BERNOULLI ACTIONS OF SOFIC GROUPS HAVE COMPLETELY POSITIVE ENTROPY

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Abstract. We prove that every Bernoulli action of a sofic group has completely positive entropy with respect to every sofic approximation net. We also prove that every Bernoulli action of a finitely generated free group has the property that each of its nontrivial factors with a finite generating partition has positive f-invariant.

1. Introduction

A probability-measure-preserving action $G \curvearrowright (X, \mu)$ of a countable amenable group is said to have completely positive entropy if each of its nontrivial factors has positive entropy in the Kolmogorov-Sinai sense. This can also be expressed by saying that every nontrivial finite partition of X has positive entropy with respect to the action. In the case of a single transformation, completely positive entropy is equivalent to being a K-automorphism by the Rokhlin-Sinai theorem, and these conditions are moreover equivalent to uniform mixing (see Section 18.2 of [6]). Rudolph and Weiss showed in [19] that actions of general countable amenable groups with completely positive entropy also possess strong mixing properties.

Bernoulli transformations are the prototypes for completely positive entropy, and it was an open question for some time whether these were the only examples until Ornstein constructed a non-Bernoulli K-automorphism in [13]. Extending the celebrated work of Ornstein in the single transformation case, Ornstein and Weiss proved that entropy is a complete invariant for Bernoulli actions of countably infinite amenable groups and that every factor of such an action is Bernoulli [14]. In particular, Bernoulli actions of countable amenable groups have completely positive entropy.

Bowen showed in [2] that the theory of dynamical measure entropy can be extended beyond the realm of amenability to the context of acting groups possessing the much weaker finite approximation property of soficity. In this case the entropy is defined with respect to a sofic approximation sequence for the group and depends in general on the choice of this sequence. In [10] Li and the author developed an operator-algebraic approach to sofic entropy that removes the assumption in Bowen's definition of a generating partition with finite Shannon entropy, and this was recast in [8] in the language of finite partitions. In accord with the amenable case, the entropy of a Bernoulli action of a countable sofic group is equal to the Shannon entropy of the base, independently of the sofic approximation sequence [2, 9]. Using this fact Bowen was able to extend the Ornstein-Weiss classification to a large class of acting groups that includes all nontorsion countable sofic groups.

It is not known whether any nontrivial Bernoulli action of a nonamenable group has the property that each of its factors is Bernoulli. On the other hand, it is a striking consequence

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of the cohomology computations in Popa's deformation-rigidity theory that many countable nonamenable groups have Bernoulli actions admitting non-Bernoulli factors [15, 18, 16, 17]. This includes infinite groups with property (T) and countable groups of the form $G \times H$ where G is infinite and H is nonamenable. Another contrast with the amenable setting is that, by a result of Ball [1], every finitely generated nonamenable group has Bernoulli actions with finite base which factor onto every Bernoulli action. This phenomenon cannot occur for amenable groups because Kolmogorov-Sinai entropy is nonincreasing under taking factors. Furthermore, Bowen showed in [5] that if G is a countable group containing the free group F_2 then every nontrivial Bernoulli action of G factors onto every other Bernoulli action of G.

In this note we prove that every Bernoulli action of a sofic group has completely positive entropy with respect to every sofic approximation net (Theorem 2.6). As a consequence, those nonamenable sofic groups that fall within the scope of the cohomology results of [15, 18, 16, 17] (e.g., $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$) admit non-Bernoulli actions with completely positive entropy, as happens in the amenable case but for completely differently reasons.

What we in fact demonstrate is that there is a positive lower bound on the local entropy of a nontrivial finite partition which is uniform over all good enough sofic approximations (Lemma 2.5). This additionally enables us to show that a Bernoulli action of a free group has the property that each of its nontrivial factors with a finite generating partition has positive f-invariant (Theorem 3.2), answering a question raised by Bowen in Section 1.2 of [3]. The f-invariant, introduced by Bowen in [4], is an entropy-type invariant for probability-measure-preserving actions of free groups which admit a generating partition with finite Shannon entropy. Unlike sofic entropy, it is defined using Shannon entropy, but it is nevertheless related to sofic entropy via a formula that involves local averaging over permutation models of the group [3]. This formula was used by Bowen to obtain a weaker version of Theorem 3.2 with "nonnegative" in place of "positive" (Corollary 1.8 of [3]). To establish Theorem 3.2 we use the formula in the same way, along with an asymptotic freeness result for random permutation models of a free group (Lemma 3.1) that follows from work of Nica [12] by a standard measure concentration argument. We remark that this asymptotic freeness is also necessary to derive Corollary 1.8 in [3], although it is not mentioned there.

