# DIMENSION, COMPARISON, AND ALMOST FINITENESS

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Abstract. We develop a dynamical version of some of the theory surrounding the Toms–Winter conjecture for simple separable nuclear C$^*$-algebras and study its connections to the C$^*$-algebra side via the crossed product. We introduce an analogue of hyperfiniteness for free actions of amenable groups on compact spaces and show that it plays the role of $\mathcal{Z}$-stability in the Toms–Winter conjecture in its relation to dynamical comparison, and also that it implies $\mathcal{Z}$-stability of the crossed product. This property, which we call almost finiteness, generalizes Matui's notion of the same name from the zero-dimensional setting. We also introduce a notion of tower dimension as a partial analogue of nuclear dimension and study its relation to dynamical comparison and almost finiteness, as well as to the dynamical asymptotic dimension and amenability dimension of Guentner, Willett, and Yu.

## 1. Introduction

Two of the cornerstones of the theory of von Neumann algebras with separable predual are the following theorems due to Murray–von Neumann [31] and Connes [5], respectively:

(i) there is a unique hyperfinite $\mathrm{II}_1$ factor,
(ii) injectivity is equivalent to hyperfiniteness.

Injectivity is a form of amenability that gives operator-algebraic expression to the idea of having an invariant mean, while hyperfiniteness means that the algebra can be expressed as the weak operator closure of an increasing sequence of finite-dimensional $^*$-subalgebras (or, equivalently, that one has local $^*$-ultrastrong approximation by such $^*$-subalgebras [12]). The basic prototype for the relation between an invariant-mean-type property and finite or finite-dimensional approximation is the equivalence between amenability and the Følner property for discrete groups, and indeed Connes's proof of (ii) draws part of its inspiration from the Day–Namioka proof of this equivalence.

In the theory of measured equivalence relations on standard probability spaces one has the following analogous pair of results, the first of which is a theorem of Dye [8] and the second of which is the Connes–Feldmann–Weiss theorem [6] (here p.m.p. stands for probability-measure-preserving):

(iii) there is a unique hyperfinite ergodic p.m.p. equivalence relation,
(iv) amenability is equivalent to hyperfiniteness.

Again amenability is defined as the existence of a suitable type of invariant mean, while hyperfiniteness means that the relation is equal a.e. to an increasing union of subrelations with finite classes. Thus among both $\mathrm{II}_1$ factors and p.m.p. equivalence relations there is a unique amenable object and it can be characterized via a finite or finite-dimensional approximation property. The two settings are furthermore linked in a direct technical way

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by the equivalence of the following three conditions for a free p.m.p. action of a countably infinite group:

(v) the orbit equivalence relation of the action is hyperfinite,
(vi) the crossed product is isomorphic to the unique hyperfinite $\mathrm{II}_1$ factor,
(vii) the group is amenable.

The implication (vii)⇒(v) was established by Ornstein and Weiss as a consequence of their Rokhlin-type tower theorem [33, 34] and can also be deduced from the Connes–Feldman–Weiss theorem. The implication (vii)⇒(vi) was established by Connes as an application of his result that injectivity implies hyperfiniteness and can also be derived in a more elementary way using the Ornstein–Weiss tower theorem (it is interesting to note however that one needs the full force of Connes's theorem in order to show that the group von Neumann algebra of an amenable group is hyperfinite).

In the type III case there is a similarly definitive theory, with the isomorphism classes being much more abundant but still classifiable in a nice way. For the present discussion however we will leave this aside, since our focus will be on amenable type II phenomena in the topological-dynamical and C*-algebraic realms, where the uniqueness in (i) and (iii) already gets replaced by a vast array of possible behaviour for which a complete classification is likely hopeless without the addition of further regularity hypotheses. In fact our principal aim has been to clarify what kind of regularity properties on the dynamical side match up, at least through analogy and one-way implications, with the key regularity properties of finite nuclear dimension, $\mathcal{Z}$-stability, and strict comparison that have helped set the stage for the dramatic advances made over the last few years in the classification program for simple separable nuclear (i.e., amenable) C*-algebras. In the process we will try to reimagine the equivalence of (v) and (vi) in the context of actions on compact metrizable spaces by introducing an analogue of hyperfiniteness and relating it to the $\mathcal{Z}$-stability of the crossed product.

For C*-algebras, the strictest and simplest technical analogue of a hyperfinite von Neumann algebra would be an AF algebra, which similarly means that the algebra can be expressed as the closure of an increasing union of finite-dimensional *-subalgebras (or, equivalently, that one has local approximation by such *-subalgebras), but with the weak operator topology replaced by the norm topology. In the 1970s, separable AF algebras were shown to be classified by their ordered $K$-theory (Elliott) as well as by related combinatorial objects called Bratteli diagrams (Bratteli). This reinforced the affinity with von-Neumann-algebraic hyperfiniteness by revealing a parallel structural tractability, however different the nature of the invariants.

What is remarkable is that the classification of AF algebras ended up being only the beginning of a much more ambitious program that was launched in the 1980s by Elliott, who realized that $C^*$-inductive limits of more general types of building blocks could be classified by ordered $K$-theory paired with traces and suggested that a similar classification might hold for even larger classes of (or perhaps even all) separable nuclear C*-algebras. The Elliott program has experienced many successes and several surprising twists over the last twenty-five years through the efforts of many researchers and has recently culminated, in the simple unital UCT case, with a definitive classification which merely assumes the abstract regularity hypothesis of finite nuclear dimension (this result combines theorems of Gong–Lin–Niu [15], Elliott–Gong–Lin–Niu [10], and White–Winter–Tikuisis [48], while also incorporating the earlier Kirchberg–Phillips classification on the purely infinite side [23, 36]). The UCT (universal coefficient theorem) is a homological condition relating $K$-theory and $KK$-theory which is possibly redundant and is automatic for crossed products of actions of countable amenable groups on compact metrizable spaces by a result of Tu [51].

That classifiability of simple separable unital C$^*$-algebras in the UCT class now boils down to the simple question of whether the nuclear dimension is finite belies the critical role that several other regularity properties have played and continue to play in classification theory. The most important among these are $\mathcal{Z}$-stability (i.e., tensorial absorption of the Jiang–Su algebra $\mathcal{Z}$), strict comparison, and tracial rank conditions. Strict comparison is a C$^*$-algebraic version of the property that the comparability of projections in a type II von Neumann algebra is determined on traces and applies more generally to positive elements in a C$^*$-algebra with respect to the relation of Cuntz subequivalence. The notion of tracial rank, which has its roots in work of Gong [9] and Popa [38] and was formalized and applied by Lin in his seminal work of the 1990s as a way to circumvent inductive limit hypotheses in the stably finite case [26, 27, 28], continues to do much of the technical legwork in classification. The simple unital projectionless C$^*$-algebra $\mathcal{Z}$ was introduced in the 1990s by Jiang and Su, who observed the parallel between its tensorial behaviour and that of the hyperfinite II$_1$ factor $R$ [20].

Winter's approach to classification, which was developed in the 2000s and has greatly impacted the course of the subject [57], made novel use of the operation of tensoring with $\mathcal{Z}$, rendering greater urgency to the problem of recognizing when a C$^*$-algebra is $\mathcal{Z}$-stable and strengthening the analogy with $R$ through the latter's use in Connes's classification work, which served as an inspiration. Winter was also the first to realize the significance of dimensional invariants based on nuclearity-type finite-dimensional approximation, among which nuclear dimension has become the most eminent, and the connection between such invariants and $\mathcal{Z}$-stability has become a centerpiece of his program. In fact, it is a conjecture of Toms and Winter that for infinite-dimensional simple separable unital nuclear C$^*$-algebras the following three conditions are equivalent:

(i) finite nuclear dimension,
(ii) $\mathcal{Z}$-stability,
(iii) strict comparison.

The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are theorems of Winter [56] and Rørdam [39], respectively. Matui and Sato proved (iii)$\Rightarrow$(ii) when the set of extremal tracial states is finite and nonempty [30], and this was later generalized to the case where the extreme traces form a nonempty compact set with finite covering dimension [24, 42, 50]. The implication (ii)$\Rightarrow$(i) was first established by Sato, White, and Winter in the case of a unique tracial state [43] and then more generally by Bosa, Brown, Sato, Tikuisis, White, and Winter when the extreme tracial states form a nonempty compact set [2]. Thus the Toms–Winter conjecture has been fully confirmed in the case that the extreme tracial states form a nonempty compact set with finite covering dimension, and in particular when there is a unique tracial state.

The goal of these notes is to promote the development of a dynamical version of this theory surrounding the Toms–Winter conjecture, including connections to the C$^*$-algebra side via the crossed product. This program requires first of all identifying the appropriate analogues of nuclear dimension, strict comparison, and $\mathcal{Z}$-stability. There is a natural dynamical version of strict comparison which has appeared in lectures of Winter and has been studied by Buck in the case $G = \mathbb{Z}$ [3] (see also [14] for an earlier application of this concept to minimal transformations of the Cantor set). In parallel with [55], we simply refer to it as *comparison*, and also define the useful higher-order notions of $m$-comparison for integers $m \geq 0$, with comparison representing the case $m = 0$ (Definition 3.2). There are also by now several analogues of nuclear dimension, including the dynamic asymptotic dimension and amenability dimension of Guentner, Willett, and Yu [17], and we will introduce here another, called *tower dimension*, whose connections to nuclear dimension and dynamical comparison are particularly stark, as shown in Sections 6 and 7. Although dynamic asymptotic dimension, amenability dimension, and tower dimension do not coincide in general, there are inequalities relating them in the finite-dimensional case, and they are all equal when the space is zero-dimensional (see Section 5).

What has been missing is a dynamical substitute for $\mathcal{Z}$-stability. We introduce here a notion of *almost finiteness* for group actions on compact metrizable spaces that will play the role of $\mathcal{Z}$-stability in the Toms–Winter conjecture and of hyperfiniteness in the p.m.p. setting. We have adopted the terminology from Matui's almost finiteness for groupoids, seeing that in the case of free actions on zero-dimensional compact metrizable spaces our definition reduces to Matui's (Section 10). As a comparison with the measure-theoretic framework, we recall that, for a free p.m.p. action $G \curvearrowright (X, \mu)$ of a countable amenable group, we can express the property of hyperfiniteness, in accordance with the original proof of Ornstein and Weiss, by saying that for every $\varepsilon > 0$ there are measurable sets $V_1, \dots, V_n \subseteq X$ and finite sets $S_1, \dots, S_n \subseteq G$ with prescribed approximate invariance (in the Følner sense) such that

(i) the sets $sV_i$ for $i = 1, \dots, n$ and $s \in S_i$ are pairwise disjoint, and
(ii) $\mu(X \setminus \bigsqcup_{i=1}^n S_i V_i) < \varepsilon$.

The pair $(V_i, S_i)$ we refer to as a *tower*, the set $V_i$ as the *base* of the tower, the set $S_i$ as the *shape* of the tower, and the sets $sV_i$ for $s \in S_i$ as the *levels* of the tower. In the definition of almost finiteness, the sets $V_i$ are replaced by open sets and the smallness of the remainder in (ii) is expressed topologically in terms of comparison with a portion of the tower levels. Note in particular that almost finiteness implies that the acting group is amenable because of the Følner requirement on the shapes of the towers. In Theorem 12.4 we prove that, for actions of countably infinite groups on compact metrizable spaces, almost finiteness implies that the crossed product is $\mathcal{Z}$-stable. As we discuss at the end of Section 12, this can be used to give new examples of classifiable crossed products for which dynamical techniques connected to nuclear dimension (such as in [17] or Section 6) are inapplicable due to finite-dimensionality requirements on the space. What is particularly novel from the classification perspective is that many of these examples can exhibit both infinite asymptotic dimension in the group and positive topological entropy in the dynamics.

It is important to point out that almost finiteness is not an analogue of $\mathcal{Z}$-stability by itself, but rather of the conjunction of $\mathcal{Z}$-stability and nuclearity. In view of classification theory, this combination (or its conjectural Toms–Winter equivalent, finite nuclear dimension) could be argued to be the true topological analogue of hyperfiniteness, as opposed to just nuclearity, which is the direct technical translation of hyperfiniteness into the realm of C$^*$-algebras and as such is an essentially measure-theoretic property. The interpretation of almost finiteness as a combination of $\mathcal{Z}$-stability and nuclearity is illustrated at a technical level in the proof of Theorem 12.4, which relies on a criterion for $\mathcal{Z}$-stability that is special to the nuclear setting, due to Hirshberg and Orovitz (Theorem 12.1). The general characterization of $\mathcal{Z}$-stability from which the Hirshberg–Orovitz result is derived (Proposition 2.3 of [55]) involves an additional approximate centrality requirement that does not seem to translate into dynamical terms, and in particular does not seem to be amenable to the kind of tiling techniques that are integral to the proof of Theorem 12.4.

Consider now the following triad of properties for a free minimal action $G \curvearrowright X$ of a countably infinite amenable group on a compact metrizable space:

(i) finite tower dimension,
(ii) almost finiteness,
(iii) comparison.

In Theorem 9.2 we establish the implication (ii)$\Rightarrow$(iii), as well as the converse (iii)$\Rightarrow$(ii) in the case that the set $E_G(X)$ of ergodic $G$-invariant Borel probability measures is finite. This precisely parallels the results of Rørdam [39] and Matui–Sato [30] mentioned above. Moreover, the argument for (iii)$\Rightarrow$(ii), like that of Matui and Sato, relies on an appeal to measure-theoretic structure, which in our case is the Ornstein–Weiss tower theorem. In our proof of (iii)$\Rightarrow$(ii) it is enough that the action have $m$-comparison for some $m \geq 0$, which is important as we also prove in Theorem 7.2 that if the covering dimension $\dim(X)$ is finite then (i) implies $m$-comparison for some $m \geq 0$, and hence comparison in the case that $E_G(X)$ is finite. Thus if $E_G(X)$ and $\dim(X)$ are both finite then we have (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii), which we record as Theorem 9.3.

In [29] Matui showed that his property of almost finiteness for groupoids has several nice consequences for the homology of the groupoid and its relation to both the topological full group and the $K$-theory of the reduced groupoid C$^*$-algebra. In particular, if the groupoid is furthermore assumed to be principal (which amounts to freeness in the case of actions) then the first homology group is canonically isomorphic to the quotient of the topological full group by the subgroup generated by the elements of finite order. Matui observes in Lemma 6.3 of [29] that the groupoids associated to free actions of $\mathbb{Z}^d$ on zero-dimensional compact metrizable spaces are almost finite. By combining the work of Szabó, Wu, and Zacharias in [47] with Theorems 10.2 and Theorem 9.3 we deduce that this also holds for free minimal actions $G \curvearrowright X$ of finitely generated nilpotent groups on zero-dimensional compact metrizable spaces with $E_G(X)$ finite (Remark 10.3).

While our results suggest that almost finiteness and comparison are full-fledged dynamical analogues of their Toms–Winter counterparts, tower dimension and its relatives unfortunately fall short on this account, despite their utility in establishing finite nuclear dimension for crossed products of large classes of actions. The problem is that tower dimension, dynamical asymptotic dimension, and amenability dimension are too much affected by the dimensionality of the acting group and too little affected by the dimensionality of the space and its interaction with the dynamics (as captured by an invariant like mean dimension). On the side of the space, if we drop the assumption of finite-dimensionality then the implication (i)$\Rightarrow$(ii) fails, even for $G = \mathbb{Z}$ (Example 12.5). One can attempt to rectify this by imposing a small diameter condition on the tower levels in the definition of tower dimension (we call the resulting invariant the *fine tower dimension*) but one would not gain anything in the effort to relate dimensional invariants to almost finiteness and comparison since finite fine tower dimension already implies that $\dim(X)$ is finite. Even more serious is the structural restriction imposed from the side of the group: the tower dimension, dynamical asymptotic dimension, and amenability dimension are always infinite whenever $G$ has infinite asymptotic dimension, which occurs frequently in the amenable case, an example being the Grigorchuk group. In contrast, a generic free minimal action of any countably infinite amenable group on the Cantor set is almost finite [4]. Given that tower dimension seems as close as we can come in dynamics to being able to formally mimic the definition of nuclear dimension, and that it connects naturally to dynamical $m$-comparison and nuclear dimension in one direction of logical implication, we will perhaps have to be content with the prospect that the Toms–Winter conjecture cannot be fully analogized within the coordinatized framework of group actions. On the other hand, it is conceivable that the tower dimension of a free minimal action $G \curvearrowright X$ is always finite when $G$ is amenable and has finite asymptotic dimension and $X$ has finite covering dimension. This is indeed what happens if we furthermore assume $G$ to be finitely generated and nilpotent (see Example 4.9).

One more curious fact worth mentioning here is the possibility, suggested by the work of Elliott and Niu [11], that for free minimal actions the small boundary property (or, alternatively, zero mean dimension) is equivalent to $\mathcal{Z}$-stability of the crossed product. Elliott and Niu showed that for free minimal $\mathbb{Z}$-actions the small boundary property (which is equivalent to zero mean dimension in this case) implies $\mathcal{Z}$-stability. The small boundary property and mean dimension are formally very different from either nuclear dimension or $\mathcal{Z}$-stability and are more akin to slow dimension growth in inductive limit C*-algebras, as demonstrated by the proof in [11], which employs arguments from an article of Toms on the equivalence of slow dimension growth and $\mathcal{Z}$-stability for unital simple ASH algebras [49].

We begin in Section 2 by laying down some basic notation and terminology used throughout the paper. In Section 3 we define (dynamical) comparison, and also more generally $m$-comparison. Section 4 introduces tower dimension and Section 5 establishes inequalities relating it to dynamical asymptotic dimension and amenability dimension. In Section 6 we show how to derive an upper bound for the nuclear dimension of the crossed product of a free action of an amenable group in terms of the tower dimension of the action and the covering dimension of the space. In Section 7 we prove that if the acting group is amenable and the tower dimension and covering dimension are both finite, with values $d$ and $c$, then the action has $((c+1)(d+1)-1)$-comparison. In Section 8 we introduce almost finiteness and in Section 9 we establish Theorem 9.2 relating it to comparison. In Section 10 we prove that, for free actions on the Cantor set, almost finiteness is equivalent to having clopen tower decompositions of the space with almost invariant shapes, so that it reduces to Matui's notion of almost finiteness in this setting. The behaviour of almost finiteness under extensions is investigated in Section 11. In Section 12 we show that almost finiteness implies $\mathcal{Z}$-stability and use it to give new examples of classifiable crossed products. Finally, in Section 13 we prove that, for free minimal actions of an amenable group on the Cantor set, almost finiteness implies that the clopen type semigroup is almost unperforated, that this almost unperforation in turn implies comparison, and that all three of these properties are equivalent when the set $E_G(X)$ of ergodic $G$-invariant Borel probability measures is finite.

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## 2. General notation and terminology

Throughout the paper $G$ is a countable discrete group.

For a compact Hausdorff space $X$, we write $C(X)$ for the unital C*-algebra of continuous complex-valued functions on $X$. For an open set $V \subseteq X$ we denote by $C_0(V)$ the C*-algebra of continuous complex-valued functions on $V$ which vanish at infinity, which can be naturally viewed as a sub-C*-algebra of $C(X)$. We write $M(X)$ for the convex set of all regular Borel probability measures on $X$, which is compact as a subset of the dual $C(X)^*$ equipped with the weak* topology. We denote the indicator function of a set $A \subseteq X$ by $1_A$. The covering dimension of $X$ is written $\dim(X)$.

Actions on compact Hausdorff spaces are always assumed to be continuous. Let $G \curvearrowright X$ be such an action. The image of a point $x \in X$ under a group element $s$ is expressed as $sx$. For $A \subseteq X$, $s \in G$, and $K \subseteq G$ we write $sA = \{sx : x \in A\}$ and $KA = \{sx : s \in K, x \in A\}$. We write $M_G(X)$ for the convex set of $G$-invariant regular Borel probability measures on $X$, which is a weak* compact subset of $M(X)$. We write $E_G(X)$ for the set of extreme points of $M_G(X)$, which are precisely the ergodic measures in $M_G(X)$.

The *chromatic number* of a family $\mathcal{C}$ of subsets of a given set is defined as the least $d \in \mathbb{N}$ such that there is a partition of $\mathcal{C}$ into $d$ subcollections each of which is disjoint.

For any of the various notions of dimension which will appear, we will add a superscript $+1$ to denote the value of the dimension plus one, so that $\dim^{+1}(X) = \dim(X) + 1$, for example. This "denormalization" serves to streamline many formulas.

## 3. Comparison and $m$-comparison

Throughout $G \curvearrowright X$ is an action on a compact metrizable space.

**Definition 3.1.** Let $m \in \mathbb{N}$. Let $A, B \subseteq X$. We write $A \prec_m B$ if for every closed set $C \subseteq A$ there exist a finite collection $\mathcal{U}$ of open subsets of $X$ which cover $C$, an $s_U \in G$ for each $U \in \mathcal{U}$, and a partition $\mathcal{U} = \mathcal{U}_0 \sqcup \cdots \sqcup \mathcal{U}_m$ such that for each $i = 0, \dots, m$ the images $s_U U$ for $U \in \mathcal{U}_i$ are pairwise disjoint subsets of $B$. When $m = 0$ we also write $A \prec B$.

Note that the relation $\prec$ is transitive, as is straightforward to check.

**Definition 3.2.** Let $m \in \mathbb{N}$. The action $G \curvearrowright X$ is said to have *$m$-comparison* if $A \prec_m B$ for all nonempty open sets $A, B \subseteq X$ satisfying $\mu(A) < \mu(B)$ for every $\mu \in M_G(X)$. When $m = 0$ we will also simply say that the action has *comparison*.

The condition of nonemptiness on $A$ and $B$ above is included so as to cover the situation when $M_G(X)$ is empty and can otherwise be dropped, as for example when $G$ is amenable.

The following lemma will be used repeatedly throughout the paper and will be needed here to verify Proposition 3.4.

