ is nonzero on some open interval I in $[0, 1]$. Since the intervals $T^m I_i$ for $i, m \in \mathbb{N}$ are pairwise disjoint, there exist $i', m' \in \mathbb{N}$ and an open subinterval $I' \subseteq I$ such that the intersection of I' and $T^m I_i$ is empty for every $i \neq i'$ and $m \neq m'$. Then, since for $m \in \mathbb{Z}$ the map T_d^m acts as $x \mapsto x^{2^m}$ on $[0,1]$, each function $g_i \circ T_d^m$ appearing in $(*)$, except possibly for $g_{i'} \circ T_d^{m'}$, restricts on I' to a nonconstant linear combination of dyadic rational powers of the variable. Evaluating $(*)$ on rational numbers in I' of the form $a/2^b$ for some integers a, b with $b \geq r$ and using the rational independence of the set Θ, we deduce that $a_0 = 0$ and the function $\sum_{m=-r}^{r} a_{i,m}(g_i \circ T_d^m)$ is zero on I' for each $i = 1, \dots, s$.

Now for a given $i = 1, \dots, s$, the functions $g_i \circ T_d^m$ for $m = -r, \dots, r$ (with the omission of $g_{i'} \circ T_d^{m'}$ if $i = i'$) are linearly independent on I' since, viewing them as linear combinations of dyadic rational powers of the variable, each contains a term which does not appear in any of the others, namely the term in $x^{2^m q_i}$ where q_i is the degree of g_i as a polynomial on $T_d^{-m} I'$. It follows that all of the $a_{i,m}$ except for $a_{i',m'}$ are zero. Furthermore, since S must fix the point 1 and the numbers $g_i(1)$ for $i \in \mathbb{N}$ are distinct and rationally independent, we in fact have $i' = j$, so that $g_j \circ T_d^l \circ S = g_j \circ T_d^{m'}$. It follows that S permutes the pairwise disjoint intervals $T_d^m I_j$ for $j, m \in \mathbb{N}$, and on the endpoints of each of these intervals S agrees with some power of T_d. But S must also preserve the order of these intervals as disjoint subsets of $[0,1]$, and so we infer the existence of a $k \in \mathbb{N}$ such that $g_j \circ T_d^l \circ S = g_j \circ T_d^{l+k}$ for all $j, l \in \mathbb{N}$, from which we conclude that $S = T_d^k$. □

Having performed the above construction for each $d = 2, 3, \dots$, we let X be the one-point compactification of the disjoint union $\coprod_{d \geq 2} X_d$, and write x_∞ for the point at infinity. Let G be the additive group of all $g \in C_{\mathbb{R}}(X)$ for which there exists an integer $m \geq 2$ such that $g|_{X_d} \in G_d$ for each $d = 1, \dots, m$ and the restriction of g to $X \setminus \bigcup_{d=1}^m X_d$ is constant and rational-valued. Then G is countable and contains the constant function 1. It is also uniformly dense in $C_{\mathbb{R}}(X)$, which implies that it is uniformly dense in $\mathrm{Aff}(M_X)$ when viewed as a subset of the latter space. We endow G with the strict ordering, i.e., the positive cone is given by

$$G_+ = \{g \in G : g(x) > 0 \text{ for all } x \in X\} \cup \{0\} = \{g \in G : \mu(g) > 0 \text{ for all } \mu \in M_X\} \cup \{0\}.$$

Then (G, G_+) is a simple dimension group by Lemma 3.1 of [6].

Set

$$\Sigma = \{g \in G : g(x) \in (0,1) \text{ for all } x \in X\} \cup \{1\} \subseteq G_+.$$

Observe that Σ is upwards directed and hereditary, and that every element of G_+ is a rational multiple of an element of Σ. It follows that Σ is a scale [5, Sect. IV.3][11, Sect. 1.4]. Hence there is a simple unital AF algebra A such that the scaled dimension group $(K_0(A), K_0(A)_+, \Sigma(A))$ is isomorphic to (G, G_+, Σ), with the class of the unit of A being associated to the constant function 1 in Σ. This isomorphism induces an affine homeomorphism from T_A, as canonically identified with the state space of $(K_0(A), K_0(A)_+, \Sigma(A))$, to M_X, as canonically identified with the state space of (G, G_+, Σ).