Section 2 contains a review of the definition of sofic measure entropy from [8] along with the main technical result of the paper, Lemma 2.5, and its consequence in the sofic case, Theorem 2.6. In Section 3 we concentrate on free groups and the f-invariant and prove Theorem 3.2.

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2. Sofic measure entropy

Let G be a discrete group. We write its identity element as e. For a map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ and a finite set $K \subseteq G$ write $\mathcal{V}(\sigma, K)$ for the set of all $v \in \{1, \dots, d\}$ such that

(i) $\sigma_{st}(v) = \sigma_s \sigma_t(v)$ for all $s, t \in F$, and
(ii) $\sigma_s(v) \neq \sigma_t(v)$ for all distinct $s, t \in F$.

A net $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}$ is called a *sofic approximation net* if $\lim_i d_i = \infty$ and $\lim_i |\mathcal{V}(\sigma_i, F)|/d_i = 1$ for all finite sets $F \subseteq G$. The group G is *sofic* if it admits a sofic approximation net. We say that a map $\sigma : G \to \mathrm{Sym}(d)$ is a sufficiently good sofic approximation for a given purpose if there are a finite set $K \subseteq G$ and $\varepsilon > 0$ such that σ satisfies $|\mathcal{V}(\sigma, K)| \geq 1-\varepsilon$ and this condition is enough for σ to perform what is required of it.

For the remainder of this section G is a sofic group and $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}$ a fixed but arbitrary sofic approximation net.

We next recall the definition of sofic measure entropy for a measure-preserving action $G \curvearrowright (X, \mathcal{B}, \mu)$ on a probability space, as formulated in [8]. When speaking about partitions of a measure space we tacitly assume that they are measurable. For a set V we write $\mathcal{P}(V)$ for the power set of V, viewed as an algebra. In the case that $V = \{1, \dots, d\}$ for some $d \in \mathbb{N}$ we simply write $\mathcal{P}_d$. For a partition ξ of X we write $\mathcal{A}(\xi)$ for the algebra generated by ξ.

Let α be a finite partition of X, F a finite subset of G, and $\delta > 0$. Write α_F for the partition $\{\bigcap_{s\in F} sA_s : A \in \alpha^F\}$ where A_s denotes the value of A at s. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. We write $\mathrm{Hom}_\mu(\alpha, F, \delta, \sigma)$ for the set of all homomorphisms $\varphi : \mathcal{A}(\alpha_F) \to \mathcal{P}_d$ such that

- (i) $\sum_{A\in\alpha} |\sigma_s\varphi(A)\Delta\varphi(sA)|/d < \delta$ for all $s \in F$, and
- (ii) $\sum_{A\in\alpha_F} \big||\varphi(A)|/d - \mu(A)\big| < \delta$.

For a partition $\xi \leq \alpha$ we write $|\mathrm{Hom}_\mu(\alpha, F, \delta, \sigma)|_\xi$ to mean the cardinality of the set of restrictions of elements of $\mathrm{Hom}_\mu(\alpha, F, \delta, \sigma)$ to ξ.

Let $\mathcal{S}$ be a subalgebra of the σ-algebra $\mathcal{B}$. Let ξ and α be finite measurable partitions of X with $\alpha \geq \xi$. Let F be a nonempty finite subset of G and $\delta > 0$. Set

$$\begin{aligned} h^\xi_{\Sigma,\mu}(\alpha, F, \delta) &= \limsup_i \frac{1}{d_i} \log |\mathrm{Hom}_\mu(\alpha, F, \delta, \sigma_i)|_\xi, \\ h^\xi_{\Sigma,\mu}(\alpha, F) &= \inf_{\delta>0} h^\xi_{\Sigma,\mu}(\alpha, F, \delta), \\ h^\xi_{\Sigma,\mu}(\alpha) &= \inf_F h^\xi_{\Sigma,\mu}(\alpha, F), \\ h^\xi_{\Sigma,\mu}(\mathcal{S}) &= \inf_\alpha h^\xi_{\Sigma,\mu}(\alpha), \\ h_{\Sigma,\mu}(\mathcal{S}) &= \sup_\xi h^\xi_{\Sigma,\mu}(\mathcal{S}) \end{aligned}$$

where the infimum in the third line is over all nonempty finite subsets of G, the infimum in the fourth line is over all finite partitions $\alpha \subseteq \mathcal{S}$ which refine ξ, and the supremum in the last line is over all finite partitions in $\mathcal{S}$. In the case that $\mathrm{Hom}_\mu(\alpha, F, \delta, \sigma_i)$ is empty for all i greater than some i_0, we put $h^\xi_{\Sigma,\mu}(\alpha, F, \delta) = -\infty$.