**Lemma 3.3.** *Let $X$ be a compact metrizable space with compatible metric $d$ and let $\Omega$ be a weak$^*$ closed subset of $M(X)$. Let $A$ be a closed subset of $X$ and $B$ an open subset of $X$ such that $\mu(A) < \mu(B)$ for all $\mu \in \Omega$. Then there exists an $\eta > 0$ such that the sets*

$$B_- = \{x \in X : d(x, X \setminus B) > \eta\},$$
$$A_+ = \{x \in X : d(x, A) \leq \eta\}$$

*satisfy $\mu(A_+) + \eta \leq \mu(B_-)$ for all $\mu \in \Omega$.*

*Proof.* Suppose that the conclusion does not hold. Then for every $n \in \mathbb{N}$ we can find a $\mu_n \in \Omega$ such that the sets

$$B_n = \{x \in X : d(x, X \setminus B) > 1/n\},$$
$$A_n = \{x \in X : d(x, A) \leq 1/n\}$$

satisfy $\mu_n(A_n) + 1/n > \mu_n(B_n)$. By the compactness of $\Omega$ there is a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ which weak$^*$ converges to some $\mu \in \Omega$. For a fixed $j \in \mathbb{N}$ we have, for every $k \geq j$,

$$\mu_{n_k}(A_{n_j}) + \frac{1}{n_k} \geq \mu_{n_k}(A_{n_k}) + \frac{1}{n_k} > \mu_{n_k}(B_{n_k}) \geq \mu_{n_k}(B_{n_j}),$$

and since $A_{n_j}$ is closed and $B_{n_j}$ is open the portmanteau theorem ([21], Theorem 17.20) then yields

$$\mu(A_{n_j}) \geq \limsup_{k \to \infty} \mu_{n_k}(A_{n_j}) \geq \liminf_{k \to \infty} \mu_{n_k}(B_{n_j}) \geq \mu(B_{n_j}).$$

Note that $B$ is equal to the increasing union of the sets $B_{n_j}$ for $j \in \mathbb{N}$, while $A$ is equal to the decreasing intersection of the sets $A_{n_j}$ for $j \in \mathbb{N}$. Thus

$$\mu(A) = \lim_{j \to \infty} \mu(A_{n_j}) \geq \lim_{j \to \infty} \mu(B_{n_j}) = \mu(B),$$

contradicting our hypothesis. □

In practice, we will use the following characterization as our effective definition of $m$-comparison, usually without saying so.

**Proposition 3.4.** *Let $m \in \mathbb{N}$. The action $G \curvearrowright X$ has $m$-comparison if and only if $A \prec_m B$ for every closed set $A \subseteq X$ and nonempty open set $B \subseteq X$ satisfying $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$.*

*Proof.* For the nontrivial direction, suppose that the action has $m$-comparison. Let $A$ be a closed subset of $X$ and $B$ a nonempty open subset of $X$ such that $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$. Fixing a compatible metric $d$ on $X$, by Lemma 3.3 there is an $\eta > 0$ such that the open set $A' = \{x \in X : d(x, A) < \eta\}$ satisfies $\mu(A') < \mu(B)$ for all $\mu \in M_G(X)$. Then $A' \prec_m B$ by $m$-comparison, and so $A \prec_m B$, as desired. □

The remainder of the section is aimed at showing that if $X$ is zero-dimensional then we can express comparison using clopen sets and clopen partitions, as asserted by Proposition 3.6.

**Proposition 3.5.** *Suppose that $X$ is zero-dimensional. Let $m \in \mathbb{N}$, and let $A$ and $B$ be clopen subsets of $X$. Then $A \prec_m B$ if and only if there exist a clopen partition $\mathcal{P}$ of $A$, an $s_U \in G$ for every $U \in \mathcal{P}$, and a partition $\mathcal{P} = \mathcal{P}_0 \sqcup \cdots \sqcup \mathcal{P}_m$ such that for each $i = 0, \ldots, m$ the images $s_U U$ for $U \in \mathcal{P}_i$ are pairwise disjoint subsets of $B$.*

*Proof.* For the nontrivial direction, suppose that $A \prec_m B$. Then there exist $j_0, \dots, j_m \in \mathbb{N}$, open sets $U_{i,j}$ for $0 \leq i \leq m$ and $1 \leq j \leq j_i$ which cover $A$, and $s_{i,j} \in G$ such that for each $i = 0, \dots, m$ the images $s_{i,j}U_{i,j}$ for $j = 1, \dots, j_i$ are pairwise disjoint subsets of $B$. By the normality of $X$ we can then find, for all $i, j$, a closed set $C_{i,j} \subseteq U_{i,j}$ such that the sets $C_{i,j}$ for all $i, j$ still cover $A$. By compactness and zero-dimensionality, for given $i, j$ we can produce finitely many clopen sets contained in $U_{i,j}$ which cover $C_{i,j}$, and so we may assume that $C_{i,j}$ is clopen by replacing it with the union of these clopen sets. We now recursively define, with respect to the lexicographic order on the pairs $i, j$,

$$A_{i,j} = \Bigg(\bigg(A \setminus \bigsqcup_{k=0}^{i-1} \bigsqcup_{l=1}^{j_i} A_{k,l}\bigg) \cap C_{i,j}\Bigg) \setminus (C_{i,1} \cup \dots \cup C_{i,j-1}).$$

These sets form a clopen partition of $A$ and for each $i = 0, \dots, m$ the images $s_{i,j}A_{i,j}$ for $j = 1, \dots, j_i$ are pairwise disjoint subsets of $B$, as desired. $\square$

**Proposition 3.6.** *Suppose that $X$ is zero-dimensional. Let $m \in \mathbb{N}$. Then the action $G \curvearrowright X$ has $m$-comparison if and only if for all nonempty clopen sets $A, B \subseteq X$ satisfying $\mu(A) < \mu(B)$ for every $\mu \in M_G(X)$ there exist a clopen partition $\mathcal{P}$ of $A$, an $s_U \in G$ for every $U \in \mathcal{P}$, and a partition $\mathcal{P} = \mathcal{P}_0 \sqcup \dots \sqcup \mathcal{P}_m$ such that for each $0 = 1, \dots, m$ the images $s_U U$ for $U \in \mathcal{P}_i$ are pairwise disjoint subsets of $B$.*

*Proof.* The forward direction is immediate from Proposition 3.5. Suppose conversely that the action satisfies the condition in the proposition statement involving clopen sets and let us establish $m$-comparison. Let $A$ be a closed subset of $X$ and $B$ a nonempty open subset of $X$ satisfying $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$. By Lemma 3.3 there exists an $\eta > 0$ such that the sets

$$B_- = \{x \in X : d(x, X \setminus B) > \eta\},$$
$$A_+ = \{x \in X : d(x, A) \leq \eta\}$$

satisfy $\mu(A_+) < \mu(B_-)$ for all $\mu \in \Omega$. By an argument as in the proof of Proposition 3.5, we can find clopen sets $A', B' \subseteq X$ such that $A \subseteq A' \subseteq A_+$ and $B_- \subseteq B' \subseteq B$, in which case $\mu(A') \leq \mu(A_+) < \mu(B_-) \leq \mu(B')$ for all $\mu \in \Omega$. It follows by our hypothesis that there exist a clopen partition $\mathcal{P}$ of $A'$, an $s_U$ for every $U \in \mathcal{P}$, and a partition $\mathcal{P} = \mathcal{P}_0 \sqcup \dots \sqcup \mathcal{P}_m$ such that for each $i = 0, \dots, m$ the images $s_U U$ for $U \in \mathcal{P}_i$ are pairwise disjoint subsets of $B'$. Since $A \subseteq A'$ and $B' \subseteq B$, we conclude (by Proposition 3.4) that the action has $m$-comparison. $\square$

## 4. Tower dimension

Throughout $G \curvearrowright X$ is a free action on a compact Hausdorff space.

**Definition 4.1.** A *tower* is a pair $(V, S)$ consisting of a subset $V$ of $X$ and a finite subset $S$ of $G$ such that the sets $sV$ for $s \in S$ are pairwise disjoint. The set $V$ is the *base* of the tower, the set $S$ is the *shape* of the tower, and the sets $sV$ for $s \in S$ are the *levels* of the tower. We say that the tower $(V, S)$ is *open* if $V$ is open, *clopen* if $V$ is clopen, and *measurable* if $V$ is measurable. A collection of towers $\{(V_i, S_i)\}_{i \in I}$ is said to *cover* $X$ if $\bigcup_{i \in I} S_i V_i = X$.

**Definition 4.2.** Let $E$ be a finite subset of $G$. A collection of towers $\{(V_i, S_i)\}_{i \in I}$ covering $X$ is *$E$-Lebesgue* if for every $x \in X$ there are an $i \in I$ and a $t \in S_i$ such that $x \in tV_i$ and $Et \subseteq S_i$.

**Definition 4.3.** The *tower dimension* $\dim_{\rm tow}(X, G)$ of the action $G \curvearrowright X$ is the least integer $d \geq 0$ with the property that for every finite set $E \subseteq G$ there is an $E$-Lebesgue collection of open towers $\{(V_i, S_i)\}_{i \in I}$ covering $X$ such that the family $\{S_i V_i\}_{i \in I}$ has chromatic number at most $d + 1$ (note that we may always take the index set $I$ to be finite in view of the compactness of $X$). If no such $d$ exists we set $\dim_{\rm tow}(X, G) = \infty$.

In the above definition one may assume, whenever convenient, that for each $i$ the identity element $e$ is contained in $S_i$ (i.e., the base $V_i$ is actually a level of the tower), for one can choose a $t \in S_i$ (assuming that $S_i$ is nonempty, as we may) and replace $S_i$ by $S_i t^{-1}$ and $V_i$ by $tV_i$.

Note that if $G$ is not locally finite then the tower dimension must be at least 1, for if $E$ is a symmetric finite subset of $G$ and $\{(V_i, S_i)\}_{i \in I}$ is an $E$-Lebesgue collection of towers for which the sets $S_i V_i$ partition $X$ then for each $i$ with $V_i \neq \emptyset$ the set $S_i$ contains $\langle E \rangle S_i$ where $\langle E \rangle$ is the subgroup of $G$ generated by $E$.

**Remark 4.4.** When $X$ is zero-dimensional we can equivalently restrict to clopen towers in Definition 4.3, for we can assume $I$ to be finite and use normality to slightly shrink the base of each of the open towers $(V_i, T_i)$ to a closed set without destroying $E$-Lebesgueness and the fact that the collection of towers covers $X$, and then use compactness and zero-dimensionality to slightly enlarge each of these closed bases to a clopen base which is contained in the corresponding original base.

**Example 4.5.** Let $\mathbb{Z} \curvearrowright X$ be a minimal action on the Cantor set. This is given by $(n, x) \mapsto T^n x$ for some transformation $T$ and is automatically free. We can decompose $X$ into clopen towers by the following standard procedure. Take a nonempty clopen set $V \subseteq X$, and consider the first return map which assigns to each $x \in V$ the smallest $n_x \in \mathbb{N}$ for which $T^{n_x} x \in V$, which is well defined by minimality. This map is continuous by the clopenness of $V$ and so there is a clopen partition $\{V_1, \dots, V_k\}$ of $V$ and integers $1 \leq n_1 < n_2 < \cdots < n_k$ such that for each $i$ the set of all points in $V$ with return time $n_i$ is equal to $V_i$. Setting $S_i = \{0, \dots, n_i - 1\}$, we thus have a collection of clopen towers $\{(V_i, S_i)\}_{i=1}^k$ such that the sets $S_i V_i$ are pairwise disjoint, and since the union $\bigsqcup_{i=1}^k S_i V_i$ is closed and $T$-invariant it must be equal to $X$ by minimality. The only problem is that this collection will not satisfy the Lebesgue condition in the definition of tower dimension. To remedy this, we produce a second collection of towers by taking the image of the original one under some power of $T$, and make sure that the numbers $n_i$ are sufficiently large. More precisely, let $E$ be a finite subset of $\mathbb{Z}$ and choose an $N > 2 \max_{n \in E} |n|$. Since the action is free, by shrinking $V$ we can force $n_1$ to be much larger than $N$, which will imply that the collection of towers $\{(V_i, S_i)\}_{i=1}^k \cup \{(V_i, S_i + N)\}_{i=1}^k$ is $E$-Lebesgue, as is easily verified. Thus the tower dimension of the action is at most 1, and hence equal to 1 by the observation following Definition 4.3.

The following is verified by taking the inverse images under the extension map $Y \to X$ of all of towers at play in the definition of tower dimension.

**Proposition 4.6.** *Let $G \curvearrowright Y$ be a free action on a compact Hausdorff space which is an extension of $G \curvearrowright X$, meaning that there is $G$-equivariant continuous surjection $Y \to X$. Then*

$$\dim_{\rm tow}(Y, G) \leq \dim_{\rm tow}(X, G).$$

**Example 4.7.** It was shown in [13] that there are free minimal $\mathbb{Z}$-actions on compact metrizable spaces such that the crossed product $C(X) \rtimes \mathbb{Z}$ fails to be $\mathcal{Z}$-stable. By Proposition 4.6 the examples given there have tower dimension at most 1 since they factor onto an odometer, which has tower dimension 1 by Example 4.5 (note that 1 is always a lower bound for the tower dimension of free $\mathbb{Z}$-actions by the observation following Definition 4.3).

We recall that the asymptotic dimension $\mathrm{asdim}(G)$ of the group $G$ [16] can be expressed as the least integer $d \geq 0$ such that for every finite set $E \subseteq G$ there exists a family $\{U_i\}_{i\in I}$ of subsets of $G$ of multiplicity at most $d+1$ with the following properties:

(i) there exists a finite set $F \subseteq G$ such that for every $i \in I$ there is a $t \in G$ with $U_i \subseteq Ft$, and

(ii) for each $t \in G$ there is an $i \in I$ for which $Et \subseteq U_i$ (Lebesgue condition).

If no such $d$ exists then $\mathrm{asdim}(G)$ is declared to be infinite. It is readily seen that the asymptotic dimension is zero if and only if the group is locally finite. The asymptotic dimension of $\mathbb{Z}^m$ for $m \in \mathbb{N}$ is equal to $m$, while the asymptotic dimension of the free group $F_m$ for $m \in \mathbb{N}$ is equal to 1. An example of a finitely generated amenable group with infinite asymptotic dimension is the Grigorchuk group [44]. See [1] for a general reference on the subject.

The following inequality is a refinement of the observation in the second paragraph following Definition 4.3, which can be rephrased as saying that $\dim_{\mathrm{tow}}(X, G)$ is nonzero whenever $\mathrm{asdim}(G)$ is nonzero.

**Proposition 4.8.** $\dim_{\mathrm{tow}}(X, G) \geq \mathrm{asdim}(G)$.

*Proof.* We may assume that $\dim_{\mathrm{tow}}(X, G)$ is finite. Let $E$ be a finite subset of $G$. Setting $d = \dim_{\mathrm{tow}}(X, G)$, we can then find a finite $E$-Lebesgue collection of open towers $\{(V_i, S_i)\}_{i\in I}$ covering $X$ such that the family $\{S_i V_i\}_{i\in I}$ has chromatic number at most $d+1$. Pick an $x \in X$. For every $i \in I$ set $L_i = \{s \in G : sx \in V_i\}$. Then the family $\bigcup_{i\in I}\{S_i t : t \in L_i\}$ of subsets of $G$ is readily seen to satisfy the conditions in the above formulation of asymptotic dimension with respect to the set $E$. □

**Example 4.9.** Let $m \in \mathbb{N}$. It follows easily from Theorems 3.8 and 4.6 of [46] that for every $d \in \mathbb{N}$ there is a constant $C > 0$ such that for every free action $\mathbb{Z}^m \curvearrowright X$ on a compact metrizable space with $\dim(X) \leq d$ one has

$$\dim^{+1}_{\mathrm{tow}}(X, \mathbb{Z}^m) \leq C \cdot \dim^{+1}(X), \tag{1}$$

and that we can also relax the hypothesis $\dim(X) \leq d$ by merely requiring that the action have the topological small boundary property with respect to $d$ ([46], Definition 3.2). The arguments in Section 7 of [47] show more generally that that if $G$ is finitely generated and nilpotent then there exists such a $C > 0$ such that (1) holds for every free action $G \curvearrowright X$ on a compact metrizable space with $\dim(X) \leq d$.

Finally, we define a variant of tower dimension which requires that the bases of the towers have small diameter.

**Definition 4.10.** The *fine tower dimension* $\dim_{\mathrm{ftow}}(X, G)$ of the action $G \curvearrowright X$ is the least integer $d \geq 0$ with the property that for every finite set $E \subseteq G$ and $\delta > 0$ there is an $E$-Lebesgue collection of open towers $\{(V_i, S_i)\}_{i\in I}$ covering $X$ such that $\mathrm{diam}(sV_i) < \delta$ for all $i \in I$ and $s \in S_i$ and the family $\{S_i V_i\}_{i\in I}$ has chromatic number at most $d+1$ (as for tower dimension we may take $I$ to be finite). If no such $d$ exists we set $\dim_{\mathrm{ftow}}(X,G) = \infty$.

**Proposition 4.11.** *One has*

$$\dim^{+1}_{\mathrm{tow}}(X,G) \leq \dim^{+1}_{\mathrm{ftow}}(X,G) \leq \dim^{+1}_{\mathrm{tow}}(X,G) \cdot \dim^{+1}(X).$$

*In particular,* $\dim_{\mathrm{ftow}}(X,G) < \infty$ *if and only if* $\dim_{\mathrm{tow}}(X,G) < \infty$ *and* $\dim(X) < \infty$.

*Proof.* The first inequality is trivial. For the second, we may suppose that $\dim^{+1}_{\mathrm{tow}}(X,G)$ and $\dim^{+1}(X)$ are both finite. Denote these numbers by $d$ and $c$, respectively. Let $E$ be a finite subset of $G$ and $\delta > 0$. Then there is a finite $E$-Lebesgue collection of towers $\{(V_i, S_i)\}_{i\in I}$ covering $X$ such that the family $\{S_i V_i\}_{i\in I}$ has chromatic number at most $d+1$. By normality we can find open sets $U_i \subseteq X$ with $\overline{U_i} \subseteq V_i$ such that the family $\{S_i U_i\}_{i\in I}$ still covers $X$ and the collection $\{(U_i, S_i)\}_{i\in I}$ is still $E$-Lebesgue. Since $X$ has covering dimension $c$, by compactness we can find for each $i$ a collection $\{V_{i,1}, \dots, V_{i,k_i}\}$ of open subsets of $V_i$ which covers $\overline{U_i}$, satisfies $\mathrm{diam}(sV_i) < \delta$ for all $s \in S_i$, and has chromatic number at most $c+1$. Then $\{(V_{i,j}, S_i) : i \in I, 1 \leq k \leq j_i\}$ is an $E$-Lebesgue collection of towers such that each level of each tower has diameter less than $\delta$ and the family $\{S_i V_i\}_{i\in I}$ has chromatic number at most $(d+1)(c+1)$. This establishes the second inequality. □

## 5. Tower dimension, amenability dimension, and dynamic asymptotic dimension

Throughout $G \curvearrowright X$ is a free action on a compact metrizable space. Our aim here is to establish inequalities connecting its tower dimension, amenability dimension, and dynamic asymptotic dimension (Theorem 5.14). We will see in particular that when the space is zero-dimensional, all of these dimensions are equal (Corollary 5.15).

The notions of amenability dimension and dynamic asymptotic dimension are due to Guentner, Willett, and Yu [17] and are recalled in Definitions 5.1 and 5.3 (these do not require freeness or metrizability). After defining amenability dimension we establish an inequality relating it to tower dimension in Theorem 5.2. We then turn to dynamic asymptotic dimension and prove some lemmas which will help us link it to tower dimension in Theorem 5.14.

Write $\Delta(G)$ for the set of probability measures on $G$, and $\Delta_d(G)$ for the set of probability measures on $G$ whose support has cardinality at most $d+1$. We view both as subsets of $\ell^1(G)$.

**Definition 5.1.** The *amenability dimension* $\dim_{\mathrm{am}}(X,G)$ of the action $G \curvearrowright X$ is the least integer $d \geq 0$ with the property that for every finite set $F \subseteq G$ and $\varepsilon > 0$ there is a continuous map $\varphi : X \to \Delta_d(G)$ such that

$$\sup_{x\in X} \|\varphi(sx) - s\varphi(x)\|_1 < \varepsilon$$

for all $s \in F$.

If $G$ is finite then every action $G \curvearrowright X$ has amenability dimension at most $|G|$, since we may construct a $G$-invariant map by sending everything in $X$ to the uniform probability measure on $G$. More generally, if $G$ is amenable we can construct an approximately invariant continuous map $\varphi : X \to \Delta(G)$ by sending everything in $X$ to the uniform probability measure on a sufficiently left invariant finite subset of $G$. However, when $G$ is infinite the cardinality of the supports of such maps will necessarily tend to infinity as the approximate invariance becomes better and better, and so to derive bounds for the amenability dimension in this case one must search for maps which are approximately equivariant for reasons other than approximate invariance. Indeed the support constraint in the definition of amenability dimension results in phenomena that are qualitatively very different from the approximate invariance we see in an amenable group and instead involve the presence of collections of towers as in the definition of tower dimension.

**Theorem 5.2.** *The action $G \curvearrowright X$ satisfies*

$$\dim_{\mathrm{am}}(X,G) \leq \dim_{\mathrm{tow}}(X,G).$$

*Proof.* We denote the induced action of $G$ on $C(X)$ by $\alpha$, that is, $\alpha_s(f)(x) = f(s^{-1}x)$ for all $s \in G$, $f \in C(X)$, and $x \in X$.

We may assume that $\dim_{\mathrm{tow}}(X,G)$ is finite, and we denote this number by $d$. Fix a compatible metric $d$ on $X$.