Now suppose we are given a *-automorphism α of A. Since the induced homeomorphism T_α of T_A is affine, the extreme boundary of M_X, which we canonically identify with X, is T_α-invariant. Moreover the connected components $\{x_\infty\}, X_2, X_3, \dots$ of X are each T_α-invariant since they are pairwise non-homeomorphic. By the classification theory for AF algebras (see [11, Sect. 1.3]) we have

$$\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \cong \mathrm{Aut}(K_0(A), K_0(A)_+, \Sigma(A)),$$

and thus, by Proposition 2.1, for each $h \in \mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ there is a sequence $\{k_d\}_{d\geq 2}$ of integers such that $T_\alpha|_{X_d} = T_d^{k_d}$ for each representative $\alpha \in \mathrm{Aut}(A)$ of the class h. Consequently $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \cong \mathbb{Z}^{\mathbb{N}}$.

Lemma 2.2. Let $T : Y \to Y$ be a homeomorphism of a compact Hausdorff space and suppose that the restriction of T to some closed invariant set $Z \subseteq Y$ has the shift on $F^{\mathbb{Z}}$ as a factor, where F is a set of cardinality 2^q for a given $q \in \mathbb{N}$. Then there is a set Ω of nonnegative norm-one functions in $C_{\mathbb{R}}(Y)$ with $|\Omega| = q$ such that for each $n \in \mathbb{N}$ the set $\Omega \cup \alpha_T\Omega \cup \cdots \cup \alpha_T^{n-1}\Omega$ is 2-equivalent to the standard basis of ℓ_1^{nq} over $\mathbb{R}$.

Proof. It suffices to show a suitable Ω exists for the shift S on $F^{\mathbb{Z}}$, for in that case we can identify Ω with its image under the equivariant embedding $C(F^{\mathbb{Z}}) \hookrightarrow C(Z)$ and then any set $\Omega' \subseteq C_{\mathbb{R}}(Y)$ consisting of nonegative norm-one lifts of each of the elements in Ω will satisfy the conclusion for the given T. For convenience we take F to be the set $\{0,1\}^{\{1,\dots,q\}}$. For each $i = 1,\dots,q$ define the function $f_i \in C_{\mathbb{R}}(F^{\mathbb{Z}})$ by $f_i((b_k)_k) = b_0(i)$ for all $(b_k)_k \in F^{\mathbb{Z}}$ and set $\Omega = \{f_1,\dots,f_q\}$. Then the collection of pairs $((f_i \circ S^k)^{-1}(0), (f_i \circ S^k)^{-1}(1))$ of subsets of $F^{\mathbb{Z}}$ for $i = 1,\dots,q$ and $k = 0,1,\dots$ is independent, in the terminology of [12]. The proof of Proposition 4 in [12] then shows that for each $n \in \mathbb{N}$ the set $\Omega \cup \alpha_S\Omega \cup \cdots \cup \alpha_S^{n-1}\Omega$ is 2-equivalent to the standard basis of ℓ_1^{nq} over $\mathbb{R}$, as desired. ☐

Lemma 2.3. Let $d \geq 2$ and let α be a *-automorphism of A such that the restriction of T_α to X_d is equal to T_d^k for some nonzero integer k. Then $\mathrm{ht}(\alpha) = \mathrm{lht}(\alpha) = \mathrm{hc}(\alpha) = \infty$.

Proof. We may assume that $k \geq 1$ by replacing α with its inverse if necessary. Let q be a positive integer. By the construction of T_d, the kth power T_d^k has a restriction to a closed invariant subset of X_d which admits the shift on 2^{kq} symbols as a factor. By Lemma 2.2 there is a set Ω of nonnegative norm-one functions in $C_{\mathbb{R}}(X)$ with $|\Omega| = kq$ such that for each $n \in \mathbb{N}$ the set $\Omega \cup \bar{\alpha}\Omega \cup \cdots \cup \bar{\alpha}^{n-1}\Omega$ is 2-equivalent to the standard basis of ℓ_1^{nkq} over $\mathbb{R}$, where $\bar{\alpha}$ is the induced automorphism $f \mapsto f \circ T_\alpha$ of $C_{\mathbb{R}}(X)$. Since $\Sigma(A)$ is uniformly dense in $C(X,[0,1])$, for each $g \in \Omega$ we can find an $f_g \in \Sigma(A)$ such that $\|f_g - g\| \leq 1/2$. Choose projections $p_g \in A$ for $g \in \Omega$ such that $[p_g] = f_g$ for each $g \in \Omega$, and set $\Upsilon = \{p_g : g \in \Omega\}$.