Definition 2.1. The measure entropy $h_{\Sigma,\mu}(X, G)$ of the action $G \curvearrowright (X, \mathcal{B}, \mu)$ with respect to Σ is defined to be $h_{\Sigma,\mu}(\mathcal{B})$.

Note that if $\mathcal{S}$ is a generating subalgebra of $\mathcal{B}$ then $h_{\Sigma,\mu}(X, G) = h_{\Sigma,\mu}(\mathcal{S})$ by Theorem 2.6 of [8], although we will not need this fact.

Definition 2.2. The action $G \curvearrowright (X, \mu)$ is said to have *completely positive entropy with respect to* Σ if each of its nontrivial factors $G \curvearrowright (Y, \nu)$ satisfies $h_{\Sigma,\nu}(Y, G) > 0$ (by nontrivial we mean that ν does not have an atom of full measure).

Lemma 2.3. *Let (X,μ) be a probability space. Let $\varepsilon > 0$. Then there is a $\beta > 0$ and an $M \in \mathbb{N}$ such that if $\xi = \{A_1, A_2\}$ and $\eta = \{B_1, B_2\}$ are two-element ordered partitions of X, V is a finite set of cardinality at least M, and $\psi : \mathcal{A}(\xi) \to \mathcal{P}(V)$ is a homomorphism, then the set of all restrictions to η of homomorphisms $\varphi : \mathcal{A}(\xi \vee \eta) \to \mathcal{P}(V)$ which restrict to ψ on ξ and satisfy*

$$\max(|\varphi(A_1 \cap B_2)|, |\varphi(A_2 \cap B_1))|) < \beta|V|$$

has cardinality at most $e^{\varepsilon|V|}$.

Proof. Let $\beta > 0$, to be determined. Let $\xi = \{A_1, A_2\}$ and $\eta = \{B_1, B_2\}$ be two-element ordered partitions of X, V a finite set, and $\psi : \mathcal{A}(\xi) \to \mathcal{P}(V)$ a homomorphism. Let Θ be the set of all restrictions to η of homomorphisms $\varphi : \mathcal{A}(\xi \vee \eta) \to \mathcal{P}(V)$ which restrict to ψ on ξ and satisfy

$$\max(|\varphi(A_1 \cap B_2)|, |\varphi(A_2 \cap B_1)|) < \beta|V|.$$

Such a restriction to η is determined by our knowledge of $\varphi(A_1 \cap B_2)$ and $\varphi(A_2 \cap B_1)$, for

$$\begin{aligned} \varphi(B_1) &= (\psi(A_1) \cup \varphi(A_2 \cap B_1)) \setminus \varphi(A_1 \cap B_2), \\ \varphi(B_2) &= (\psi(A_2) \cup \varphi(A_1 \cap B_2)) \setminus \varphi(A_2 \cap B_1). \end{aligned}$$

Therefore $|\Theta|$ is bounded above by $2 \cdot \sum_{k=0}^{\lfloor \beta|V| \rfloor} \binom{|V|}{k}$. It follows by Stirling's formula that if β is sufficiently small as a function of ε then there is an $M \in \mathbb{N}$ such that $|\Theta| \leq e^{\varepsilon|V|}$ whenever $|V| \geq M$. □

Lemma 2.4. *Let $G \curvearrowright (X,\mu)$ be a measure-preserving action on a probability space. Let α be a finite partition of X. Let E and F be nonempty finite subsets of G with $e \in F$, and let $\delta > 0$. Then for every good enough sofic approximation $\sigma : G \to \mathrm{Sym}(d)$ one has*

$$\mathrm{Hom}_\mu(\alpha, FE, \delta, \sigma) \subseteq \mathrm{Hom}_\mu(\alpha_E, F, 3|E||\alpha|^{|E|}\delta, \sigma).$$

Proof. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Let $\varphi \in \mathrm{Hom}_\mu(\alpha, FE, \delta, \sigma)$. Then for all $A \in \alpha$, $s \in E$, and $t \in F$ we have, assuming that σ is a good enough sofic approximation and noting that $e \in F$ implies $s \in EF$,

$$\begin{aligned} |\sigma_t\varphi(sA)\Delta\varphi(tsA)| &\leq |\sigma_t\varphi(sA)\Delta\sigma_t\sigma_s\varphi(A)| + |\sigma_t\sigma_s\varphi(A)\Delta\sigma_{ts}\varphi(A)| + |\sigma_{ts}\varphi(A)\Delta\varphi(tsA)| \\ &< \delta d + \delta d + \delta d = 3\delta d. \end{aligned}$$