Let $F$ be a finite subset of $G$ and let $\varepsilon > 0$. In order to verify the condition in the definition of amenability dimension we may assume that $F^{-1} = F$ by replacing $F$ with $F \cup F^{-1}$, and also that $e \in F$. Choose an integer $n > 1$ such that $(d+1)(d+2)/n < \varepsilon$. By the definition of tower dimension, there is a finite $F^n$-Lebesgue collection of towers $\{(V_i, S_i)\}_{i \in I}$ such that $\{S_i V_i\}_{i \in I}$ is a cover of $X$ with chromatic number at most $d+1$.

By the $F^n$-Lebesgue condition and a simple compactness argument we can find a $\delta > 0$ such that for every $x \in X$ there are an $i \in I$ and a $t \in S_i$ such that $d(x, X \setminus tV_i) > \delta$ and $F^n t \subseteq S_i$. For every $i \in I$ and $t \in S_i$ define the function $\hat{g}_{i,t} \in C(X)$ by

$$\hat{g}_{i,t}(x) = \min\{1, \delta^{-1} d(x, X \setminus tV_i)\}.$$

For every $i \in I$ set

$$g_i = \max_{t \in S_i} \alpha_{t^{-1}}(\hat{g}_{i,t}),$$

and note that for $t \in S_i$ the support of the function $\alpha_t(g_i)$ is contained in $tV_i$.

Let $i \in I$. Set $B_{i,n} = \bigcap_{t \in F^n} tS_i$ and $B_{i,0} = G \setminus \bigcap_{t \in F} tS_i$. For $k = 1, \dots, n-1$ set

$$B_{i,k} = \Bigg( \bigcap_{t \in F^k} tS_i \Bigg) \setminus \bigcap_{s \in F^{k+1}} tS_i.$$

The sets $B_{i,k}$ for $k = 0, \dots, n$ form a partition of $G$, and for all $t \in F$ we have

(i) $tB_{i,0} \subseteq B_{i,0} \cup B_{i,1}$,
(ii) $tB_{i,k} \subseteq B_{i,k-1} \cup B_{i,k} \cup B_{i,k+1}$ for every $k = 1, \dots, n-1$,
(iii) $tB_{i,n} \subseteq B_{i,n-1} \cup B_{i,n}$.

For each $t \in G$ take $k$ such that $t \in B_{i,k}$ and define the function

$$\hat{h}_{i,t} = \frac{k}{n} \alpha_t(g_i)$$

in $C(X)$, and note that $|\hat{h}_{i,t}(sx) - \hat{h}_{i,s^{-1}t}(x)| \leq 1/n$ for all $x \in X$ and $s \in F$.

Now set $H = \sum_{i \in I} \sum_{t \in G} \hat{h}_{i,t}$. By our choice of $\delta$, for every $x \in X$ there is an $i \in I$ and a $t \in S_i$ such that $d(x, X \setminus tV_i) > \delta$ and $F^n t \subseteq S_i$, in which case $t \in B_{i,n}$ and hence, in view of the definition of $g_i$,

$$\hat{h}_{i,t}(x) = \alpha_t(g_i)(x) \geq \hat{g}_{i,t}(x) = 1.$$

This shows that $H \geq 1$. Setting

$$h_{i,t} = H^{-1}\hat{h}_{i,t}$$

for every $i \in I$ and $t \in G$, we then define a continuous map $\varphi : X \to \Delta_d(G)$ by

$$\varphi(x)(t) = \sum_{i \in I} h_{i,t}(x)$$

for $x \in X$ and $t \in G$.

Since for each $x \in X$ the set of all $i \in I$ such that $x \in S_i V_i$ has cardinality at most $d+1$, for $s \in F$ the difference between the values of $H$ at $x$ and $sx$ is at most $(d+1)/n$. Since $H \geq 1$, it follows that the difference between the values of $H^{-1}$ at $x$ and $sx$ is also at most $(d+1)/n$. Consequently for every $x \in X$, $s \in F$, and $t \in G$ we have

$$\begin{aligned} |h_{i,t}(sx) - h_{i,s^{-1}t}(x)| &\leq H(sx)^{-1}\big|\hat{h}_{i,t}(sx) - \hat{h}_{i,s^{-1}t}(x)\big| \\ &\qquad + \big|H(sx)^{-1} - H(x)^{-1}\big|\hat{h}_{i,s^{-1}t}(x) \\ &\leq \frac{d+2}{n}, \end{aligned}$$

while $h_{i,t}(sx) = h_{i,s^{-1}t}(x) = 0$ whenever $x \notin S_i V_i$. Using again the fact that for each $x \in X$ the set of all $i \in I$ such that $x \in S_i V_i$ has cardinality at most $d+1$, it follows that for every $x \in X$ and $s \in F$ we have

$$\begin{aligned} \|\varphi(sx) - s\varphi(x)\|_1 &= \sum_{t \in G} |\varphi(sx)(t) - \varphi(x)(s^{-1}t)| \\ &\leq \sum_{t \in G}\sum_{i \in I} |h_{i,t}(sx) - h_{i,s^{-1}t}(x)| \\ &\leq (d+1)\bigg(\frac{d+2}{n}\bigg) < \varepsilon, \end{aligned}$$

from which we conclude that $\dim_{\mathrm{am}}(X,G) \leq d$. □

**Definition 5.3.** The *dynamic asymptotic dimension* $\mathrm{dad}(X,G)$ of the action $G \curvearrowright X$ is the least integer $d \geq 0$ with the property that for every finite set $E \subseteq G$ there are a finite set $F \subseteq G$ and an open cover $\mathcal{U}$ of $X$ of cardinality $d+1$ such that, for all $x \in X$ and $s_1, \dots, s_n \in E$, if the points $x, s_1 x, s_2 s_1 x, \dots, s_n \cdots s_1 x$ are contained in a common member of $\mathcal{U}$ then $s_n \cdots s_1 \in F$.

**Definition 5.4.** Let $E$ be a finite subset of $G$. An open cover $\mathcal{U}$ of $X$ is said to be *$E$-Lebesgue* if for every $x \in X$ there is an $1 \leq i \leq n$ such that $Ex \subseteq U_i$.

**Remark 5.5.** The above definition should not be confused with the $E$-Lebesgue condition for a collection of towers. Given a collection of towers $\mathcal{T} = \{(V_i, S_i\}_{i \in I}$ such that the family $\mathcal{V} = \{S_i V_i\}_{i \in I}$ covers $X$, if $\mathcal{T}$ is $E$-Lebesgue then $\mathcal{V}$ is $E$-Lebesgue, but not conversely. For example, if $E$ contains an element of infinite order then there is no tower $(V, S)$ such that the singleton $\{(V, S)\}$ is $E$-Lebesgue and $SV = X$, although $\{X\}$ is an $E$-Lebesgue cover of $X$. For collections of towers the $E$-Lebesgue condition involves the way in which each tower is coordinatized by its shape, while no such coordinatization is at play when dealing with members of an arbitrary cover.

The following is part of Corollary 4.2 in [17].

**Proposition 5.6.** *In Definition 5.3 the open cover $\mathcal{U}$ can be chosen to be $E$-Lebesgue.*

We next record some lemmas that will allow us to establish the inequality $\dim^{+1}_{\mathrm{ftow}}(X, G) \leq \mathrm{dad}^{+1}(X, G) \cdot \dim^{+1}(X)$ in Theorem 5.14.

**Definition 5.7.** Let $G \curvearrowright X$ be a free action on a compact metric space. A *castle* is a finite collection of towers $\{(V_i, S_i)\}_{i \in I}$ such that the sets $S_i V_i$ for $i \in I$ are pairwise disjoint. The *levels* of the castle are the sets $sV_i$ for $i \in I$ and $s \in S_i$. We say that the castle is *open* if each of the towers is open.

**Definition 5.8.** For sets $W \subseteq X$ and $E \subseteq G$ we write $R_{W,E}$ for the equivalence relation on $W$ under which two points $x$ and $y$ are equivalent if there exist $s_1, \dots, s_n \in E \cup E^{-1} \cup \{e\}$ such that $y = s_n \cdots s_1 x$ and $s_k \cdots s_1 x \in W$ for $k = 1, \dots, n-1$. Note that $R_{W,E}$ is symmetric because the set $E \cup E^{-1} \cup \{e\}$ is symmetric.

For an equivalence relation $R$ on a set $Z$ and an $A \subseteq Z$ we write $[A]_R$ for the saturation of $A$, i.e., the set of all $x \in Z$ for which there exists a $y \in A$ such that $xRy$. For sets $W, A \subseteq X$ we write $\partial_A W$ for the boundary of $A \cap W$ as a subset of the set $A$ equipped with the relative topology.

We will use without comment the following properties of covering dimension for a metrizable space $Y$. The second and third are consequences of the fact that covering dimension and large inductive dimension coincide in the metrizable setting. See [32] for more information.

(i) If $A$ is a closed subset of $Y$ then $\dim(A) \leq \dim(Y)$.
(ii) For every open set $U \subseteq Y$ and closed set $C \subseteq U$ there exists an open set $V \subseteq Y$ with $C \subseteq V \subseteq U$ and $\dim(\partial V) < \dim(Y)$.
(iii) If $\{C_1, \dots, C_n\}$ is a closed covering of $Y$ then $\dim(Y) \leq \max_{i=1,\dots,n} \dim(C_i)$.

**Lemma 5.9.** *Let $A$ be a nonempty closed subset of $X$ and let $\delta > 0$. Then there is a finite collection $\{B_1, \dots, B_n\}$ of pairwise disjoint relatively open subsets of $A$ of diameter less than $\delta$ such that the set $\bigsqcup_{j=1}^n B_j$ is dense in $A$ and $\dim(\partial_A B_j) < \dim(A)$ for every $j = 1, \dots, n$.*

*Proof.* By compactness there exists a finite open cover $\{U_1, \dots, U_n\}$ of $X$ whose members each have diameter less than $\delta$, and by normality we can find closed sets $C_j \subseteq U_j$ such that the collection $\{C_1, \dots, C_n\}$ is also a cover of $X$. Relativizing to $A$, we can then find for each $j = 1, \dots, n$ a relatively open subset $V_j$ of $A$ such that $C_j \cap A \subseteq V_j \subseteq U_j \cap A$ and $\dim(\partial_A V_j) < \dim(A)$. Now recursively define $B_1 = V_1$ and $B_j = V_j \setminus (\overline{V_1} \cup \dots \cup \overline{V_{j-1}})$ for $j = 2, \dots, n$. Then the set $B = \bigsqcup_{j=1}^n B_j$ is dense in $A$ and for every $j = 1, \dots, n$ we have

$$\begin{aligned} \dim(\partial_A B_j) &\leq \max\{\dim(\partial_A V_j), \dim(\partial_A \overline{V_{j-1}}), \dots, \dim(\partial_A \overline{V_1})\} \\ &\leq \max\{\dim(\partial_A V_j), \dim(\partial_A V_{j-1}), \dots, \dim(\partial_A V_1)\} \\ &< \dim(A). \end{aligned}$$

□

**Lemma 5.10.** *Let $E$ be a finite subset of $G$ with $E^{-1} = E$ and $e \in E$. Let $C$ be a closed subset of $X$, and suppose that there is a finite set $F \subseteq G$ such that, for all $x \in C$ and*

*$s_1, \dots, s_m \in E$, if $s_k \cdots s_1 x \in C$ for all $k = 1, \dots, m$ then $s_m \cdots s_1 \in F$. Let $A$ be a closed subset of $C$. Then $[A]_{R_{C,E}}$ is closed.*

*Proof.* Let $x$ be a point in $X$ which is the limit of some sequence $\{x_n\}$ in $[A]_{R_{C,E}}$, and let us show that $x \in [A]_{R_{C,E}}$. Since $F$ is finite we can assume, by passing to a subsequence, that there are $s_1, \dots, s_m \in E$ and $a_n \in A$ such that for every $n$ we have $x_n = s_m \cdots s_1 a_n$ and $s_k \cdots s_1 a_n \in C$ for $k = 1, \dots, m$. By the continuity of the action, we have $a_n = s_1^{-1} \cdots s_m^{-1} x_n \to s_1^{-1} \cdots s_m^{-1} x$ as $n \to \infty$. Writing $a = s_1^{-1} \cdots s_m^{-1} x$, which belongs to $A$ since $A$ is closed, we then have $x = s_m \cdots s_1 a$, and also $s_k \cdots s_1 a \in C$ for $k = 1, \dots, m$ since $C$ is closed. Thus $x$ and $a$ are $R_{C,E}$-equivalent, so that $x \in [A]_{R_{C,E}}$. We conclude that $[A]_{R_{C,E}}$ is closed. □

**Lemma 5.11.** *Let $E$ be a finite subset of $G$ with $E^{-1} = E$ and $e \in E$. Let $\delta > 0$. Let $U$ be an open subset of $X$ and $F$ a finite subset of $G$ such that, for all $x \in U$ and $s_1, \dots, s_m \in E$, if $s_k \cdots s_1 x \in U$ for all $k = 1, \dots, m$ then $s_m \cdots s_1 \in F$. Let $C$ be a nonempty closed subset of $X$ such that $C \subseteq U$. Let $A$ be a closed subset of $\overline{U}$ with $A = [A]_{R_{\overline{U},E}}$. Then there are an open set $W \subseteq X$ with $C \subseteq W \subseteq \overline{W} \subseteq U$, an open castle $\{(V_i, S_i)\}_{i \in I}$, and sets $O_i \subseteq V_i$ such that*

*(i)* $\mathrm{diam}(sV_i) < \delta$ *for all $i \in I$ and $s \in S_i$,*
*(ii)* $\bigsqcup_{i \in I} S_i V_i \subseteq W$,
*(iii)* $[tx]_{R_{W,E}} = S_i x$ *for every $i \in I$, $t \in S_i$, and $x \in O_i$,*
*(iv) the set* $(A \cap \overline{W}) \setminus \bigsqcup_{i \in I} S_i O_i$ *is closed and has dimension strictly less than* $\dim(A)$.

*Proof.* Take an open set $W_0 \subseteq X$ with $C \subseteq W_0 \subseteq \overline{W_0} \subseteq U$. Then we can find a relatively open subset $W_1$ of $A$ with $A \cap C \subseteq W_1 \subseteq A \cap W_0$ such that $\dim(\partial_A W_1) < \dim(A)$. Now take an open set $W \subseteq W_0$ such that $W_1 = A \cap W$ and $C \subseteq W$, and note that $\overline{W} \subseteq \overline{W_0} \subseteq U$. Set

$$X_0 = A \setminus [\partial_A W_1]_{R_{\overline{W},E}}$$

Since $A$ is closed the set $\partial_A W_1$ is closed, and thus by Lemma 5.10 the set $[\partial_A W_1]_{R_{\overline{W},E}}$ is closed, so that $X_0$ is relatively open in $A$. Moreover, since $[\partial_A W_1]_{R_{\overline{W},E}}$ is contained in $F \partial_A W_1$ we have

$$\dim([\partial_A W_1]_{R_{\overline{W},E}}) \leq \dim(F \partial_A W_1) = \dim(\partial_A W_1) < \dim(A). \tag{2}$$

Define a map $\varphi : X_0 \to \mathscr{P}(F)$ (the power set of $F$) by

$$\varphi(x) = \{s \in F : sx \in [x]_{R_{W,E}}\},$$

which by freeness is determined by the equation $\varphi(x)x = [x]_{R_{W,E}}$. Let us verify that $\varphi$ is continuous. Let $x \in X_0$. Since $W$ is open and the action is continuous, we can find a relatively open subset $V$ of $X_0$ containing $x$ such that $\varphi(x) \subseteq \varphi(y)$ for every $y \in V$. Suppose that there exists a sequence $\{x_n\}$ in $X_0$ converging to $x$ such that $\varphi(x) \neq \varphi(x_n)$ for every $n$. We may assume, by passing to a subsequence, that there is a $t \in F$ such that $t \notin \varphi(x)$ and $t \in \varphi(x_n)$ for every $n$. Since the cardinality of each equivalence class of $R$ is bounded above by $|F|$, we can also assume, by passing to a further subsequence, that there are $s_1, \dots, s_m \in E$ such that $s_m \cdots s_1 = t$ and $s_k \cdots s_1 x_n \in W$ for $k = 1, \dots, m$. Then by the continuity of the action we have $s_k \cdots s_1 x \in \overline{W}$ for $k = 1, \dots, m$. Now if it were the case that $s_k \cdots s_1 x \notin W$ for some $1 \leq k \leq m$, then since $s_k \cdots s_1 x \in [A]_{R_{\overline{U},E}} = A$ and

$s_k \cdots s_1 x_n \in [A]_{R_{\overline{U},E}} \cap W = A \cap W$ for every $n$ it would follow that $s_k \cdots s_1 x \in \partial_A W_1$ and hence $x \in [\partial_A W_1]_{R_{\overline{W},E}}$, contradicting the membership of $x$ in $X_0$. Therefore $s_k \cdots s_1 x \in W$ for $k = 1, \dots, m$, showing that $t \in \varphi(x)$, a contradiction. We conclude from this that $\varphi$ is constant on some open neighbourhood of $x$, and hence that $\varphi$ is continuous on $X_0$, as desired.

Enumerate the subsets of $F$ containing $e$ as $S_1, \dots, S_q$. Recursively define subsets $O_1, \dots, O_q$ of $X_0$ by setting $O_1 = \varphi^{-1}(S_1)$ and, for $i = 2, \dots, q$,

$$O_i = \varphi^{-1}(S_i) \setminus \big(\overline{S_{i-1}\varphi^{-1}(S_{i-1})} \cup \cdots \cup \overline{S_1\varphi^{-1}(S_1)}\big).$$

The sets $S_i O_i$ for $i = 1, \dots, q$ are pairwise disjoint because $R_{W,E}$ is an equivalence relation. Note also that each set $S_i O_i$ is contained in $X_0$ since $[A]_{R_{W,E}} \subseteq [A]_{R_{\overline{U},E}} = A$ and $[x]_{R_{W,E}} \subseteq [\partial_A W_1]_{R_{\overline{W},E}}$ for every $x \in [\partial_A W_1]_{R_{\overline{W},E}}$. Moreover, for every $i = 1, \dots, q$ we have, using the relative openness of $\varphi^{-1}(S_i)$ in $A$ and the closedness of $A$ and appealing to (2),

$$\begin{aligned}\dim(\partial_A S_i \varphi^{-1}(S_i)) &= \dim(S_i \partial_A \varphi^{-1}(S_i)) \\ &= \dim(\partial_A \varphi^{-1}(S_i)) \leq \dim([\partial_A W_1]_{R_{\overline{W},E}}) < \dim(A)\end{aligned}$$

and hence

$$\begin{aligned}\dim(\partial_A O_i) &\leq \max\big(\dim(\partial_A \varphi^{-1}(S_i)), \dim(\partial_A \overline{S_{i-1}\varphi^{-1}(S_{i-1})}), \dots, \dim(\partial_A \overline{S_1\varphi^{-1}(S_1)})\big) \\ &\leq \max\big(\dim(\partial_A \varphi^{-1}(S_i)), \dim(\partial_A S_{i-1}\varphi^{-1}(S_{i-1})), \dots, \dim(\partial_A S_1\varphi^{-1}(S_1))\big) \\ &< \dim(A).\end{aligned}$$

We thus have a castle $\{(O_i, S_i)\}_{1 \leq i \leq q}$ with the following properties:

(i) $\bigsqcup_{i=1}^q S_i O_i \subseteq X_0 \subseteq \bigsqcup_{i=1}^q \overline{S_i O_i}$,
(ii) $[tx]_{R_{W,E}} = S_i x$ for every $i = 1, \dots, q$, $t \in S_i$, and $x \in O_i$,
(iii) $\dim(\partial_A O_i) < \dim(A)$ for every $i = 1, \dots, q$.

By Lemma 5.9 and uniform continuity there is a family $\{B_1, \dots, B_n\}$ of pairwise disjoint relatively open subsets of $A$ such that the diameter of $sB_j$ is less than $\delta$ for every $j = 1, \dots, n$ and $s \in F$, the set $\bigsqcup_{j=1}^n B_j$ is dense in $A$, and $\dim(\partial_A B_j) < \dim(A)$ for every $j = 1, \dots, n$. Replacing the castle $\{(O_i, S_i)\}_{1 \leq i \leq q}$ (the details of whose construction we don't care about) with the castle $\{(O_i \cap B_j, S_i)\}_{1 \leq i \leq q, 1 \leq j \leq n}$ and relabeling, we may assume that, in addition to satisfying (i) to (iii), the castle $\{(O_i, S_i)\}_{1 \leq i \leq q}$ has the property that all of its levels have diameter less than $\delta$ (to see that condition (iii) still holds observe that $\dim(\partial_A(O_i \cap B_j)) \leq \dim(\partial_A O_i \cup \partial_A B_j) \leq \max\{\dim(\partial_A O_i), \dim(\partial_A B_j)\} < \dim(A)$).

Next let $1 \leq i \leq q$ and $s \in S_i$ and let us show that $sO_i$ is relatively open in $X_0$. Suppose to the contrary that there exists a sequence $\{y_n\}$ in $X_0 \setminus sO_i$ which converges to some $y \in sO_i$. Set $x = s^{-1}y \in O_i$. Then there are $s_1, \dots, s_m \in E$ such that $s = s_m \dots s_1$ and $s_k \cdots s_1 x \in W$ for every $k = 1, \dots, m$. For every $n$ set $x_n = s^{-1}y_n = s_1^{-1} \cdots s_m^{-1} y_n$, and note that $x_n \to x$ by the continuity of the action. Since $W$ is open we may assume, by passing to subsequences, that for each $n$ we have $s_k \cdots s_1 x_n \in W$ for every $k = 1, \dots, m$, which means that $x_n$ and $y_n$ are $R_{W,E}$-equivalent. Since $y_n$ belongs to $A$ but not $[\partial_A W]_{R_{W,E}}$, this implies that $x_n \in [A]_{R_{W,E}} \subseteq [A]_{R_{\overline{U},E}} = A$ and $x_n \notin [\partial_A W]_{R_{W,E}}$. Therefore $x_n \in X_0$ for every $n$. Since $x_n = s^{-1}y_n \notin O_i$ for every $n$ and $x_n \to x$, and $O_i$ is relatively open in $X_0$ by the continuity of $\varphi$, we thus arrive at a contradiction. We therefore conclude that $sO_i$ is relatively open in

$X_0$. It follows that for each $i = 1, \dots, q$ and $s \in S_i$ we can find an open subset $V_{i,s}$ of $W$ which contains $sO_i$ and has diameter less than $\delta$ so that the sets $V_{i,s}$ are pairwise disjoint. For each $i = 1, \dots, q$ set $V_i = \bigcap_{s \in S_i} s^{-1} V_{i,s}$ Then $\{(V_i, S_i)\}_{1 \leq i \leq q}$ is an open castle such that $O_i \subseteq V_i$ for every $i = 1, \dots, q$ and

(iv) $\dim(sV_i) < \delta$ for all $i = 1, \dots, q$ and $s \in S_i$,
(v) $\bigsqcup_{i=1}^{q} S_i V_i \subseteq W$.