Let $\varphi : A_{\mathrm{sa}} \to C_{\mathbb{R}}(X)$ be the contractive real linear map given by evaluation on tracial states, and note that $\varphi \circ \alpha|_{A_{\mathrm{sa}}} = \bar{\alpha} \circ \varphi$. Given an $n \in \mathbb{N}$ and real scalars $c_{g,i}$ for $g \in \Omega$ and $i = 0,\dots,n-1$, we then have

$$\begin{aligned}
\frac{1}{4}\sum_{i=0}^{n-1}\sum_{g\in\Omega}|c_{g,i}| &\leq \Big\|\sum_{i=0}^{n-1}\sum_{g\in\Omega}c_{g,i}\bar{\alpha}^i(f_g)\Big\| - \Big\|\sum_{i=0}^{n-1}\sum_{g\in\Omega}c_{g,i}\bar{\alpha}^i(f_g - g)\Big\| \\
&\leq \Big\|\sum_{i=0}^{n-1}\sum_{g\in\Omega}c_{g,i}\bar{\alpha}^i(f_g)\Big\| = \Big\|\varphi\Big(\sum_{i=0}^{n-1}\sum_{g\in\Omega}c_{g,i}\alpha^i(p_g)\Big)\Big\| \\
&\leq \Big\|\sum_{i=0}^{n-1}\sum_{g\in\Omega}c_{g,i}\alpha^i(p_g)\Big\| \leq \sum_{i=0}^{n-1}\sum_{g\in\Omega}|c_{g,i}|,
\end{aligned}$$

so that $\Upsilon \cup \alpha\Upsilon \cup \cdots \cup \alpha^{n-1}\Upsilon$ is 4-equivalent to the standard basis of ℓ_1^{nkq} over $\mathbb{R}$ and hence 8-equivalent to the standard basis of ℓ_1^{nkq} over $\mathbb{C}$. Since q is an arbitrary positive integer, we thus obtain the result by Lemma 3.1 of [9] and Lemma 3.2 of [10]. ☐

Lemma 2.4. Let B be a unital AF algebra, D a finite-dimensional C^*-algebra, and $\Phi : D \to B$ a unital *-homomorphism. Let γ be a scale-preserving order automorphism of $K_0(B)$ such that γ restricts to the identity on $\Phi_*(K_0(D))$. Then there exists a *-automorphism α of B such that $\alpha_* = \gamma$ and α restricts to the identity on $\Phi(D)$.

Proof. By Elliott's classification theorem for AF algebras there exists a *-automorphism β of B such that $\beta_* = \gamma$. Then $\Phi_* = (\beta \circ \Phi)_*$, and so there exists a unitary $u \in B$ such that $\Phi = (\mathrm{Ad}\, u) \circ \beta \circ \Phi$. The *-automorphism $\alpha = (\mathrm{Ad}\, u) \circ \beta$ then has the desired properties. □

Theorem 2.5. The *-automorphisms α of A which are not approximately inner satisfy $\mathrm{ht}(\alpha) = \mathrm{lht}(\alpha) = \mathrm{hc}(\alpha) = \infty$ and form a dense open subset of $\mathrm{Aut}(A)$.

Proof. Denote by Γ the complement in $\mathrm{Aut}(A)$ of the set of approximately inner *-automorphisms. Then Γ is open and consists of all $\alpha \in \mathrm{Aut}(A)$ such that for some $d \geq 2$ the restriction of T_α to X_d is not the identity, i.e., is of the form T_d^k for some integer $k \neq 0$. By Lemma 2.3 we have $\mathrm{ht}(\alpha) = \mathrm{lht}(\alpha) = \mathrm{hc}(\alpha) = \infty$ for all $\alpha \in \Gamma$.

We now argue that Γ is dense in $\mathrm{Aut}(A)$. Suppose we are given $\alpha \in \mathrm{Aut}(A) \setminus \Gamma$. Let B be a finite-dimensional unital C^*-subalgebra of A and $F \subseteq B$ a finite set of projections which generate $K_0(B)$. Then there exists a $d \geq 2$ such that for each $p \in F$ the class $[p] \in K_0(A)$, viewed as a function on X, is constant on X_d. Let θ be the scale-preserving order automorphism of $K_0(A)$ such that θ restricts to T_d on X_d and to the identity on $X \setminus X_d$. By Lemma 2.4 there exists a $\beta \in \mathrm{Aut}(A)$ such that $\beta_* = \theta$ and β restricts to the identity on B. Setting $\gamma = \alpha \circ \beta$ we have $\gamma|_B = \alpha|_B$ while $\gamma \in \Gamma$, and so we conclude that Γ is dense in $\mathrm{Aut}(A)$, completing the proof. □

Finally we point out that there are also inner *-automorphisms α of A satisfying $\mathrm{ht}(\alpha) = \mathrm{lht}(\alpha) = \mathrm{hc}(\alpha) = \infty$. Indeed every unital C^*-algebra that is not of type I possesses such *-automorphisms [3, 10].

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