Define the set

$$D = S_1 \partial_A O_1 \cup \cdots \cup S_q \partial_A O_q,$$

which can be written as $\partial_A(S_1 O_1) \cup \cdots \cup \partial_A(S_q O_q)$ by the relative openness of each $O_i$ in $A$ and thus by (i) satisfies $X_0 \setminus D \subseteq \bigsqcup_{i=1}^{q} S_i O_i$. Using (iii) we have

$$\begin{aligned} \dim(D) &\leq \max\{\dim(S_1 \partial_A O_1), \dots, \dim(S_q \partial_A O_q)\} \\ &\leq \max\{\dim(\partial_A O_1), \dots, \dim(\partial_A O_q)\} \\ &< \dim(A). \end{aligned} \tag{3}$$

Finally, set $A' = (A \cap \overline{W}) \setminus \bigsqcup_{i=1}^{q} S_i O_i$, which is relatively closed in $A$ and hence closed in $X$. Since $X_0 \setminus D \subseteq \bigsqcup_{i=1}^{q} S_i O_i$ and $A \cap \overline{W} = \overline{A \cap W} = \overline{W_1}$, we have

$$\begin{aligned} A' \subseteq (A \cap \overline{W}) \setminus (X_0 \setminus D) &\subseteq \partial_A W_1 \cup (W_1 \setminus X_0) \cup D \\ &\subseteq [\partial_A W_1]_{R_{\overline{W},E}} \cup D \end{aligned}$$

and hence, using (2) and (3) and the fact that $[\partial_A W_1]_{R_{\overline{W},E}}$ and $D$ are relatively closed in $A$,

$$\dim(A') \leq \max\{\dim([\partial_A W_1]_{R_{\overline{W},E}}), \dim(D)\} < \dim(A).$$

This completes the verification of the required properties. □

**Lemma 5.12.** *Let $E$, $\delta$, $U$, $F$, and $C$ be as in the statement of Lemma 5.11. Then there are a nonnegative integer $d \leq \dim(X)$ and for each $j = 1, \dots, d+1$ an open castle $\{(V_i, S_i)\}_{i \in I_j}$ and sets $O_i \subseteq V_i$ such that*

*(i) $C \subseteq \bigcup_{j=1}^{d+1} \bigsqcup_{i \in I_j} S_i O_i \subseteq \bigcup_{j=1}^{d+1} \bigsqcup_{i \in I_j} S_i V_i \subseteq U$,*
*(ii) $[x]_{R_{C,E}} \subseteq S_i x$ for every $i \in I$, $t \in S_i$, and $x \in tO_i \cap C$, and*
*(iii) for every $j = 1, \dots, d+1$ one has $\operatorname{diam}(sV_i) < \delta$ for all $i \in I_j$ and $s \in S_i$.*

*Proof.* Take an open set $U_0$ with $C \subseteq U_0 \subseteq \overline{U_0} \subseteq U$ and set $A_0 = [\overline{U_0}]_{R_{\overline{U_0},E}}$. Then $A_0$ is closed by Lemma 5.10, and $A_0 = [A_0]_{R_{\overline{U_0},E}}$. Thus by Lemma 5.11 there is an open set $U_1$ with $C \subseteq U_1 \subseteq \overline{U_1} \subseteq U_0$, an open castle $\{(V_i, S_i)\}_{i \in I_1}$, and sets $O_i \subseteq V_i$ such that

(i) $\operatorname{diam}(sV_i) < \delta$ for all $i \in I_1$ and $s \in S_i$,
(ii) $\bigsqcup_{i \in I_1} S_i V_i \subseteq U_1$,
(iii) $[tx]_{R_{U_1,E}} = S_i x$ for every $i \in I_1$, $t \in S_i$, and $x \in O_i$, and
(iv) the set $A_0' = (A_0 \cap \overline{U_0}) \setminus \bigsqcup_{i \in I_1} S_i O_i$ is closed and satisfies

$$\dim(A_0') < \dim(A_0).$$

Note that (iii) implies that for every $i \in I_1$, $t \in S_i$, and $x \in tO_i \cap C$ we have

$$[x]_{R_{C,E}} \subseteq [x]_{R_{U_1,E}} = S_i x.$$

Set $A_1 = [A_0']_{R_{\overline{U_0},E}}$, which is closed by Lemma 5.10, and observe that since $A_1 \subseteq FA_0'$ we have

$$\dim(A_1) \leq \dim(FA_0') = \dim(A_0') < \dim(A_0).$$

Apply Lemma 5.11 again, this time using $A_1$ and $U_1$, to get an open set $U_2$ with $C \subseteq U_2 \subseteq \overline{U_2} \subseteq U_1$, an open castle $\{(V_i, S_i)\}_{i\in I_2}$, and sets $O_i \subseteq V_i$ such that

(i) $\operatorname{diam}(sV_i) < \delta$ for all $i \in I_2$ and $s \in S_i$,
(ii) $\bigsqcup_{i\in I_2} S_i V_i \subseteq U_2$,
(iii) $[tx]_{R_{U_2,E}} = S_i x$ for every $i \in I_1$, $t \in S_i$, and $x \in O_i$, and
(iv) the set $A_1' = (A_1 \cap \overline{U_1}) \setminus \bigsqcup_{i\in I_2} S_i O_i$ is closed and satisfies
$$\dim(A_1') < \dim(A_1).$$

Note that (iii) implies that for every $i \in I_2$, $t \in S_i$, and $x \in tO_i \cap C$ we have

$$[x]_{R_{C,E}} \subseteq [x]_{R_{U_2,E}} = S_i x.$$

Set $A_2 = [A_1']_{R_{A_1,E}}$, which is closed by Lemma 5.10, and observe that since $A_2 \subseteq FA_1'$ we have

$$\dim(A_2) \leq \dim(FA_1') = \dim(A_1') < \dim(A_1).$$

Continue this procedure by recursively applying Lemma 5.11 to produce at the $j$th stage sets $U_j$ and $A_j$, an open castle $\{(V_i, S_i)\}_{i\in I_j}$, and sets $O_i \subseteq V_i$ as above, until we reach the point that $\dim(A_{d+1}) = -1$ for some $d \leq \dim(X)$. The castles $\{(V_i, S_i)\}_{i\in I_j}$ and sets $O_i$ then satisfy the requirements of the lemma. $\square$

**Lemma 5.13.** *Let $E$ be a finite subset of $G$ and let $\{U_1, \dots, U_n\}$ be an $E$-Lebesgue open cover of $X$. Then there exist closed sets $C_i \subseteq U_i$ such that $\{C_1, \dots, C_n\}$ is an $E$-Lebesgue cover of $X$.*

*Proof.* For $i = 1, \dots, n$ write $V_i$ for the set of all $x \in X$ such that $Ex \subseteq U_i$. This is an open set since $U_i$ is open and the action is continuous. Because $\{U_1, \dots, U_n\}$ is $E$-Lebesgue, the collection $\{V_1, \dots, V_n\}$ covers $X$. By normality we can find closed sets $D_i \subseteq V_i$ such that $\{D_1, \dots, D_n\}$ still covers $X$. For each $i$ define $C_i = \bigcup_{s\in E} s^{-1} D_i$, which is a closed subset of $U_i$. Then $\{C_1, \dots, C_n\}$ is an $E$-Lebesgue cover of $X$ of the required kind. $\square$

**Theorem 5.14.** *The action $G \curvearrowright X$ satisfies*

$$\begin{aligned} \operatorname{dad}^{+1}(X, G) &\leq \dim_{\mathrm{am}}^{+1}(X, G) \leq \dim_{\mathrm{tow}}^{+1}(X, G) \\ &\leq \dim_{\mathrm{ftow}}^{+1}(X, G) \leq \operatorname{dad}^{+1}(X, G) \cdot \dim^{+1}(X). \end{aligned}$$

*Proof.* The first inequality follows from Theorem 4.11 of [17], as pointed out in Remark 4.14 of that paper. The second inequality is Theorem 5.2. The third inequality is trivial.

It remains to establish the last inequality. For this we may assume that $\operatorname{dad}(X, G)$ and $\dim(X)$ are both finite. Let $E$ be a finite subset of $G$ with $E^{-1} = E$ and $e \in E$. By Proposition 5.6 there are a finite set $F \subseteq G$ and an $E$-Lebesgue open cover $\{U_0, \dots, U_d\}$ of $X$ with $d \leq \operatorname{dad}(X, G)$ such that, for all $j = 0, \dots, d$, if $x \in U_j$ and $s_1, \dots, s_n \in E$ satisfy $s_k \cdots s_1 x \in U_j$ for all $k = 1, \dots, n$ then $s_n \cdots s_1 \in F$. By Lemma 5.13 there exist closed sets $C_j \subseteq U_j$ for $j = 0, \dots, d$ such that $\{C_0, \dots, C_d\}$ is an $E$-Lebesgue cover of $X$. By

Lemma 5.12, for every $j = 0, \dots, d$ there are a collection of open towers $\{(V_i, S_i)\}_{i\in I_j}$ with chromatic number at most $\dim(X) + 1$ and levels of diameter less than $\delta$ and sets $O_i \subseteq V_i$ such that $C_j \subseteq \bigcup_{j=0}^d \bigsqcup_{i\in I_j} S_i O_i \subseteq \bigcup_{j=0}^d \bigsqcup_{i\in I_j} S_i V_i \subseteq U_j$ and

$$[x]_{R_{C_j,E}} \subseteq S_i x \tag{4}$$

for every $i \in I_j$, $t \in S_i$, and $x \in tO_i \cap C_j$. Note that the collection of open towers $\{(V_i, S_i)\}_{i\in I_j, 0\leq j\leq d}$ has chromatic number at most $(d + 1)(\dim(X) + 1)$.

Now let $x \in X$. Since the cover $\{C_0, \dots, C_d\}$ is $E$-Lebesgue, there is a $0 \leq j \leq d$ such that $Ex \subseteq C_j$. Since $x \in C_j$ there are $i \in I_j$, $t \in S_i$, and $y \in O_i$ such that $x = ty$. By (4) and our choice of $j$, for every $s \in E$ we have $sty \in [ty]_{R_{C_j,E}} \subseteq S_i y$ so that $Ety \subseteq S_i y$ and hence $Et \subseteq S_i$. This shows that the collection of open towers $\{(V_i, S_i)\}_{i\in I_j, 0\leq j\leq d}$ is $E$-Lebesgue. We have thus verified that $\dim^{+1}_{\rm ftow}(X, G) \leq {\rm dad}^{+1}(X, G) \cdot \dim^{+1}(X)$, as desired. □

**Corollary 5.15.** *Suppose that $X$ is zero-dimensional. Then the action $G \curvearrowright X$ satisfies*

$$\dim_{\rm tow}(X, G) = \dim_{\rm ftow}(X, G) = {\rm dad}(X, G) = \dim_{\rm am}(X, G).$$

## 6. Tower dimension and nuclear dimension

Let $G \curvearrowright X$ be a free action on a compact Hausdorff space. We write $C(X) \rtimes_\lambda G$ for the associated reduced crossed product. In Section 8 of [17], Guentner, Willett, and Yu showed that

$$\dim^{+1}_{\rm nuc}(C(X) \rtimes_\lambda G) \leq {\rm dad}^{+1}(X, G) \cdot \dim^{+1}(X).$$

By Theorem 5.14 this implies that

$$\dim^{+1}_{\rm nuc}(C(X) \rtimes_\lambda G) \leq \dim^{+1}_{\rm tow}(X, G) \cdot \dim^{+1}(X). \tag{5}$$

We will give here a shorter direct proof of (5) in order to illustrate the formal affinity between tower dimension and nuclear dimension. This can be seen as a distillation of the arguments in Section 8 of [17] into their simplest combinatorial form.

First we recall the definition of nuclear dimension [58]. We use the abbreviation *c.p.c.* for "completely positive contractive". A map $\varphi : A \to B$ between C*-algebras is *order zero* if it preserve orthogonality, that is, $\varphi(a_1)\varphi(a_2) = 0$ for all $a_1, a_2 \in A$ satisfying $a_1a_2 = 0$.

**Definition 6.1.** The *nuclear dimension* $\dim_{\rm nuc}(A)$ of a C*-algebra $A$ is the least integer $d \geq 0$ such that for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there are finite-dimensional C*-algebras $B_0, \dots, B_d$ and linear maps

$$A \xrightarrow{\varphi} B_0 \oplus \cdots \oplus B_d \xrightarrow{\psi} A$$

such that $\varphi$ is c.p.c., $\psi|_{B_i}$ is c.p.c. and order zero for each $i = 0, \dots, d$, and

$$\|\psi \circ \varphi(a) - a\| < \varepsilon$$

for every $a \in \Omega$. If no such $d$ exists then we set $\dim_{\rm nuc}(A) = \infty$.

Let $(V, S)$ be an open tower. Write $A_{V,S}$ for the $C^*$-subalgebra of $C(X) \rtimes_\lambda G$ generated by the sets $u_s C_0(V) u_t^*$ for $s, t \in S$. Denoting by $M_T$ the matrix algebra with entries indexed by pairs in $T\times T$ and by $\{e_{s,t}\}_{s,t\in T}$ the matrix units of $M_T$, there is a canonical isomorphism $M_T\otimes C_0(V)\to A_{V,S}$ determined by

$$e_{s,t}\otimes f\mapsto u_s f u_t^*$$

for $s,t\in S$ and $f\in C_0(V)$.

**Theorem 6.2.** *The action $G\curvearrowright X$ satisfies*

$$\dim_{\rm nuc}^{+1}(C(X)\rtimes_\lambda G)\leq \dim_{\rm tow}^{+1}(X,G)\cdot\dim^{+1}(X).$$

*Proof.* We denote the induced action of $G$ on $C(X)$ by $\alpha$, that is, $\alpha_s(f)(x)=f(s^{-1}x)$ for all $s\in G$, $f\in C(X)$, and $x\in X$.

We may assume that $\dim_{\rm tow}(X,G)$ is finite. For brevity we denote this quantity by $d$. Let $\Omega$ be a finite subset of $C(X)\rtimes G$ and $\varepsilon>0$. In order to verify the existence of the desired maps in the definition of nuclear dimension which approximately factorize the identity map on $C(X)\rtimes G$ to within $\varepsilon$ on the set $\Omega$, we may assume that $\Omega=\{fu_s : f\in\Upsilon,\ s\in F\}$ where $\Upsilon$ is a finite set of functions in $C(X)$ and $F$ is a finite subset of $G$ satisfying $F^{-1}=F$ and $e\in F$.

Let $n$ be an integer greater than 1, to be determined. By the definition of tower dimension, there is a finite $F^n$-Lebesgue collection of open towers $\{(V_i,S_i)\}_{i\in I}$ covering $X$ such that the family $\{S_iV_i\}_{i\in I}$ has chromatic number at most $d+1$. For convenience we may also assume that for each $i$ the set $S_i$ contains $e$, for if necessary we can choose a $t\in S_i$ and replace $S_i$ by $S_it^{-1}$ and $V_i$ by $tV_i$.

We next construct functions $g_i$ as in the proof of Theorem 5.2. The $F^n$-Lebesgue condition and a simple compactness argument produce a $\delta>0$ such that for every $x\in X$ there are an $i\in I$ and a $t\in S_i$ for which $d(x,X\setminus tV_i)>\delta$ and $F^nt\subseteq S_i$. For every $i\in I$ and $t\in S_i$ define $\hat g_{i,t}\in C(X)$ by

$$\hat g_{i,t}(x)=\min\{1,\delta^{-1}d(x,X\setminus tV_i)\},$$

and set

$$g_i=\max_{t\in S_i}\alpha_{t^{-1}}(\hat g_{i,t}).$$

Then for every $i\in I$ and $t\in S_i$ the support of $\alpha_t(g_i)$ is contained in $tV_i$, and $\alpha_t(g_i)\geq\hat g_{i,t}$, which implies that $\sum_{i\in I}\sum_{t\in S_i}\alpha_t(g_i)\geq 1$.

Let $i\in I$. Set $B_{i,n}=\bigcap_{t\in F^n}tS_i$ and $B_{i,0}=S_i\setminus\bigcap_{t\in F}tS_i$. For $k=1,\dots,n-1$ set

$$B_{i,k}=\Bigg(\bigcap_{t\in F^k}tS_i\Bigg)\setminus\bigcap_{t\in F^{k+1}}tS_i.$$

The sets $B_{i,k}$ for $k=0,\dots,n$ form a partition of $S_i$, and for all $s\in F$ we have

(i) $sB_{i,k}\subseteq B_{i,k-1}\cup B_{i,k}\cup B_{i,k+1}$ for every $k=1,\dots,n-1$,
(ii) $sB_{i,n}\subseteq B_{i,n-1}\cup B_{i,n}$.

Since for each $t\in S_i$ the function $\alpha_t(g_i)$ is supported in the tower level $tV_i$, it follows that the function

$$\hat h_i=\sum_{k=0}^n\sum_{t\in B_{i,k}}\frac{k}{n}\alpha_t(g_i)$$

satisfies $\sup_{x\in X} |\hat{h}_i(s^{-1}x) - \hat{h}_i(x)| \leq 1/n$ for every $s \in F$. Put $H = \sum_{i\in I} \hat{h}_i$. As in the proof of Theorem 5.2, our choice of $\delta$ entails that $H \geq 1$, and so for every $i$ we can set $h_i = H^{-1}\hat{h}_i$, which gives us a partition of unity $\{h_i\}_{i\in I}$ in $C(X)$.

Let $s \in F$. Let $x \in X$. The collection of all $i \in I$ such that $x \in S_iV_i$ has cardinality at most $d+1$, and so the difference between the values of $H$ at $x$ and $s^{-1}x$ is at most $(d+1)/n$. Since $H \geq 1$, it follows that the difference between the values of $H^{-1}$ at $x$ and $s^{-1}x$ is also at most $(d+1)/n$. We then get, for every $i$,

$$
\begin{aligned}
\|u_s h_i - h_i u_s\| &= \|u_s h_i u_s^{-1} - h_i\| \\
&= \sup_{x\in X} |h_i(s^{-1}x) - h_i(x)| \\
&\leq \sup_{x\in X} H(s^{-1}x)^{-1}\big|\hat{h}_i(s^{-1}x) - \hat{h}_i(x)\big| \\
&\qquad\qquad + \sup_{x\in X} \big|H(s^{-1}x)^{-1} - H(x)^{-1}\big|\hat{h}(x) \\
&\leq \frac{d+2}{n}.
\end{aligned} \tag{6}
$$

Since the collection $\{S_iV_i\}_{i\in I}$ has chromatic number at most $d+1$, there is a partition $I_0, \dots, I_d$ of $I$ such that for every $k = 0, \dots, d$ the collection $\{S_iV_i\}_{i\in I_k}$ is disjoint. For each $k = 0, \dots, d$ set $q_k = \sum_{i\in I_k} h_i$.

For $i \in I$ we write $A_i$ for the $C^*$-subalgebra of $C(X) \rtimes G$ generated by the sets $u_s C_0(V_i) u_t^*$ for $s, t \in S_i$. Let $k \in \{0, \dots, d\}$ and set $A_k = \bigoplus_{i\in I_k} A_i$. Since the $A_i$ for $i \in I_k$ are pairwise orthogonal as sub-C$^*$-algebras of $C(X) \rtimes G$, we can view $A_k$ as a C$^*$-subalgebra of $C(X) \rtimes G$. Since $A_i \cong M_{S_i} \otimes C_0(V_i)$ for every $i$ (as explained prior to the statement of the theorem), the nuclear dimension of $A_i$ is at most $\dim(X)$, as one can verify by a straightforward partition of unity argument using the formulation of covering dimension in term of the chromatic numbers of open covers (see the proof of Proposition 3.4 in [25]). Noting that $f u_s q_k = u_s \alpha_s(f) q_k \in A_k$ for every $f \in \Upsilon$ and $s \in F$ since $\hat{h}_i$ vanishes on $B_{i,0}$ for each $i$, we can thus find finite-dimensional $C^*$-algebras $D_{k,0}, \dots, D_{k,m_k}$ with $m_k \leq \dim(X)$, a c.p.c. map $\theta_k : A_k \to D_{k,0} \oplus \cdots \oplus D_{k,m_k}$, and a map $\psi_k : D_{k,0} \oplus \cdots \oplus D_{k,m_k} \to A_k \subseteq C(X) \rtimes G$ whose restriction to each summand is c.p.c. and order zero such that

$$
\|\psi_k \circ \theta_k(f u_s q_k) - f u_s q_k\| < \frac{\varepsilon}{2(d+1)} \tag{7}
$$

for all $f \in \Upsilon$ and $s \in F$. By Arveson's extension theorem we can extend $\theta_k$ to a c.p.c. map $C(X) \rtimes G \to D_{k,0} \oplus \cdots \oplus D_{k,m_k}$, which we will again call $\theta_k$. Define the c.p.c. map $\varphi_k : C(X) \rtimes G \to D_{k,0} \oplus \cdots \oplus D_{k,m_k}$ by

$$
\varphi_k(a) = \theta_k(q_k^{1/2} a q_k^{1/2}).
$$

Now define the maps

$$
C(X) \rtimes G \xrightarrow{\varphi} \bigoplus_{k=0}^{d} D_{k,0} \oplus \cdots \oplus D_{k,m_k} \xrightarrow{\psi} C(X) \rtimes G
$$

by $\varphi = \bigoplus_{k=0}^{d} \varphi_k$ and

$$
\psi(a_0, \dots, a_d) = \psi_0(a_0) + \cdots + \psi_d(a_d).
$$

Then $\varphi$ is c.p.c. and the restriction of $\psi$ to each $D_{k,j}$ is c.p.c. and order zero. Since $m_0+\cdots+m_d \leq \dim^{+1}(X,G)\cdot\dim^{+1}(X)$, to obtain the desired upper bound on $\dim_{\rm nuc}(C(X)\rtimes G)$ it remains to verify that $\|\psi\circ\varphi(fu_s)-fu_s\|<\varepsilon$ for all $f\in\Upsilon$ and $s\in F$.

By a straightforward functional calculus argument that uses a polynomial approximation to the function $x\mapsto x^{1/2}$ on $[0,1]$, we see from (6) that if $n$ is small enough relative to $d$ then for each $i\in I$, $f\in\Upsilon$, and $s\in F$ we will have

$$\|h_i^{1/2}fu_sh_i^{1/2}-fu_sh_i\|<\frac{\varepsilon}{2(d+1)}.$$

Let $s\in F$ and $k\in\{0,\dots,d\}$. Since for every $i\in I_k$ the element $h_i^{1/2}fu_sh_i^{1/2}-fu_sh_i$ belongs to $A_i$ and the sub-C*-subalgebras $A_i$ for $i\in I_k$ are pairwise orthogonal, we get

$$\|q_k^{1/2}fu_sq_k^{1/2}-fu_sq_k\|=\max_{i\in I_k}\|h_i^{1/2}fu_sh_i^{1/2}-fu_sh_i\|<\frac{\varepsilon}{2(d+1)}.$$

Using (7) this yields

$$\begin{aligned}\|\psi_k\circ\varphi_k(fu_s)-fu_sq_k\| &\leq \|\psi_k\circ\theta_k(q_k^{1/2}fu_sq_k^{1/2}-fu_sq_k)\|\\ &\qquad+\|\psi_k\circ\theta_k(fu_sq_k)-fu_sq_k\|\\ &<\|h_k^{1/2}fu_sq_k^{1/2}-fu_sq_k\|+\frac{\varepsilon}{2(d+1)}<\frac{\varepsilon}{d+1},\end{aligned}$$

whence

$$\begin{aligned}\|\psi\circ\varphi(fu_s)-fu_s\| &= \left\|\sum_{k=0}^{d}(\psi_k\circ\varphi_k(fu_s)-fu_sq_k)\right\|\\ &\leq\sum_{k=0}^{d}\|\psi_k\circ\varphi_k(fu_s)-fu_sq_k\|\\ &<(d+1)\cdot\frac{\varepsilon}{d+1}=\varepsilon,\end{aligned}$$

as desired. □

## 7. Tower dimension and comparison

We aim here to establish Theorem 7.2.

**Lemma 7.1.** *Suppose that $G$ is amenable. Let $G\curvearrowright X$ be a free action with tower dimension $d<\infty$. Let $K$ be a finite subset of $G$ and $\delta>0$. Then there is a finite collection $\{(V_i,S_i)\}_{i\in I}$ of open towers covering $X$ such that $S_i$ is $(K,\delta)$-invariant for every $i\in I$ and the family $\{S_iV_i\}_{i\in I}$ has chromatic number at most $d+1$.*

*Proof.* By the main theorem of [7] there exist nonempty $(K,\delta)$-invariant finite sets $F_1,\dots,F_n\subseteq G$ and sets $C_1,\dots,C_n\subseteq G$ such that

$$G=\bigsqcup_{k=1}^{n}\bigsqcup_{c\in C_k}F_kc.$$

Set $F = F_1F_1^{-1} \cup \cdots \cup F_nF_n^{-1}$. By our tower dimension hypothesis there is a finite $F$-Lebesgue collection of open towers $\{(V_i, T_i)\}_{i\in I}$ covering $X$ such that the family $\{S_iV_i\}_{i\in I}$ has chromatic number at most $d+1$. For each $i \in I$ set

$$T_i' = \bigcup \{F_kc : 1 \leq k \leq n,\ c \in C_k, \text{ and } F_kc \subseteq T_i\},$$
$$T_i'' = \bigcap_{s\in F} s^{-1}T_i.$$

Let $i \in I$. Let $x \in T_i \setminus T_i'$. Take $1 \leq k \leq n$ and $c \in C_k$ such that $x \in F_kc \cap T_i$. Then there exists a $y \in F_kc \cap (G \setminus T_i)$. We have $x = sc$ and $y = tc$ for some $s, t \in F_k$, whence $ts^{-1}x = y \notin T_i$, which shows that $x \notin T_i''$ since $ts^{-1} \in F$. We conclude from this that $T_i'' \subseteq T_i'$. It follows by the $F$-Lebesgue condition that the towers $(V_i, T_i')$ for $i \in I$ cover $X$.

Finally, for each $i \in I$ write $T_i' = \bigsqcup_{j\in J_i} S_j$ where each $S_j$ has the form $F_kc$ for some $1 \leq k \leq n$ and $c \in C_k$. Then the collection of open towers $\{(V_i, S_j)\}_{i\in I, j\in J_i}$ covers $X$, each of its shapes is $(K, \delta)$-invariant, and the family $\{S_iV_i\}_{i\in I}$ has chromatic number at most $d+1$, as desired. □

**Theorem 7.2.** *Suppose that $G$ is amenable. Let $X$ be a compact metric space with covering dimension $c < \infty$. Let $G \curvearrowright X$ be a free action with tower dimension $d < \infty$. Then the action has $((c+1)(d+1)-1)$-comparison.*

*Proof.* Let $A$ be a closed subset of $X$ and $B$ an open subset of $X$ such that $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$. By Lemma 3.3 we can find an $\eta > 0$ such that the sets

$$B_- = \{x \in X : d(x, X \setminus B) > \eta\},$$
$$A_+ = \{x \in X : d(x, A) \leq \eta\}$$

satisfy $\mu(A_+) + \eta \leq \mu(B_-)$ for all $\mu \in M_G(X)$.

We claim that there are a finite set $K \subseteq G$ and a $\delta > 0$ such that if $F$ is a nonempty $(K, \delta)$-invariant finite subset of $G$ then for all $x \in X$ we have

$$\frac{1}{|F|}\sum_{s\in F} 1_{A_+}(sx) + \frac{\eta}{2} \leq \frac{1}{|F|}\sum_{s\in F} 1_{B_-}(sx). \tag{8}$$

Suppose that this is not possible. Then there exists a Følner sequence $\{F_n\}$ and a sequence $\{x_n\}$ in $X$ such that, writing $\mu_n$ for the probability measure $(1/|F_n|)\sum_{s\in F_n} \delta_{sx}$, we have

$$\mu_n(A_+) + \frac{\eta}{2} > \mu_n(B_-)$$

for all $n$. By passing to a subsequence we may assume that the sequence $\{\mu_n\}$ converges to some $\mu \in M(X)$, and the Følner property implies that $\mu$ is $G$-invariant, as is easily verified. Since $B_-$ is open and $A_+$ is closed, the portmanteau theorem yields

$$\mu(B_-) + \frac{\eta}{2} \leq \liminf_{n\to\infty} \mu_n(B_-) + \frac{\eta}{2} \leq \limsup_{n\to\infty} \mu_n(A_+) + \eta \leq \mu(A_+) + \eta,$$

contradicting our choice of $\eta$. The desired $K$ and $\delta$ thus exist.

By Lemma 7.1 there are a finite collection $\{(V_i, T_i)\}_{i\in I}$ of open towers covering $X$ and a partition $I = I_0 \sqcup \cdots \sqcup I_d$ such that for every $i \in I$ the shape $T_i$ is $(K, \delta)$-invariant and for every $j = 0, \ldots, d$ the sets $T_iV_i$ for $i \in I_j$ are pairwise disjoint. By normality we can find for every $i \in I$ a closed set $V_i' \subseteq V_i$ such that the sets $T_iV_i'$ for $i \in I$ still cover $X$. Using the formulation of covering dimension in term of the chromatic numbers of open covers, we can then find, for each $i \in I$, a finite collection $\mathcal{U}_i$ of open subsets of $V_i$ such that

(i) the collection $\mathcal{U}_i$ covers $V_i'$,
(ii) each of the sets $sU$ for $s \in T_i$ and $U \in \mathcal{U}_i$ has diameter less than $\eta$, and
(iii) there is a partition $\mathcal{U}_i = \mathcal{U}_{i,0} \sqcup \cdots \sqcup \mathcal{U}_{i,c}$ such that the collection $\mathcal{U}_{i,j}$ is disjoint for each $j$.

For convenience we reindex the collection of towers $\{(U, T_i)\}_{i \in I, U \in \mathcal{U}_i}$ as $\{(U_j, S_j)\}_{j \in J}$. Then the shapes $S_j$ are all $(K, \delta)$-invariant and, setting $m = (c+1)(d+1) - 1$, there is a partition $J = J_0 \sqcup \cdots \sqcup J_m$ such that for each $k = 0, \dots, m$ the sets $S_j U_j$ for $j \in J_k$ are pairwise disjoint.

Let $0 \leq k \leq m$ and $j \in J_k$. Since the levels of the tower $(U_j, S_j)$ all have diameter less than $\eta$, if $sU_j \cap A \neq \emptyset$ for some $s \in S_j$ then $sU_j \subseteq A_+$, and so by (8) the sets

$$S_{j,1} = \{s \in S_j : sU_j \cap A \neq \emptyset\},$$
$$S_{j,2} = \{s \in S_j : sU_j \cap B_- \neq \emptyset\}$$

must satisfy $|S_{j,1}|/|S_j| + \eta/2 \leq |S_{j,2}|/|S_j|$ and hence $|S_{j,1}| \leq |S_{j,2}|$. We can thus find an injection $\varphi_j : S_{j,1} \to S_{j,2}$. Now the sets $sU_j$ for $s \in S_{j,1}$ and $j \in J$ cover $A$, while for each $k = 0, \dots, m$ the pairwise disjoint sets $\varphi(s)U_j = (\varphi(s)s^{-1})sU_j$ for $j \in J_k$ and $s \in S_{j,1}$ are contained in $B$ since the levels of the tower $(U_j, S_j)$ all have diameter less than $\eta$. This verifies that $A \prec_m B$, as desired. □

## 8. Almost finiteness

We begin by recalling the following notion of castle from Definition 5.7.

**Definition 8.1.** Let $G \curvearrowright X$ be a free action on a compact metric space. A *castle* is a finite collection of towers $\{(V_i, S_i)\}_{i \in I}$ such that the sets $S_i V_i$ for $i \in I$ are pairwise disjoint. The *levels* of the castle are the sets $sV_i$ for $i \in I$ and $s \in S_i$. We say that the castle is *open* if each of the towers is open, and *clopen* if each of the towers is clopen.

**Definition 8.2.** We say that a free action $G \curvearrowright X$ on a compact metric space is *almost finite* if for every $n \in \mathbb{N}$, finite set $K \subseteq G$, and $\delta > 0$ there are

(i) an open castle $\{(V_i, S_i)\}_{i \in I}$ whose shapes are $(K, \delta)$-invariant and whose levels have diameter less than $\delta$,
(ii) sets $S_i' \subseteq S_i$ such that $|S_i'| < |S_i|/n$ and
$$X \setminus \bigsqcup_{i \in I} S_i V_i \prec \bigsqcup_{i \in I} S_i' V_i.$$

**Remark 8.3.** Observe in the context of Definition 8.2 that if we have sets $S_i' \subseteq S_i$ satisfying

$$X \setminus \bigsqcup_{i \in I} S_i V_i \prec \bigsqcup_{i \in I} S_i' V_i$$

then any other sets $S_i'' \subseteq S_i$ with $|S_i''| \geq |S_i'|$ will similarly satisfy

$$X \setminus \bigsqcup_{i \in I} S_i V_i \prec \bigsqcup_{i \in I} S_i'' V_i.$$

since the relation $\prec$ is transitive and $\bigsqcup_{i\in I} S'_i V_i \prec \bigsqcup_{i\in I} S''_i V_i$. The latter follows from the fact that for each $i$ we have $S'_i V_i \prec S''_i V_i$, which can be witnessed by taking an injection $\varphi : S'_i \to S''_i$ and considering the open collections $\{sV_i : s \in S'_i\}$ and $\{\varphi(s)V_i : s \in S'_i\}$, the first of which partitions $S'_i V_i$ and the second of which partitions the subset $\varphi(S'_i)V_i$ of $S''_i V_i$.

**Remark 8.4.** Almost finiteness does not pass to extensions, the obstruction being the diameter condition. For example, the minimal actions in [13] factor onto an odometer, which is almost finite, but are not themselves almost finite by Theorem 12.4, since their crossed product fails to be $\mathcal{Z}$-stable. See however Theorem 11.6.

**Example 8.5.** Every free $\mathbb{Z}^m$-action on a zero-dimensional compact metrizable space is almost finite. This was established in Lemma 6.3 of [29] in the language of groupoids, whose translation to Definition 8.2 is discussed in the first paragraph of Section 10.

The following was shown in [4].

**Theorem 8.6.** *Let $G$ be a countable amenable group. Then a generic free minimal action of $G$ on the Cantor set is almost finite.*

The following two facts are simple consequences of Definition 8.2.

**Proposition 8.7.** *Almost finiteness is preserved under inverse limits of free actions.*

**Proposition 8.8.** *Let $G \curvearrowright X$ be a free action on a compact metrizable space, and suppose that $G$ can be expressed as a union of an increasing sequence $G_1 \subseteq G_2 \subseteq \dots$ of subgroups such that the restriction action $G_n \curvearrowright X$ is almost finite for every $n$. Then the action $G \curvearrowright X$ is almost finite.*

**Problem 8.9.** Let $G \curvearrowright X$ be a uniquely ergodic free minimal action of a countable amenable group on the Cantor set. Must it be almost finite?

## 9. Almost finiteness and comparison

In Theorem 9.2 we relate almost finiteness and comparison. By combining this with Theorem 7.2 we are then able to give a connection between tower dimension, almost finiteness, and comparison, which we record as Theorem 9.3.

**Lemma 9.1.** *Let $X$ be a compact metrizable space and let $\Omega$ be a weak$^*$ closed subset of $M(X)$. Let $A$ be a closed subset of $X$ such that $\mu(A) = 0$ for all $\mu \in \Omega$, and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that*

$$\mu(\{x \in X : d(x, A) \le \delta\}) < \varepsilon$$

*for all $\mu \in \Omega$.*

*Proof.* Suppose that the conclusion does not hold. Then for every $n \in \mathbb{N}$ we can find a $\mu_n \in \Omega$ such that the set $A_n = \{x \in X : d(x, A) \le 1/n\}$ satisfies $\mu_n(A_n) \ge \varepsilon$. By the compactness of $\Omega$ there is a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ which weak$^*$ converges to some $\mu \in \Omega$. For a fixed $j \in \mathbb{N}$ we have $\mu_{n_k}(A_{n_j}) \ge \varepsilon$ for every $k \ge j$, and since $A_{n_j}$ is closed the portmanteau theorem then yields

$$\mu(A_{n_j}) \ge \limsup_{k\to\infty} \mu_{n_k}(A_{n_j}) \ge \varepsilon.$$

As $A$ is closed it is equal to the intersection of the decreasing sequence of sets $A_{n_j}$, and so $\mu(A) = \lim_{j\to\infty} \mu(A_{n_j}) \ge \varepsilon$, in contradiction to our hypothesis. $\square$

**Theorem 9.2.** *Suppose that $G$ is amenable. Let $G \curvearrowright X$ be a free minimal action and consider the following conditions:*

*(i) the action is almost finite,*
*(ii) the action has comparison,*
*(iii) the action has $m$-comparison for all $m \geq 0$,*
*(iv) the action has $m$-comparison for some $m \geq 0$.*

*Then (i)⇒(ii)⇒(iii)⇒(iv), and if $E_G(X)$ is finite then all four conditions are equivalent.*

*Proof.* (i)⇒(ii). Let $A$ be a closed subset of $X$ and $B$ an open subset of $X$ such that $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$. We aim to show that $A \prec B$, which will establish (ii). By Lemma 3.3 there exists an $\eta > 0$ such that $\mu(A) + \eta \leq \mu(B)$ for all $\mu \in M_G(X)$. As the set $B \setminus A$ must be nonempty, we can pick a $y \in B \setminus A$. By Lemma 9.1 there is a $\kappa > 0$ such that the closed ball $C = \{x \in X : d(x, y) \leq \kappa\}$ is contained in $B \setminus A$ and satisfies $\mu(C) \leq \eta/2$ for all $\mu \in M_G(X)$. By minimality the open ball $C_- = \{x \in X : d(x, y) < \kappa/2\}$ satisfies $\mu(C_-) > 0$ for all $\mu \in M_G(X)$, and so by Lemma 3.3 (taking $A = \emptyset$ and $B = C_-$ there) there is a $\theta > 0$ such that $\mu(C_-) \geq \theta$ for all $\mu \in M_G(X)$.

Set $\tilde{B} = B \setminus C$. Then for all $\mu \in M_G(X)$ we have

$$\mu(\tilde{B}) = \mu(B) - \mu(C) \geq \mu(A) + \frac{\eta}{2} > \mu(A)$$

and so by Lemma 3.3 there exists an $\eta' > 0$ with $\eta' \leq \eta$ such that the sets

$$\begin{aligned} B_- &= \{x \in X : d(x, X \setminus \tilde{B}) > \eta'\}, \\ A_+ &= \{x \in X : d(x, A) \leq \eta'\} \end{aligned}$$

satisfy $\mu(A_+) + \eta' \leq \mu(B_-)$ for all $\mu \in M_G(X)$. Note that each of the sets $A_+$ and $B_-$ is disjoint from $C$.

We claim that there are a finite set $K \subseteq G$ and a $\delta > 0$ such that if $F$ is a nonempty $(K, \delta)$-invariant finite subset of $G$ then for all $x \in X$ the following both hold:

$$\frac{1}{|F|} \sum_{s \in F} 1_{A_+}(sx) + \frac{\eta'}{2} \leq \frac{1}{|F|} \sum_{s \in F} 1_{B_-}(sx), \tag{9}$$

$$\frac{1}{|F|} \sum_{s \in F} 1_{C_-}(sx) \geq \frac{\theta}{2}. \tag{10}$$

Suppose to the contrary that this is not possible. Then we can find a Følner sequence $\{F_n\}$ and a sequence $\{x_n\}$ in $X$ such that, writing $\mu_n$ for the probability measure $(1/|F_n|) \sum_{s \in F_n} \delta_{sx}$, one of the following holds:

(i) $\mu_n(A_+) + \eta'/2 > \mu_n(B_-)$ for all $n$,
(ii) $\mu_n(C_-) < \theta/2$ for all $n$.

Suppose first that (i) holds. By passing to a subsequence we may assume that the sequence $\{\mu_n\}$ converges to some $\mu \in M(X)$, and it is readily verified using the Følner property that $\mu$ is $G$-invariant. Since $B_-$ is open and $A_+$ is closed, the portmanteau theorem then yields

$$\mu(B_-) + \frac{\eta'}{2} \leq \liminf_{n \to \infty} \mu_n(B_-) + \frac{\eta'}{2} \leq \limsup_{n \to \infty} \mu_n(A_+) + \eta' \leq \mu(A_+) + \eta',$$

a contradiction. If on the other hand (ii) holds, then as before we may assume that $\{\mu_n\}$ converges to some $\mu \in M_G(X)$, and since $C_-$ is open the portmanteau theorem yields

$$\mu(C_-) \leq \liminf_{n\to\infty} \mu_n(C_-) \leq \frac{\theta}{2},$$

a contradiction. We may thus find the desired $K$ and $\delta$.

Set $\varepsilon = \min\{\eta', \kappa/2\}$ and choose an integer $n > 3/\theta$. Then by almost finiteness there are

(i) an open castle $\{(V_i, S_i)\}_{i\in I}$ whose shapes are $(K, \delta)$-invariant and whose levels have diameter less than $\varepsilon$, and

(ii) sets $S_i' \subseteq S_i$ such that $|S_i'| < |S_i|/n$ and the set $D := \bigsqcup_{i\in I} S_i V_i$ satisfies

$$X \setminus D \prec \bigsqcup_{i\in I} S_i' V_i.$$

Let $i \in I$. Since the levels of the towers all have diameter less than both $\eta'$ and $\kappa/2$, by (9) and (10) the sets

$$S_{i,1} = \{s \in S_i : sV_i \cap A \neq \emptyset\},$$
$$S_{i,2} = \{s \in S_i : sV_i \cap B_- \neq \emptyset\},$$
$$S_{i,3} = \{s \in S_i : sV_i \cap C_- \neq \emptyset\}$$

satisfy

$$\frac{|S_{i,1}|}{|S_i|} + \frac{\eta'}{2} \leq \frac{|S_{i,2}|}{|S_i|} \quad \text{and} \quad \frac{|S_{i,3}|}{|S_i|} \geq \frac{\theta}{2}$$

so that $|S_{i,1}| \leq |S_{i,2}|$ and $|S_{i,3}| \geq |S_i'|$. We can thus find injective maps $\varphi_i : S_{i,1} \to S_{i,2}$ and $\psi_i : S_i' \to S_{i,3}$.

Since $X \setminus D \prec \bigsqcup_{i\in I} S_i' V_i$ we can find a finite collection $\mathcal{U}$ of open subsets of $X$ which cover $X \setminus D$ and a $t_U \in G$ for each $U \in \mathcal{U}$ such that the images $t_U U$ for $U \in \mathcal{U}$ are pairwise disjoint subsets of $\bigsqcup_{i\in I} S_i' V_i$. For all $U \in \mathcal{U}$, $i \in I$, and $s \in S_i'$ write $W_{U,i,s}$ for the (possibly empty) open set $U \cap t_U^{-1} s V_i$. These open sets cover $X \setminus D$, and so in particular cover $(X \setminus D) \cap A$, and the images $\psi_i(t_U) W_{U,i,s}$ for $U \in \mathcal{U}$, $i \in I$, and $s \in S_i'$ are pairwise disjoint subsets of $B \cap \bigsqcup_{i\in I} S_{i,3} V_i$. At the same time, the open sets $sV_i$ for $i \in I$ and $s \in S_{i,1}$ cover $D \cap A$, while the images $\varphi_i(s) V_i = (\varphi_i(s) s^{-1}) s V_i$ for $i \in I$ and $s \in S_{i,1}$ are pairwise disjoint subsets of $B \cap \bigsqcup_{i\in I} S_{i,2} V_i$. Since the sets $\bigsqcup_{i\in I} S_{i,3} V_i$ and $\bigsqcup_{i\in I} S_{i,2} V_i$ are disjoint, we have thus verified that $A \prec B$.

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv). Trivial.

Now suppose that $E_G(X)$ is finite and let us verify (iv)$\Rightarrow$(i). We thus suppose that there is an $m \in \mathbb{N}$ such that the action has $m$-comparison. We may assume that $G$ is infinite, for otherwise minimality implies that $X$ consists of a single orbit, in which case the action is obviously almost finite. Write $E_G(X) = \{\mu_1, \dots, \mu_q\}$ and set $\mu = (1/q) \sum_{k=1}^{q} \mu_k \in M_G(X)$. Let $K$ be a finite subset of $G$, $\delta > 0$, and $n \in \mathbb{N}$. Put $\varepsilon = 1/(4nq(m+1))$. Choose an integer $N > 1/\varepsilon$. Since $G$ is infinite and $m \cdot 4q\varepsilon < 1$ we can find a finite set $K' \subseteq G$ with $K \subseteq K$ and a $\delta' > 0$ with $\delta' \leq \delta$ such that every nonempty $(K', \delta')$-invariant finite set $F \subseteq G$ has large enough cardinality so that it has $m$ pairwise disjoint subsets of equal cardinality $\kappa$ satisfying $2q\varepsilon < \kappa/|F| < 4q\varepsilon$.

Since the action is free, by the Ornstein–Weiss tower theorem (as formulated in Theorem 4.46 of [22]) there exists a finite collection $\{(M_i, T_i)\}_{i\in I}$ of measurable towers such that the sets $T_iM_i$ for $i \in I$ are pairwise disjoint, $\mu(\bigsqcup_{i\in I} T_iM_i) \geq 1 - \varepsilon/(2q)$, and $T_i$ is $(K', \delta')$-invariant for every $i$. By regularity we can find closed sets $C_i \subseteq M_i$ with $\mu(M_i \setminus C_i)$ small enough to ensure that $\mu(\bigsqcup_{i\in I} T_iC_i) \geq 1 - \varepsilon/q$. Then by compactness we can find open sets $V_i \supseteq C_i$ such that the sets $T_iV_i$ for $i \in I$ are pairwise disjoint.

Let $i \in I$. By our choice of $K'$ and $\delta'$ we can find pairwise disjoint sets $S_{i,0}, \dots, S_{i,m} \subseteq T_i$ all having the same cardinality $\kappa$ satisfying $2q\varepsilon < \kappa/|T_i| < 4q\varepsilon$. Set $T_i' = S_{i,0} \sqcup \cdots \sqcup S_{i,m}$. Then

$$|T_i'| = (m+1)\kappa < 4(m+1)q\varepsilon|T_i| = \frac{1}{n}|T_i|. \tag{11}$$

Set $A = X \setminus \bigsqcup_{i\in I} T_iV_i$ and $B = \bigsqcup_{i\in I} S_{i,0}V_i$. Then for every $\nu \in M_G(X)$ we have, since $\nu$ is convex combination of the measures $\mu_1, \dots, \mu_q$,

$$\nu(A) \leq \max_{k=1,\dots,q} \mu_k(A) \leq q\mu(A) \leq \varepsilon,$$

from which we get $\nu(\bigsqcup_{i\in I} T_iV_i) \geq 1 - \varepsilon \geq 1/2$ and hence

$$\nu(B) \geq \sum_{i\in I} \frac{|S_{i,0}|}{|T_i|}\nu(T_iV_i) > 2q\varepsilon\nu\biggl(\bigsqcup_{i\in I} T_iV_i\biggr) \geq \frac{1}{4n(m+1)} \geq \varepsilon.$$

Since $A$ is closed and $B$ is open we thus have $A \prec_m B$ by our $m$-comparison hypothesis. We can therefore find a finite collection $\mathcal{U}$ of open subsets of $X$ which cover $A$, an $s_U \in G$ for each $U \in \mathcal{U}$, and a partition $\mathcal{U} = \mathcal{U}_0 \sqcup \cdots \sqcup \mathcal{U}_m$ such that for each $i = 0, \dots, m$ the images $s_UU$ for $U \in \mathcal{U}_i$ are pairwise disjoint subsets of $B$.

For each $i \in I$ and $j = 0, \dots, m$ choose a bijection $\varphi_{i,j} : S_{i,0} \to S_{i,j}$. For $U \in \mathcal{U}$, $i \in I$, and $t \in S_{i,0}$ write $W_{U,i,t}$ for the open set $U \cap s_U^{-1}tV_i$. For a fixed $U$, the sets $W_{U,i,t}$ for $i \in I$ and $t \in S_{i,0}$ partition $U$. Moreover, writing $j_U$ for the $j$ such that $U \in \mathcal{U}_j$, the sets $\varphi_{i,j_U}(t)t^{-1}s_UW_{U,i,t}$ over all $U \in \mathcal{U}$, $i \in I$, and $t \in S_{i,0}$ are pairwise disjoint and contained in $\bigsqcup_{i\in I} T_i'V_i$. This shows that

$$A \prec \bigsqcup_{i\in I} T_i'V_i.$$

Combined with (11) and the fact that the levels of the towers $(M_i, T_i)$ and hence also of the towers $(V_i, T_i)$ can be chosen to have as small a diameter as we wish (by measurably partitioning each base $M_i$ to create finer towers), this verifies almost finiteness. $\square$

Combining Theorems 7.2 and 9.2 yields:

**Theorem 9.3.** *Suppose that $G$ is amenable. Let $G \curvearrowright X$ be a free minimal action on a compact metrizable space such that $E_G(X)$ is finite. Consider the following conditions:*

*(i)* $\dim_{\mathrm{tow}}(X, G) < \infty$ *and dim*$(X) < \infty$,
*(ii)* $\dim_{\mathrm{ftow}}(X, G) < \infty$,
*(iii) the action is almost finite,*
*(iv) the action has comparison.*

*Then (i)⇔(ii)⇒(iii)⇔(iv).*

The implication (ii)⇒(iii) in Theorem 9.3 cannot be reversed, as the following examples show. The obstruction in both cases is infinite-dimensionality, whether in the space (Example 9.4) or in the group (Example 9.5).

**Example 9.4.** Let $\{\theta_k\}$ be a sequence of rationally independent numbers in $[0, 1)$. Consider the product action $\mathbb{Z} \overset{\alpha}{\curvearrowright} \prod_{k=0}^{\infty} X_k$ whose zeroeth factor is the odometer action $\mathbb{Z} \curvearrowright \{0, 1\}^{\mathbb{N}}$ and whose $k$th factor for $k \geq 1$ is the action $(n, z) \mapsto e^{2\pi i n\theta_k} z$ on $\mathbb{T}$. This action is free. It is uniquely ergodic since each factor is uniquely ergodic (as is well known) and the factors are mutually disjoint (because no two of them, when viewed as measure-preserving actions with respect to the unique invariant Borel probability measure on each, have a common eigenvalue except for 1). It is minimal since the unique invariant Borel probability measure on $\prod_{k=0}^{\infty} X_k$, i.e., the product of the unique invariant Borel probability measures on the factors, has full support. It is also almost finite. To see this, first note that the odometer action $\mathbb{Z} \curvearrowright \{0, 1\}^{\mathbb{N}}$ is almost finite since for every $n \in \mathbb{N}$ the clopen set $\{0\}^{\{1,\dots,n\}} \times \{0, 1\}^{\{n+1,n+2,\dots\}}$ is the base of a tower with shape $\{0, 1, \dots, 2^n - 1\}$ whose levels partition $\{0, 1\}^{\mathbb{N}}$. Now for $m \geq 1$ we can view $\mathbb{Z} \curvearrowright \prod_{k=0}^{m} X_k$ as an extension of $\mathbb{Z} \curvearrowright \prod_{k=0}^{m-1} X_k$ via the natural projection map, and so it follows by Theorem 11.6 and induction that $\mathbb{Z} \curvearrowright \prod_{k=0}^{m} X_k$ is almost finite for every $m \geq 0$. One can alternatively derive this conclusion by combining the fact that the odometer action has tower dimension 1 (Example 4.5) with Proposition 4.6 (tower dimension is nonincreasing under taking extensions) and (i)⇒(iii) of Theorem 9.3. It follows finally by Proposition 8.7 that $\alpha$, being the inverse limit of the actions $\mathbb{Z} \curvearrowright \prod_{k=0}^{m} X_k$, is almost finite. This example shows that, for free minimal actions of $\mathbb{Z}$, almost finiteness does not imply finite tower dimension, since the latter implies that the space has finite covering dimension, which is not the case here.

**Example 9.5.** By Proposition 4.8, a necessary condition for a free action $G \curvearrowright X$ to have finite tower dimension is that the group $G$ have finite asymptotic dimension, which fails for many amenable groups, such as the Grigorchuk group. Since every countably infinite amenable group admits almost finite free minimal actions by Theorem 8.6, this gives many examples of almost finite free minimal actions which fail to have finite tower dimension.

## 10. Disjointness in tower closures and almost finiteness in dimension zero

In [29] Matui introduced a notion of almost finiteness for second countable étale groupoids with compact zero-dimensional unit spaces. We show in Theorem 10.2 that when the groupoid arises from a free action $G \curvearrowright X$ on a zero-dimensional compact metrizable space, our notion of almost finiteness coincides with Matui's, justifying our use of the terminology. What we in fact prove is that the action is almost finite (in the sense of Definition 8.2) if and only if for every finite set $K \subseteq G$ and $\delta > 0$ there is a clopen castle (Definition 8.1) whose shapes are $(K, \delta)$-invariant and whose levels partition $X$ (a clopen castle whose levels partition $X$ will be called a *clopen tower decomposition* of $X$). That this characterization is equivalent to Matui's almost finiteness is recorded as Lemma 5.3 in [45].

The following lemma will be useful in establishing not only Theorem 10.2 but also Theorem 12.4.

**Lemma 10.1.** *In Definition 8.2 we may equivalently require each tower $(V_i, S_i)$ to have the additional property that the sets $s\overline{V_i}$ for $s \in S_i$ are pairwise disjoint.*

*Proof.* Let $G \curvearrowright X$ be a free action which is almost finite. If $G$ is finite, then by taking $n > |G|$, $K = G$, and $\delta < |G|^{-1}$ in Definition 8.2 we are guaranteed the existence of an open castle $\{(V_i, S_i)\}_{i \in I}$ such that each shape is equal to $G$, every level has diameter smaller than $\delta$, and $\bigsqcup_{i \in I} S_i V_i = X$. It follows that every $V_i$ is clopen and so we obtain the assertion of the lemma. We may thus assume that $G$ is infinite.

Let $K$ be a finite subset of $G$, $n \in \mathbb{N}$, and $\delta > 0$. Since $G$ is infinite, there exists a finite set $K' \subset G$ with $K \subseteq K'$ and a $\delta' > 0$ with $\delta' \leq \delta$ such that every $(K', \delta')$-invariant nonempty finite subset of $G$ has cardinality greater than $2n$. By almost finiteness there exist

(i) an open castle $\{(V_i, S_i)\}_{i \in I}$ whose shapes are $(K', \delta')$-invariant and whose levels have diameter less than $\delta$,
(ii) sets $S'_i \subseteq S_i$ with $|S'_i| < |S_i|/(2n)$ such that

$$X \setminus \bigsqcup_{i \in I} S_i V_i \prec \bigsqcup_{i \in I} S'_i V_i.$$

For each $i$ the set $S_i$ has cardinality greater than $2n$ by our choice of $K'$ and $\delta'$, and so by setting $S''_i = S'_i \cup \{s\}$ for some arbitrarily chosen $s \in S_i \setminus S'_i$ we will have $|S''_i| < |S_i|/n$. Given a $\mu \in M_G(X)$, from (ii) we have $\mu(X \setminus \bigsqcup_{i \in I} S_i V_i) \leq \mu(\bigsqcup_{i \in I} S'_i V_i)$, which in particular implies that $\mu(V_i) > 0$ for at least one $i \in I$, and hence that

$$\mu\bigg(X \setminus \bigsqcup_{i \in I} S_i V_i\bigg) < \mu\bigg(\bigsqcup_{i \in I} S''_i V_i\bigg).$$

It follows by Lemma 3.3 there is an $\eta > 0$ such that the sets

$$B = \bigg\{x \in X : d\bigg(x, X \setminus \bigsqcup_{i \in I} S''_i V_i\bigg) > \eta\bigg\},$$
$$A = \bigg\{x \in X : d\bigg(x, X \setminus \bigsqcup_{i \in I} S_i V_i\bigg) \leq \eta\bigg\}$$

satisfy $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$. By uniform continuity we can then find an $\eta' > 0$ such that the open sets

$$U_i = \{x \in X : d(x, X \setminus V_i) > \eta'\}$$

for $i \in I$ satisfy $X \setminus \bigsqcup_{i \in I} S_i U_i \subseteq A$ and $B \subseteq \bigsqcup_{i \in I} S''_i U_i$. Then for every $\mu \in M_G(X)$ we have

$$\mu\bigg(X \setminus \bigsqcup_{i \in I} S_i U_i\bigg) \leq \mu(A) < \mu(B) \leq \mu\bigg(\bigsqcup_{i \in I} S''_i U_i\bigg).$$

Since the action is almost finite, it has comparison by Theorem 9.2, and so we deduce that

$$X \setminus \bigsqcup_{i \in I} S_i U_i \prec \bigsqcup_{i \in I} S''_i U_i$$

Therefore the open castle $\{(U_i, S_i)\}_{i \in I}$ and the sets $S''_i \subseteq S_i$ witness the definition of almost finiteness with respect to $n$, $K$, and $\delta$, and for each $i \in I$ the inclusion $\overline{U_i} \subseteq V_i$ implies that the sets $s\overline{U_i}$ for $s \in S_i$ are pairwise disjoint, as desired. □

**Theorem 10.2.** *A free action $G \curvearrowright X$ on a zero-dimensional compact metric space is almost finite if and only if for every finite set $K \subseteq G$ and $\delta > 0$ there is a clopen castle whose shapes are $(K, \delta)$-invariant and whose levels partition $X$.*

*Proof.* The only issue in establishing the backward implication is arranging for the small diameter condition in the definition of almost finiteness, and this can be done by observing that for every $\varepsilon > 0$ and clopen tower $(V, S)$ we can use uniform continuity to find a clopen partition $\{V_i\}_{i\in I}$ of $V$ such that the levels of the clopen castle $\{(V_i, S)\}_{i\in I}$, which partition $SV$, all have diameter less than $\varepsilon$.

For the forward implication, suppose that the action is almost finite. Let $K$ be a finite subset of $G$ and $\delta > 0$. Take an $n \in \mathbb{N}$ such that $2/n \leq \delta/2$. By Lemma 10.1 there are

(i) an open castle $\{(V_i, S_i)\}_{i\in I}$ with $(K, \delta/2)$-invariant shapes such that for each $i$ the sets $s\overline{V_i}$ for $s \in S_i$ are pairwise disjoint,

(ii) sets $S'_i \subseteq S_i$ such that $|S'_i| < |S_i|/n$ and the set $D := X \setminus \bigsqcup_{i\in I} S_i V_i$ satisfies
$$D \prec \bigsqcup_{i\in I} S'_i V_i.$$

Since the sets $s\overline{V_i}$ for $i \in I$ and $s \in S_i$ are closed, by uniform continuity we can find, for each $i$, an open set $V'_i \supseteq \overline{V_i}$ such that the sets $sV'_i$ for $i \in I$ and $s \in S_i$ are pairwise disjoint. Then, using compactness and zero-dimensionality, for each $i \in I$ we can cover $\overline{V_i}$ with finitely many clopen subsets of $V'_i$, and so by replacing $V_i$ with the union of these clopen sets we may assume that each of the sets $V_i$ is clopen. Note in particular that the set $D = X \setminus \bigsqcup_{i\in I} S_i V_i$, which is now clopen, still satisfies $D \prec \bigsqcup_{i\in I} S'_i V_i$, since each new $V_i$ contains the original one. By Proposition 3.5 we can then find a clopen partition $\mathfrak{U}$ of $D$ and elements $t_U \in G$ for $U \in \mathfrak{U}$ such that the images $t_U U$ for $U \in \mathfrak{U}$ are pairwise disjoint subsets of $\bigsqcup_{i\in I} S'_i V_i$. We may assume, by splitting each tower $(V_i, S_i)$ into finitely many towers having the same shape $S_i$ and with bases forming a suitable clopen partition of $V_i$, that for every $U \in \mathfrak{U}$, $i \in I$, and $s \in S'_i$ such that $sV_i \cap t_U U \neq \emptyset$ we in fact have $sV_i \subseteq t_U U$. By replacing $\mathfrak{U}$ with the clopen refinement consisting of the sets of the form $t_U^{-1} sV_i$ where $sV_i$ is a tower level which is contained in $t_U U$ for some $U \in \mathfrak{U}$, we may now also assume that for every $U \in \mathfrak{U}$ there are an $i_U \in I$ and an $s_U \in S'_{i_U}$ such that $t_U U = s_U V_{i_U}$.

Let $i \in I$. Set $S''_i = \{t_U^{-1} s_U : U \in \mathfrak{U} \text{ and } s_U \in S'_i\}$. Note that the map $U \mapsto t_U^{-1} s_U$ from $\{U \in \mathfrak{U} : i_U = i\}$ to $S'_i$ is injective, for if $t_U^{-1} s_U = t_{U'}^{-1} s_{U'}$ for $U$ and $U'$ in the domain then $U = t_U^{-1} s_U V_i = t_{U'}^{-1} s_{U'} V_i = U'$. Thus $|S''_i| \leq |S'_i|$. Define $\tilde{S}_i = S_i \sqcup S''_i$. Then for every $t \in K$ we have
$$\begin{aligned}|t\tilde{S}_i \Delta \tilde{S}_i| &\leq |tS_i \Delta S_i| + |tS''_i| + |S''_i| \\ &< \frac{\delta}{2}|S_i| + 2|S'_i| \\ &\leq \left(\frac{\delta}{2} + \frac{2}{n}\right)|S_i| \\ &\leq \delta|\tilde{S}_i|,\end{aligned}$$
showing that $\tilde{S}_i$ is $(K, \delta)$-invariant. Therefore $\{(V_i, \tilde{S}_i)\}_{i\in I}$ is clopen tower decomposition of $X$ with $(K, \delta)$-invariant shapes, as desired. □

**Remark 10.3.** Matui showed in [29] that almost finiteness for a second countable étale groupoid $G$ with compact zero-dimensional unit space has several implications for the homology groups $H^n(G)$ and their relation to both the topological full group $[\![G]\!]$ and the

$K$-theory of the reduced groupoid C*-algebra of $G$. In particular, if the groupoid is principal and almost finite then there is a canonical isomorphism $H^1(G) \cong [\![G]\!]/N$ where $N$ is the subgroup generated by the elements of finite order (see Section 7 of [29]). As Matui observes in Lemma 6.3 of [29], the groupoid associated to a free action of $\mathbb{Z}^m$ on a zero-dimensional compact metrizable space is almost finite. By Theorem 10.2, Example 4.9, and Theorem 9.3, we see that this is also the case for every free minimal action $G \curvearrowright X$ of a finitely generated nilpotent group on a zero-dimensional compact metrizable space with $E_G(X)$ finite.

## 11. Almost finiteness and extensions

As noted in Remark 8.4, almost finiteness does not pass to extensions in general. We will show however in Theorem 11.6 that an extension $G \curvearrowright Y$ of an almost finite free action $G \curvearrowright X$ is again almost finite whenever $E_G(Y)$ and $\dim(Y)$ are both finite. To this end we will employ the following notions of coarse almost finiteness and $m$-almost finiteness.

**Definition 11.1.** We say that a free action $G \curvearrowright X$ on a compact metric space is *coarsely almost finite* if for every $n \in \mathbb{N}$, finite set $K \subseteq G$, and $\delta > 0$ there are

(i) a collection $\{(V_i, S_i)\}_{i\in I}$ of open towers with $(K,\delta)$-invariant shapes such that $\{S_i\overline{V_i}\}_{i\in I}$ is a castle,

(ii) sets $S_i' \subseteq S_i$ such that $|S_i'| < |S_i|/n$ and

$$X \setminus \bigsqcup_{i\in I} S_iV_i \prec \bigsqcup_{i\in I} S_i'V_i.$$

An almost finite free action is coarsely almost finite by Lemma 10.1.

**Definition 11.2.** Let $m \in \mathbb{N}$. We say that a free action $G \curvearrowright X$ on a compact metric space is *$m$-almost finite* if for every $n \in \mathbb{N}$, finite set $K \subseteq G$, and $\delta > 0$ there are

(i) a collection $\{(V_i, S_i)\}_{i\in I}$ of open towers with $(K,\delta)$-invariant shapes such that $\operatorname{diam}(sV_i) < \delta$ for every $i \in I$ and $s \in S_i$ and the family $\{S_iV_i\}_{i\in I}$ has chromatic number at most $m+1$,

(ii) sets $S_i' \subseteq S_i$ such that $|S_i'| < |S_i|/n$ and

$$X \setminus \bigsqcup_{i\in I} S_iV_i \prec \bigsqcup_{i\in I} S_i'V_i.$$

The following is easily verified by taking the inverse images under the extension $Y \to X$ of all of the sets at play in the definition of coarse almost finiteness.

**Proposition 11.3.** *If $Y \to X$ is an extension of free actions of $G$ and $G \curvearrowright X$ is coarsely almost finite, then $G \curvearrowright Y$ is coarsely almost finite.*

**Lemma 11.4.** *Suppose that $X$ has covering dimension $d < \infty$ and let $G \curvearrowright X$ be a free action which is coarsely almost finite. Then the action is $d$-almost finite.*

*Proof.* Let $n \in \mathbb{N}$, and let $K$ be a finite subset of $G$ and $\delta > 0$. By coarse almost finiteness there are

(i) a collection $\{(V_i, S_i)\}_{i\in I}$ of open towers with $(K,\delta)$-invariant shapes such that $\{S_i\overline{V_i}\}_{i\in I}$ is a castle, and

(ii) sets $S_i' \subseteq S_i$ such that $|S_i'| < |S_i|/n$ and

$$X \setminus \bigsqcup_{i\in I} S_i V_i \prec \bigsqcup_{i\in I} S_i' V_i.$$

Since the towers $S_i\overline{V_i}$ for $i \in I$ are pairwise disjoint, for each $i$ we can find an open set $U_i \supseteq V_i$ so that the towers $S_iU_i$ for $i \in I$ are still pairwise disjoint. Since $X$ has covering dimension $d$, for each $i$ we can find a collection $\{V_{i,1},\dots,V_{i,k_i}\}$ of open subsets of $U_i$ which covers $\overline{V_i}$, satisfies $\operatorname{diam}(sV_{i,j}) < \delta$ for every $j = 1,\dots,k_i$, and has chromatic number at most $d+1$. The collection of towers $\{(V_{i,j}, S_i) : i \in I,\, 1 \le j \le k_i\}$ then fulfills the requirements in the definition of $d$-almost finiteness. □

The proof of the following is essentially the same as for (i)⇒(ii) of Theorem 9.2, which is the case $m = 0$. We leave the details to the reader.

**Lemma 11.5.** *Let $G \curvearrowright X$ be a free action which is $m$-almost finite. Then the action has $m$-comparison.*

**Theorem 11.6.** *Let $G \overset{\alpha}{\curvearrowright} X$ be an almost finite free action and let $G \overset{\beta}{\curvearrowright} Y$ be an extension of $\alpha$ such that $E_G(Y)$ is finite and $\dim(Y) < \infty$. Then $\beta$ is almost finite.*

*Proof.* Since $\alpha$ is almost finite it is coarsely almost finite, and so by Proposition 11.3 the action $\beta$ is coarsely almost finite. Consequently $\beta$ has $m$-comparison by Lemmas 11.5 and 11.4. We then conclude by Theorem 9.2 that $\beta$ is almost finite. □

## 12. Almost finiteness and $\mathcal{Z}$-stability

We show here in Theorem 12.4 that, assuming $G$ is infinite, the reduced crossed product $C(X) \rtimes_\lambda G$ of an almost finite free minimal action $G \curvearrowright X$ on a compact metrizable space is $\mathcal{Z}$-stable. The argument uses tiling technology as in the proof of Theorem 5.3 of [4]. Note that since almost finiteness implies that $G$ is amenable, the reduced and full crossed products coincide in this case, although we will not need this fact.

Recall that *c.p.c.* stands for "completely positive contractive", and that a map $\varphi : A \to B$ between C*-algebras is *order-zero* if $\varphi(a_1)\varphi(a_2) = 0$ for all $a_1, a_2 \in A$ satisfying $a_1a_2 = 0$. We write $\precsim$ for the relation of Cuntz subequivalence.

In order to verify $\mathcal{Z}$-stability we will use the following result of Hirshberg and Orovitz (Theorem 4.1 of [19]).

**Theorem 12.1.** *Let $A$ be a simple separable unital nuclear C*-algebra not isomorphic to $\mathbb{C}$. Suppose that for every $n \in \mathbb{N}$, finite set $\Omega \subseteq A$, $\varepsilon > 0$, and nonzero positive element $a \in A$ there exists an order-zero c.p.c. map $\varphi : M_n \to A$ such that*

*(i) $1 - \varphi(1) \precsim a$,*
*(ii) $\|[b, \varphi(c)]\| < \varepsilon$ for all $b \in \Omega$ and norm-one $c \in M_n$.*

*Then $A$ is $\mathcal{Z}$-stable.*

The following is the Ornstein–Weiss quasitiling theorem [34]. See Theorem 4.36 of [22] for this precise formulation. For $0 \le \eta \le 1$, we say that a collection $\{A_i\}$ of subsets of a finite set $E$ is *$\eta$-disjoint* if there exist sets $A_i' \subseteq A_i$ with $|A_i'| \ge (1-\eta)|A_i|$ such that the collection $\{A_i'\}$ is disjoint, and that it *$\eta$-covers* $E$ if $|\bigcup_i A_i| \ge \eta|E|$.

**Theorem 12.2.** *Let $0 < \beta < \frac{1}{2}$ and let $n \in \mathbb{N}$ be such that $(1 - \beta/2)^n < \beta$. Then whenever $e \in T_1 \subseteq T_2 \subseteq \cdots \subseteq T_n$ are finite subsets of a group $G$ such that $|\partial_{T_{i-1}} T_i| \leq (\beta/8)|T_i|$ for $i = 2, \ldots, n$, for every $(T_n, \beta/4)$-invariant nonempty finite set $E \subseteq G$ there exist $C_1, \ldots, C_n \subseteq G$ such that*

*(i)* $\bigcup_{i=1}^n T_i C_i \subseteq E$, *and*
*(ii) the collection of right translates* $\bigcup_{i=1}^n \{T_i c : c \in C_i\}$ *is $\beta$-disjoint and $(1-\beta)$-covers $E$.*

**Lemma 12.3.** *Let $G \curvearrowright X$ be an action on a compact metrizable space. Let $A$ be a closed subset of $X$ and $B$ an open subset of $X$ such that $A \prec B$. Let $f, g : X \to [0,1]$ be continuous functions such that $f = 0$ on $X \setminus A$ and $g = 1$ on $B$. Then there is a $v \in C(X) \rtimes_\lambda G$ such that $v^* g v = f$.*

*Proof.* As $A \prec B$ there exist open sets $U_1, \ldots, U_n \subseteq X$ such that $A \subseteq \bigcup_{i=1}^n U_i$ and an $s_i \in G$ for each $i = 1, \ldots, n$ such that the images $s_i U_i$ for $i = 1, \ldots, n$ are pairwise disjoint subsets of $B$. In the same way that one constructs a partition of unity subordinate to a given open cover, we can produce, for each $i = 1, \ldots, n$, a continuous function $h_i : X \to [0,1]$ with $h_i = 0$ on $X \setminus U_i$ so that $0 \leq \sum_{i=1}^n h_i \leq 1$ and $\sum_{i=1}^n h_i = 1$ on $A$. Set $v = \sum_{i=1}^n u_{s_i}(f h_i)^{1/2}$.

Denote by $\alpha$ the induced action of $G$ on $C(X)$, that is, $\alpha_s(f)(x) = f(s^{-1}x)$ for all $s \in G$, $f \in C(X)$, and $x \in X$. Since $\alpha_{s_i}(h_i^{1/2})\alpha_{s_j}(h_j^{1/2}) = 0$ for $i \neq j$ and $g$ dominates $\alpha_{s_i}(h_i^{1/2})$ for every $i$, we have

$$\begin{aligned} v^* g v &= \bigg( \sum_{i=1}^n (h_i f)^{1/2} u_{s_i}^* \bigg) g \bigg( \sum_{i=1}^n u_{s_i} (f h_i)^{1/2} \bigg) \\ &= \bigg( \sum_{i=1}^n u_{s_i}^* \alpha_{s_i}(f^{1/2}) \alpha_{s_i}(h_i^{1/2}) \bigg) g \bigg( \sum_{i=1}^n \alpha_{s_i}(h_i^{1/2}) \alpha_{s_i}(f^{1/2}) u_{s_i} \bigg) \\ &= \sum_{i=1}^n u_{s_i}^* \alpha_{s_i}(f h_i) u_{s_i} \\ &= \sum_{i=1}^n u_{s_i}^* \alpha_{s_i}(f h_i) u_{s_i} = \sum_{i=1}^n f h_i = f, \end{aligned}$$

as desired. □

**Theorem 12.4.** *Suppose that $G$ is infinite. Let $G \curvearrowright X$ be a free minimal action which is almost finite. Then $C(X) \rtimes_\lambda G$ is $\mathcal{Z}$-stable.*

*Proof.* As before we denote the induced action of $G$ on $C(X)$ by $\alpha$, that is, $\alpha_s(f)(x) = f(s^{-1}x)$ for all $s \in G$, $f \in C(X)$, and $x \in X$.

Let $n \in \mathbb{N}$. Let $\Upsilon$ be a finite subset of the unit ball of $C(X)$, $F$ a symmetric finite subset of $G$ containing $e$, and $\varepsilon > 0$. Let $a$ be a nonzero positive element of $C(X) \rtimes G$. We will show the existence of a map $\varphi : M_n \to C(X) \rtimes_\lambda G$ as in Theorem 12.1 where the finite set $\Omega$ there is taken to be $\Upsilon \cup \{u_s : s \in F\}$. Since $C(X) \rtimes_\lambda G$ is generated as a C*-algebra by the unit ball of $C(X)$ and the unitaries $u_s$ for $s \in G$, we will thereafter be able to conclude by Theorem 12.1 that $C(X) \rtimes_\lambda G$ is $\mathcal{Z}$-stable.

By Lemma 7.9 in [37] we may assume that $a \in C(X)$. Then we can find an $x_0 \in X$ and a $\theta > 0$ such that $a$ is strictly positive on the closed ball of radius $3\theta$ centred at $x_0$. We may therefore assume that $a$ is a $[0,1]$-valued function which takes value 1 on all points within distance $2\theta$ from $x_0$ and value 0 at all points at distance at least $3\theta$ from $x_0$. Write $O$ for the open ball of radius $\theta$ centred at $x_0$. Minimality implies that the sets $sO$ for $s \in G$ cover $X$, and so by compactness there is a finite set $D \subseteq G$ such that $D^{-1}O = X$.

Let $0 < \kappa < 1$, to be determined. Choose an integer $Q > n^2/\varepsilon$. Take a $\beta > 0$ which is small enough so that if $T$ is a nonempty finite subset of $G$ which is sufficiently invariant under left translation by $F^Q$ then for every set $T' \subseteq T$ with $|T'| \geq (1 - n\beta)|T|$ one has

$$\Bigg| \bigcap_{s \in F^Q} sT' \Bigg| \geq (1 - \kappa)|T|.$$

Choose an $L \in \mathbb{N}$ large enough so that $(1 - \beta/2)^L < \beta$. Since $G$ is amenable by the almost finiteness of the action, there exist finite subsets $e \in T_1 \subseteq T_2 \subseteq \cdots \subseteq T_L$ of $G$ such that $|\partial_{T_{l-1}} T_l| \leq (\beta/8)|T_l|$ for $l = 2, \dots, L$. By the previous paragraph, we may also assume that for each $l$ the set $T_l$ is sufficiently invariant under left translation by $F^Q$ so that

$$\Bigg| \bigcap_{s \in F^Q} sT \Bigg| \geq (1 - \kappa)|T_l| \tag{12}$$

for every $T \subseteq T_l$ satisfying $|T| \geq (1 - n\beta)|T_l|$.

By the uniform continuity of functions in $\Upsilon \cup \Upsilon^2$ and the uniform continuity of the transformations $x \mapsto tx$ of $X$ for $t \in T_L$, there is an $\eta > 0$ such that if $d(x, y) < \eta$ then $|f(tx) - f(ty)| < \varepsilon/(4n^2)$ for all $f \in \Upsilon \cup \Upsilon^2$ and $t \in T_L$. Let $\mathcal{U} = \{U_1, \dots, U_M\}$ be an open cover of $X$ whose members all have diameter less that $\eta$. Let $\eta' > 0$ be a Lebesgue number for $\mathcal{U}$ which is no larger than $\theta$.

Let $E$ be a finite subset of $G$ containing $T_K$ and let $\delta > 0$ be such that $\delta \leq \beta/4$. Since $G$ is infinite, we may enlarge $E$ and shrink $\delta$ as necessary so as to guarantee that the cardinality of every nonempty $(E, \delta)$-invariant finite set $S \subseteq G$ is large enough to satisfy

$$\Bigg( \sum_{l=1}^{L} |T_l| \Bigg) Mn \leq \beta |S|. \tag{13}$$

Since the action is almost finite, by Lemma 10.1 we can find

(i) nonempty open sets $V_1, \dots, V_K \subseteq X$ and nonempty $(E, \delta)$-invariant finite sets $S_1, \dots, S_K \subseteq G$ such that the family $\{(\overline{V_k}, S_k)\}_{k=1}^K$ is a castle with levels of diameter less than $\eta'$, and

(ii) sets $S_k' \subseteq S_k$ such that $|S_k'|/|S_k| < 1/(4|D|^2)$ and

$$X \setminus \bigsqcup_{k=1}^{K} S_k V_k \prec \bigsqcup_{k=1}^{K} S_k' V_k. \tag{14}$$

Let $k \in \{1, \dots, K\}$. Since $S_k$ is $(T_L, \beta/4)$-invariant, by Theorem 12.2 and our choice of the sets $T_1, \dots, T_L$ we can find $C_{k,1}, \dots, C_{k,L} \subseteq S_k$ such that the collection $\{T_l c : l = 1, \dots, L,\ c \in C_{k,l}\}$ is $\beta$-disjoint and $(1-\beta)$-covers $S_k$. By $\beta$-disjointness, for every $l = 1, \dots, L$ and $c \in C_{k,l}$

we can find a $T_{k,l,c} \subseteq T_l$ satisfying $|T_{k,l,c}| \geq (1-\beta)|T_l|$ so that the collection of sets $T_{k,l,c}c$ for $l = 1, \dots, L$ and $c \in C_{k,l}$ is disjoint.

Since $\eta'$ is a Lebesgue number for $\mathcal{U}$ and the levels of the tower $(V_k, S_k)$ have diameter less than $\eta'$, for each $l = 1, \dots, L$ there is a partition

$$C_{k,l} = C_{k,l,1} \sqcup C_{k,l,2} \sqcup \cdots \sqcup C_{k,l,M}$$

such that $cV_k \subseteq U_m$ for all $m = 1, \dots, M$ and $c \in C_{k,l,m}$. For each $l$ and $m$ choose pairwise disjoint subsets $C^{(1)}_{k,l,m}, \dots, C^{(n)}_{k,l,m}$ of $C_{k,l,m}$ such that each has cardinality $\lfloor |C_{k,l,m}|/n \rfloor$. For each $i = 2, \dots, n$ choose a bijection

$$\Lambda_{k,i} : \bigsqcup_{l,m} C^{(1)}_{k,l,m} \to \bigsqcup_{l,m} C^{(i)}_{k,l,m}$$

which sends $C^{(1)}_{k,l,m}$ to $C^{(i)}_{k,l,m}$ for all $l, m$. Also, define $\Lambda_{k,1}$ to be the identity map from $\bigsqcup_{l,m} C^{(1)}_{k,l,m}$ to itself, and write $\Lambda_{k,i,j}$ for the composition $\Lambda_{k,i} \circ \Lambda^{-1}_{k,j}$.

Now consider for each $j = 1, \dots, n$ and $c \in C^{(j)}_{k,l,m}$ the set $T'_{k,l,c} := \bigcap_{i=1}^n T_{k,l,\Lambda_{k,i,j}(c)}$, which satisfies

$$|T'_{k,l,c}| \geq (1 - n\beta)|T_l|. \tag{15}$$

since each $T_{k,l,\Lambda_{k,i}(c)}$ is a subset of $T_l$ with cardinality at least $(1-\beta)|T_l|$. Set

$$B_{k,l,c,Q} = \bigcap_{s \in F^Q} sT'_{k,l,c}$$

and for $q = 0, \dots, Q-1$ put

$$B_{k,l,c,q} = F^{Q-q} B_{k,l,c,Q} \setminus F^{Q-q-1} B_{k,l,c,Q}.$$

Then the sets $B_{k,l,c,0}, \dots, B_{k,l,c,Q}$ partition $F^Q B_{k,l,c,Q}$, which is a subset of $T'_{k,l,c}$. For $s \in F$ it is clear that

$$sB_{k,l,c,Q} \subseteq B_{k,l,c,Q-1} \cup B_{k,l,c,Q}, \tag{16}$$

while for $q = 1, \dots, Q-1$ we have

$$sB_{k,l,c,q} \subseteq B_{k,l,c,q-1} \cup B_{k,l,c,q} \cup B_{k,l,c,q+1}, \tag{17}$$

for if we are given a $t \in B_{k,l,c,q}$ then $st \in F^{Q-q+1} B_{k,l,c,Q}$, while if $st \in F^{Q-q-2} B_{k,l,c,Q}$ then $t \in F^{Q-q-1} B_{k,l,c,Q}$ since $F$ is symmetric, contradicting the membership of $t$ in $B_{k,l,c,q}$.

We view $C(X) \rtimes_\lambda G$ as being canonically included in the crossed product $B(X) \rtimes_\lambda G$ of the action induced by $G \curvearrowright X$ on the C*-algebra $B(X)$ of bounded Borel functions on $X$. Since the sets $s\overline{V_k}$ for $k = 1, \dots, K$ and $s \in S_k$ are closed and pairwise disjoint, for each $k$ we can find an open set $U_k \supseteq \overline{V_k}$ such that the sets $sU_k$ for $k = 1, \dots, K$ and $s \in S_k$ are pairwise disjoint. We define a linear map $\psi : M_n \to B(X) \rtimes_\lambda G$ by declaring it on the standard matrix units $\{e_{ij}\}_{i,j=1}^n$ of $M_n$ to be given by

$$\psi(e_{ij}) = \sum_{k=1}^K \sum_{l=1}^L \sum_{m=1}^M \sum_{c \in C^{(j)}_{k,l,m}} \sum_{t \in T'_{k,l,c}} u_{t\Lambda_{k,i,j}(c)c^{-1}t^{-1}} 1_{tcU_k}$$

and extending linearly.

For each $k = 1, \dots, K$ choose a continuous function $h_k : X \to [0,1]$ such that $h_k = 1$ on $\overline{V_k}$ and $h_k = 0$ on $X \setminus U_k$. Recalling that $\alpha$ denotes the induced action of $G$ on $C(X)$, for all $k$, $l$, and $m$, all $1 \leq i, j \leq n$, and all $c \in C^{(j)}_{k,l,m}$ we set

$$h_{k,l,c,i,j} = \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q}} \frac{q}{Q} u_{t\Lambda_{k,i,j}(c)c^{-1}t^{-1}} \alpha_{tc}(h_k).$$

Define a linear map $\varphi : M_n \to C(X) \rtimes_\lambda G$ by setting

$$\varphi(e_{ij}) = \sum_{k=1}^{K} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{c \in C^{(j)}_{k,l,m}} h_{k,l,c,i,j} \tag{18}$$

and extending linearly. Note that if we put

$$h = \sum_{k=1}^{K} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} \sum_{c \in C^{(i)}_{k,l,m}} h_{k,l,c,i,i}.$$

then $h$ is a continuous function taking values in $[0,1]$ which commutes with the image of $\psi$, and we have

$$\varphi(b) = h\psi(b)$$

for all $b \in M_n$, which shows that $\varphi$ is an order-zero c.p.c. map.

We now verify condition (ii) in Theorem 12.1 for the elements of the set $\{u_s : s \in F\}$. Let $1 \leq i, j \leq n$. For $s \in F$ we have

$$\begin{aligned} u_s h_{k,l,c,i,j} u_s^{-1} - h_{k,l,c,i,j} &= \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q}} \frac{q}{Q} u_{st\Lambda_{k,i,j}(c)c^{-1}(st)^{-1}} \alpha_{stc}(h_k) \\ &\qquad - \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q}} \frac{q}{Q} u_{t\Lambda_{k,i,j}(c)c^{-1}t^{-1}} \alpha_{tc}(h_k), \end{aligned}$$

and so in view of (16) and (17) we obtain

$$\| u_s h_{k,l,c,i,j} u_s^{-1} - h_{k,l,c,i,j} \| \leq \frac{1}{Q} < \frac{\varepsilon}{n^2}.$$

Since the element $a = u_s h_{k,l,c,i,j} u_s^{-1} - h_{k,l,c,i,j}$ satisfies $a^* a \leq 1_{F^Q B_{k,l,c,Q} c U_k}$ and $aa^* \leq 1_{F^Q B_{k,l,\Lambda(c),Q} c U_k}$ and the sets $F^Q B_{k,l,c,Q} c U_k$ are pairwise disjoint for all $k$, $l$, and $c$, this yields

$$\| u_s \varphi(e_{ij}) u_s^{-1} - \varphi(e_{ij}) \| = \max_{k,l,c} \| u_s h_{k,l,c,i,j} u_s^{-1} - h_{k,l,c,i,j} \| < \frac{\varepsilon}{n^2}$$

and hence, for every norm-one $b = (b_{ij}) \in M_n$,

$$\begin{aligned} \| [u_s, \varphi(b)] \| &= \| u_s \varphi(b) u_s^{-1} - \varphi(b) \| \\ &\leq \sum_{i,j=1}^{n} \| u_s \varphi(b_{ij}) u_s^{-1} - \varphi(b_{ij}) \| < n^2 \cdot \frac{\varepsilon}{n^2} = \varepsilon. \end{aligned}$$

Next we verify condition (ii) in Theorem 12.1 for the functions in $\Upsilon$. Let $1 \leq i, j \leq n$. Let $f \in \Upsilon \cup \Upsilon^2$. Let $1 \leq k \leq K$ and $1 \leq l \leq L$. Let $c \in C^{(j)}_{k,l,m}$. Since the elements $t\Lambda_{k,i,j}(c)$ for $t \in T'_{k,l,c}$ are distinct, we have

$$h^*_{k,l,c,i,j} f h_{k,l,c,i,j} = \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q}} \frac{q^2}{Q^2} \alpha_{tc\Lambda_{k,i,j}(c)^{-1}t^{-1}}(f) \alpha_{tc}(h_k^2) \tag{19}$$

and similarly

$$f h^*_{k,c,i,j} h_{k,c,i,j} = \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q}} \frac{q^2}{Q^2} f \alpha_{tc}(h_k^2) \tag{20}$$

Now let $x \in V_k$. Since $\Lambda_{k,i,j}(c)x$ and $cx$ both belong to $U_m$ by our definition of $C_{k,l,m}$, we have $d(\Lambda_{k,i,j}(c)x, cx) < \eta$. It follows that for every $t \in T_l$ we have

$$|f(t\Lambda_{k,i,j}(c)x) - f(tcx)| < \frac{\varepsilon}{4n^2},$$

in which case

$$\begin{aligned} \|\alpha_{tc\Lambda_{k,i,j}(c)^{-1}t^{-1}}(f) - f\| &= \|\alpha_{c^{-1}t^{-1}}(\alpha_{tc\Lambda_{k,i,j}(c)^{-1}t^{-1}}(f) - f)\| \\ &= \sup_{x \in V_k} |f(t\Lambda_{k,i,j}(c)x) - f(tcx)| \\ &< \frac{\varepsilon}{3n^2}. \end{aligned}$$

Using (19) and (20) this gives us

$$\begin{aligned} &\|h^*_{k,l,c,i,j} f h_{k,l,c,i,j} - f h^*_{k,l,c,i,j} h_{k,l,c,i,j}\| \\ &\qquad = \max_{q=1,\dots,Q} \max_{t \in B_{k,l,c,q}} \frac{q^2}{Q^2} \|(\alpha_{tc\Lambda_{k,i,j}(c)^{-1}t^{-1}}(f) - f)\alpha_{tc}(h_k^2)\| \\ &\qquad < \frac{\varepsilon}{3n^2}. \end{aligned} \tag{21}$$

Set $w = \varphi(e_{ij})$ for brevity. Let $f \in \Upsilon$. Since the functions $h_{k,l,c,i,j}$ for $1 \leq k \leq K$, $1 \leq l \leq L$, $1 \leq m \leq M$, and $c \in C^{(j)}_{k,l,m}$ have pairwise disjoint supports, we infer from (21) that

$$\|w^* g w - g w^* w\| < \frac{\varepsilon}{3n^2}$$

for $g$ equal to either $f$ or $f^2$. It follows that

$$\|w^* f^2 w - f w^* f w\| \leq \|w^* f^2 w - f^2 w^* w\| + \|f(f w^* w - w^* f w)\| < \frac{2\varepsilon}{3n^2}$$

and hence

$$\begin{aligned} \|fw - wf\|^2 &= \|(fw - wf)^*(fw - wf)\| \\ &= \|w^* f^2 w - f w^* f w + f w^* w f - w^* f w f\| \\ &\leq \|w^* f^2 w - f w^* f w\| + \|(f w^* w - w^* f w) f\| \\ &< \frac{2\varepsilon}{3n^2} + \frac{\varepsilon}{3n^2} = \frac{\varepsilon}{n^2}. \end{aligned}$$

Therefore for every norm-one $b = (b_{ij}) \in M_n$ we have

$$\|[f, \varphi(b)]\| \leq \sum_{i,j=1}^{n} \|[f, \varphi(b_{ij})]\| < n^2 \cdot \frac{\varepsilon}{n^2} = \varepsilon.$$

To complete the proof, let us now show that $1 - \varphi(1) \precsim a$. By enlarging $E$ and shrinking $\delta$ if necessary we may assume that the sets $S_1, \dots, S_K$ are sufficiently left invariant so that for every $k = 1, \dots, K$ there is an $R_k \subseteq S_k$ such that the set $\{s \in R_k : Ds \subseteq S_k\}$ has cardinality at least $|S_k|/2$. Let $1 \leq k \leq K$. Let $R'_k$ be a maximal subset of $R_k$ with the property that the sets $Ds$ for $s \in R'_k$ are pairwise disjoint. Observe that if $s, t \in R_k$ satisfy $Ds \cap Dt \neq \emptyset$ then $s \in D^{-1}Dt$, which shows that $|R'_k| \geq |R_k|/|D^{-1}D| \geq |S_k|/(2|D|^2)$. Since $D^{-1}O = X$, for each $s \in R'_k$ there is a $t \in D$ such that $tsV_k$ intersects $O$, which implies that the function $a$ takes the constant value 1 on $tsV_k$ since the diameter of the latter set is less than $\theta$ and $a$ takes value 1 at all points within distance $\theta$ of the set $O$. Therefore the set $S^\sharp_k$ of all $t \in S_k$ such that $a$ takes the constant value 1 on $tV_k$ has cardinality at least $|S_k|/(2|D|^2)$. Set

$$S''_k = \bigsqcup_{l=1}^{L} \bigsqcup_{m=1}^{M} \bigsqcup_{i=1}^{n} \bigsqcup_{c \in C^{(i)}_{k,l,m}} B_{k,l,c,Q}c.$$

Since $|B_{k,l,c,Q}| \geq (1 - \kappa)|T_l| \geq (1 - \kappa)|T_{k,l,m}|$ by (12) and (15), using (13) we obtain

$$\begin{aligned}
|S''_k| &\geq \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} \sum_{c \in C^{(i)}_{k,l,m}} |B_{k,l,c,Q}| \\
&\geq (1 - \kappa) \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} |T_{k,l,m}||C^{(i)}_{k,l,m}| \\
&\geq (1 - \kappa) \sum_{l=1}^{L} \sum_{m=1}^{M} |T_{k,l,m}|(|C_{k,l,m}| - n) \\
&\geq (1 - \kappa)\bigg( \bigg| \bigsqcup_{l=1}^{L} T_{k,l,m}C_{k,l} \bigg| - Mn \sum_{l=1}^{L} |T_l| \bigg) \\
&\geq (1 - \kappa)(1 - 2\beta)|S_k|,
\end{aligned}$$

and so if $\kappa$ and $\beta$ are small enough we have

$$|S_k \setminus S''_k| \leq \frac{|S_k|}{4|D|^2} \leq \frac{|S^\sharp_k|}{2}.$$

Choose an injection $f_k : S_k \setminus S''_k \to S^\sharp_k$. Now since for each $k$ the set $Q_k = S^\sharp_k \setminus f_k(S_k \setminus S''_k)$ satisfies

$$|Q_k| \geq |S^\sharp_k| - |S_k \setminus S''_k| \geq \frac{|S^\sharp_k|}{2} \geq \frac{|S_k|}{4|D|^2},$$

we deduce, in view of (14) and Remark 8.3, that

$$X \setminus \bigsqcup_{k=1}^{K} S_k V_k \prec \bigsqcup_{k=1}^{K} Q_k V_k.$$

We can thus find an open cover $\{U_1, \dots, U_r\}$ of $X \setminus \bigsqcup_{k=1}^{K} S_k V_k$ and $s_1, \dots, s_r \in G$ such that $s_1 U_1, \dots, s_r U_r$ are disjoint subsets of $\bigsqcup_{k=1}^{K} Q_k V_k$. Then the open sets $f_k(s) V_k = (f_k(s) s^{-1}) s V_k$ for $k = 1, \dots, K$ and $s \in S_k \setminus S_k''$ together with $s_1 U_1, \dots, s_r U_r$ form a disjoint collection whose union is contained in $\bigsqcup_{k=1}^{K} S_k^\sharp V_k$, which shows that

$$X \setminus \bigsqcup_{k=1}^{K} S_k'' V_k \prec \bigsqcup_{k=1}^{K} S_k^\sharp V_k.$$

Since the function $1 - \varphi(1)$ is supported on $X \setminus \bigsqcup_{k=1}^{K} S_k'' V_k$ and $a$ takes the constant value 1 on $\bigsqcup_{k=1}^{K} S_k^\sharp V_k$, it follows by Lemma 12.3 that there is a $v \in C(X) \rtimes_\lambda G$ satisfying $v^* a v = 1 - \varphi(1)$. This shows that $1 - \varphi(1) \precsim a$. $\square$

**Example 12.5.** In [13] examples were given of free minimal actions $\mathbb{Z} \curvearrowright X$ on compact metrizable spaces such that the crossed product $C(X) \rtimes_\lambda G$ fails to be $\mathcal{Z}$-stable. As mentioned in Example 4.7, these examples have finite tower dimension since they factor onto an odometer. By Theorem 12.4, they fail to be almost finite.

Using Theorem 12.4 we can give some new examples of classifiable crossed products, as we now demonstrate. Let us write $\mathscr{C}$ for the class of simple separable unital C*-algebras having finite nuclear dimension and satisfying the UCT. This class is classified by the Elliott invariant (ordered $K$-theory paired with traces) as a consequence of the work of Elliott–Gong–Lin–Niu [10], Gong–Lin–Niu [15], Tikuisis–White–Winter [48] in the stably finite case and of Kirchberg [23] and Phillips [36] in the purely infinite case. Moreover, the stable finite C*-algebras in the class $\mathscr{C}$ are ASH algebras of topological dimension at most 2. What is particularly novel in the examples below from the perspective of classification theory is that one can combine infinite asymptotic dimension in the group with positive topological entropy in the dynamics. For a general reference on entropy for actions of amenable groups see Chapter 9 of [22].

**Proposition 12.6.** *Suppose that $G$ is infinite, residually finite, and amenable. Let $r \in [0, \infty]$. Then there exists a uniquely ergodic free minimal action $G \curvearrowright X$ on the Cantor set which is almost finite and has topological entropy $r$, and the crossed product $C(X) \rtimes G$ belongs to the class $\mathscr{C}$.*

*Proof.* As $G$ is residually finite we can find a decreasing sequence $N_1 \supseteq N_2 \supseteq \dots$ of finite-index normal subgroups of $G$ such that $\bigcap_{k=1}^{\infty} N_k = \{e\}$. Then for each $k$ we have the surjective homomorphism $G/N_{k+1} \to G/N_k$ given on cosets by $sN_{k+1} \mapsto sN_k$. Form the inverse limit $Y$ of the sequence $G/N_1 \leftarrow G/N_2 \leftarrow \cdots$, which as a topological space is a Cantor set since $G$ is infinite. Then we have the free minimal action $G \curvearrowright Y$ arising from the actions $(s, tN_k) \mapsto stN_k$ of $G$ on each $G/N_k$. This is an example of a profinite action, which by definition is an inverse limit of actions on finite sets. It has a unique $G$-invariant Borel probability measure $\mu$, namely the one induced from the uniform probability measures on the quotients $G/N_k$. The p.m.p. action $G \curvearrowright (Y, \mu)$, being compact, has zero measure entropy. By [54] there is a Følner sequence $\{F_k\}$ for $G$ such that for each $k$ the set $F_k$ is a complete set of representatives for the quotient of $G$ by $N_k$, from which it follows that the action $G \curvearrowright Y$ is almost finite, since the clopen partition of $Y$ corresponding to $N_k$ is the set of levels of a single clopen tower with shape $F_k$.

By the Jewett–Krieger theorem [53, 41] there is a uniquely ergodic minimal action $G \curvearrowright Z$ on the Cantor set whose invariant measure $\kappa$ gives a Bernoulli action with entropy $r$. Then the p.m.p. actions $G \curvearrowright (Y, \mu)$ and $G \curvearrowright (Z, \nu)$ are disjoint, which implies that the action $G \curvearrowright Y \times Z$ is uniquely ergodic and minimal, as is readily seen. Moreover, the action $G \curvearrowright Y \times Z$ is free because the action $G \curvearrowright Y$ is free, has entropy $r$ by the additivity of topological entropy and the variational principle, and is almost finite by Theorem 11.6 since it factors onto the almost finite action $G \curvearrowright Y$ and the space $Z$ is zero-dimensional.

By Theorem 12.4, the crossed product $C(X) \rtimes_\lambda G$ is $\mathcal{Z}$-stable. Since the action is free, minimal, and uniquely ergodic, the crossed product also has a unique tracial state, given by composing the canonical conditional expectation $C(X) \rtimes_\lambda G \to C(X)$ with integration with respect to the unique $G$-invariant Borel probability measure on $X$. It follows by [43] that $C(X) \rtimes_\lambda G$ has finite nuclear dimension. Since $C(X) \rtimes_\lambda G$ satisfies the UCT [51], we conclude that $C(X) \rtimes_\lambda G$ belongs to the class $\mathscr{C}$. □

We note finally that profinite actions of countably infinite residually finite amenable groups, such as the action $G \curvearrowright Y$ in the proof of Proposition 12.6, are already known to be classifiable (see Remark 5.3 in [47]).

## 13. The type semigroup and almost unperforation

Let $G \curvearrowright X$ be an action on a zero-dimensional compact Hausdorff space. Write $\alpha$ for the induced action on $C(X)$, that is, $\alpha_s(f)(x) = f(s^{-1}x)$ for all $s \in G$, $f \in C(X)$, and $x \in X$. On the space $C(X, \mathbb{Z}_{\geq 0})$ of continuous functions on $X$ with values in the set $\mathbb{Z}_{\geq 0}$ of nonnegative integers we define an equivalence relation by declaring that $f \sim g$ if there are $h_1, \dots, h_n \in C(X, \mathbb{Z}_{\geq 0})$ and $s_1, \dots, s_n \in G$ such that $\sum_{i=1}^n h_i = f$ and $\sum_{i=1}^n \alpha_{s_i}(h_i) = g$ (transitivity is not immediately obvious but is readily checked). Write $S(X, G)$ for the quotient $C(X, \mathbb{Z}_{\geq 0})/\sim$. This is an Abelian semigroup under the operation $[f] + [g] = [f + g]$, which is easily seen to be well defined. We moreover endow $S(X, G)$ with the algebraic order, that is, for $a, b \in S(X, G)$ we declare that $a \leq b$ whenever there exists a $c \in S(X, G)$ such that $a + c = b$. The ordered Abelian semigroup $S(X, G)$ is called the *(clopen) type semigroup* of the action. Note that, in view of Proposition 3.5, comparison can be expressed in this language by saying that, for all clopen sets $A, B \subseteq X$, if $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$ then $[1_A] \leq [1_B]$.

One can equivalently define $S(X, G)$ by considering the collection of clopen subsets of $X \times \mathbb{N}$ of the form $\bigsqcup_{i=1}^n A_i \times \{i\}$ for some $n \in \mathbb{N}$ (the *bounded* subsets of $X \times \mathbb{N}$) and quotienting by the relation of equidecomposability, under which $\bigsqcup_{i=1}^n A_i \times \{i\} \sim \bigsqcup_{i=1}^m B_i \times \{i\}$ if for each $i = 1, \dots, n$ there exist a clopen partition $\{A_{i,1}, \dots, A_{i,J_i}\}$ of $A_i$, $s_{i,1}, \dots, s_{i,J_i} \in G$, and $k_{i,1}, \dots, k_{i,J_i} \in \{1, \dots, m\}$ such that

$$\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{J_i} s_{i,j} A_{i,j} \times \{k_{i,j}\} = \bigsqcup_{i=1}^{m} B_i \times \{i\}.$$

The semigroup operation is the concatenation

$$\left[\bigsqcup_{i=1}^{n} A_i \times \{i\}\right] + \left[\bigsqcup_{i=1}^{m} B_i \times \{i\}\right] = \left[\left(\bigsqcup_{i=1}^{n} A_i \times \{i\}\right) \sqcup \left(\bigsqcup_{i=n+1}^{n+m} B_i \times \{i\}\right)\right].$$

The isomorphism with the first construction can be seen by considering for each function $f \in C(X, \mathbb{Z}_{\geq 0})$ the decomposition $f = \sum_{i=1}^{n} 1_{A_i}$ where $A_i = \{x \in X : f(x) \geq i\}$ and $n = \max_{x \in X} f(x)$ and associating to $f$ the subset $\bigsqcup_{i=1}^{n} A_i \times \{i\}$ of $X \times \{1, \dots, n\}$.

The idea of a type semigroup originates in Tarski's work on amenability (see [52]) and has variants depending on the type of action (e.g., on an ordinary set, on a measure space, or on a zero-dimensional compact space) and the types of sets use in the definition equidecomposability (e.g., arbitrary, measurable, or clopen). The clopen version was studied in [40].

Note that every measure $\mu$ in $M_G(X)$ induces a state on $S(X, G)$ given by $[f] \mapsto \mu(f)$. When the action is minimal, this gives a bijection from measures in $M_G(X)$ to states on $S(X, G)$ by the proof of Lemma 5.1 in [40].

An ordered Abelian semigroup $A$ is said to be *almost unperforated* if, for all $a, b \in A$ and $n \in \mathbb{N}$, the inequality $(n+1)a \leq nb$ implies $a \leq b$.

**Lemma 13.1.** *Let $G \curvearrowright X$ be a free minimal action on the Cantor set such that $S(X, G)$ is almost unperforated. Then the action has comparison.*

*Proof.* Let $A$ and $B$ be clopen subsets of $X$ such that $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$. By Lemma 5.1 of [40], every state $\sigma$ on $S(X, G)$ gives rise to a measure $\mu$ in $M_G(X)$ satisfying $\mu(A) = \sigma([1_A])$ for every clopen set $A \subseteq X$. Thus $\sigma([1_A]) < \sigma([1_B])$ for every state $\sigma$ on $S(X, G)$. Since the action is minimal, $X$ is covered by finitely many translates of $B$, so that $[1_A] \leq m[1_B]$ for some $m \in \mathbb{N}$. It follows by Proposition 2.1 of [35] that there exists an $n \in \mathbb{N}$ for which $(n+1)[1_A] \leq n[1_B]$. Almost unperforation then yields $[1_A] \leq [1_B]$, establishing comparison. ☐

The following is a generalization of Theorem 9.2(i)⇒(ii). It shows that, for free minimal actions on the Cantor set, almost finiteness implies a stable version of comparison.

**Lemma 13.2.** *Let $G \curvearrowright X$ be a free minimal action on the Cantor set which is almost finite. Let $f, g \in C(X, \mathbb{Z}_{\geq 0})$ be such that $\mu(f) < \mu(g)$ for every $\mu \in M_G(X)$. Then $[f] \leq [g]$.*

*Proof.* Write $f = \sum_{j=1}^{n} 1_{A_j}$ and $g = \sum_{k=1}^{m} 1_{B_k}$ where $A_j = \{x \in X : f(x) \geq j\}$ and $B_j = \{x \in X : g(x) \geq k\}$, with $n = \max_{x \in X} f(x)$ and $m = \max_{x \in X} g(x)$.

Let $K$ be a finite subset of $G$ and $\delta > 0$, both to be determined. By Theorem 10.2, there is a clopen castle $\{(V_i, S_i)\}_{i \in I}$ with $(K, \delta)$-invariant shapes such that $\bigsqcup_{i \in I} S_i V_i = X$. By replacing each tower $(V_i, S_i)$ with finitely many thinner towers with the same shape $S_i$ and with bases forming a clopen partition of $V_i$, we may assume that every level of every tower in the castle is either contained in or disjoint from $A_j$ for every $j = 1, \dots, n$, and also either contained in or disjoint from $B_k$ for every $k = 1, \dots, m$. For each $i$ set $E_{i,j} = \{s \in S_i : sV_i \subseteq A_j\}$ for every $j$ and $F_{i,k} = \{s \in S_i : sV_i \subseteq B_k\}$ for every $k$. An argument by contradiction using a weak* cluster point as in the proof of Theorem 7.2 shows that we can choose $K$ and $\delta$ so that for every $i$ we have

$$\frac{1}{|S_i|}\sum_{j=1}^{n}|E_{i,j}| \leq \frac{1}{|S_i|}\sum_{k=1}^{m}|F_{i,k}|,$$

in which case we can find a bijection $\bigsqcup_{j=1}^{n} E_{i,j}\times\{j\} \to \bigsqcup_{k=1}^{m} F_{i,k}\times\{k\}$, which we write as $(s,j)\mapsto(t_{i,s,j},k_{i,s,j})$. We then have

$$f = \sum_{i\in I}\sum_{j=1}^{n}\sum_{s\in E_{i,j}} 1_{sV_i}$$

and

$$\sum_{i\in I}\sum_{j=1}^{n}\sum_{s\in E_{i,j}} \alpha_{t_{i,s,j}s^{-1}}(1_{sV_i}) = \sum_{i\in I}\sum_{j=1}^{n}\sum_{s\in E_{i,j}} 1_{t_{i,s,j}V_i} \leq g,$$

showing that $[f]\leq[g]$, as desired. ☐

The following adds almost unperforation to the conditions in Theorem 9.2 in the case that the space is the Cantor set.

**Theorem 13.3.** *Let $G \curvearrowright X$ be a free minimal action on the Cantor set and consider the following conditions:*

*(i) the action is almost finite,*
*(ii) $S(X,G)$ is almost unperforated,*
*(iii) the action has comparison,*
*(iv) the action has $m$-comparison for all $m\in\mathbb{N}$,*
*(v) the action has $m$-comparison for some $m\in\mathbb{N}$.*

*Then (i)⇒(ii)⇒(iii)⇒(iv)⇒(v), and if $E_G(X)$ is finite then all five conditions are equivalent.*

*Proof.* (i)⇒(ii). Let $f,g\in C(X,\mathbb{Z}_{\geq 0})$ be such that $(n+1)[f]<n[g]$ for some $n\in\mathbb{N}$. Then for every $\mu\in M_G(X)$ we have $(n+1)\mu(f)<n\mu(g)$ and hence $\mu(f)<\mu(g)$. It follows by Lemma 13.2 that $[f]\leq[g]$, establishing almost unperforation.

(ii)⇒(iii). This is Lemma 13.1.

(iii)⇒(iv)⇒(v). Trivial.

(v)⇒(i). This is a special case of Theorem 9.2(iv)⇒(i). ☐

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