Writing $\alpha = \{A_1, \dots, A_n\}$ we thus have, for every $t \in F$ and $\omega : E \to \{1, \dots, n\}$,

$$\begin{aligned} \bigg|\sigma_t\varphi\bigg(\bigcap_{s\in E} sA_{\omega(s)}\bigg)\Delta\varphi\bigg(t\bigcap_{s\in E} sA_{\omega(s)}\bigg)\bigg| &= \bigg|\bigcap_{s\in E}\sigma_t\varphi(sA_{\omega(s)})\Delta\bigcap_{s\in E}\varphi(tsA_{\omega(s)})\bigg| \\ &\leq \sum_{s\in E}|\sigma_t\varphi(sA_{\omega(s)})\Delta\varphi(tsA_{\omega(s)})| \\ &< 3|E|\delta d \end{aligned}$$

and hence $\sum_{A\in\alpha_E}|\sigma_t\varphi(A)\Delta\varphi(tA)| < 3|E||\alpha|^{|E|}\delta d$.

Finally, since $(\alpha_E)_F = \alpha_{FE}$ we have $\sum_{A\in(\alpha_E)_F}\big||\varphi(A)|/d - \mu(A)\big| < \delta$, and so we conclude that $\varphi \in \mathrm{Hom}_\mu(\alpha_E, F, 3|E||\alpha|^{|E|}\delta, \sigma)$, yielding the inclusion of the lemma statement. □

The fact that the exponential lower bound in the following lemma is uniform over all sufficiently good sofic approximations is not necessary for establishing the completely positive entropy of Bernoulli actions in Theorem 2.6, but it will be crucial for deriving the analogous conclusion for the f-invariant in Theorem 3.2. In the proof we will use the following terminology and notation. For $d \in \mathbb{N}$ and a subset V of $\{1, \dots, d\}$ write π_V for the map $\mathcal{A}(\{1, \dots, d\}) \to \mathcal{P}(V)$ given by $W \mapsto W \cap V$. Given a $Q \subseteq \mathrm{Sym}(d)$, we say that a set $V \subseteq \{1, \dots, d\}$ is *Q-separated* if $Qv \cap Qw = \emptyset$ for all distinct $v, w \in V$.

Lemma 2.5. *Let $G \curvearrowright (X, \mu) = (X_0, \mu_0)^G$ be a Bernoulli action. Let ξ be a finite partition of X such that $H(\xi) > 0$. Then there is a $\lambda > 0$, a finite set $Q \subseteq G$, a $\theta > 0$, and a $d_0 \in \mathbb{N}$ such that*

$$|\mathrm{Hom}_\mu(\rho, F, \delta, \sigma)|_\xi \geq e^{\lambda d}$$

for all finite partitions ρ of X refining ξ, finite sets $F \subseteq G$, $\delta > 0$, and maps $\sigma : G \to \mathrm{Sym}(d)$ for which $d \geq d_0$ and $|\mathcal{V}(\sigma, Q)| \geq 1 - \theta$.

Proof. Let $\sigma : G \to \mathrm{Sym}(d)$ be a map for some $d \in \mathbb{N}$. Since the Shannon entropy $H(\cdot)$ is subadditive with respect to taking joins and ξ is a join of two-element partitions, there is a two-element partition ξ' of X such that $\xi' \leq \xi$ and $H(\xi') > 0$. Then $|\mathrm{Hom}_\mu(\rho, F, \delta, \sigma)|_\xi \geq |\mathrm{Hom}_\mu(\rho, F, \delta, \sigma)|_{\xi'}$ for all finite partitions ρ of X refining ξ, finite sets $F \subseteq G$, and $\delta > 0$, and so we may assume that ξ itself is a two-element partition.

Write $\xi = \{B_1, B_2\}$. Take an $\varepsilon > 0$ such that

$$H(\xi) - 2\varepsilon > \frac{H(\xi)}{2}.$$

Let $\beta > 0$ and $M \in \mathbb{N}$ be as given by Lemma 2.3 with respect to ε. Take two-element ordered partition $\eta = \{C_1, C_2\}$ such that C_1 and C_2 are finite unions of cylinder sets over some finite set $K \subseteq G$ and $\max(\mu(B_1 \cap C_2), \mu(B_2 \cap C_1))$ is less than $\beta/4$ and also small enough to ensure that

$$H(\eta) - 2\varepsilon \geq \frac{H(\eta)}{2}.$$

Let $\rho = \{D_1, \dots, D_m\}$ be a partition refining ξ and F a finite subset of G containing e, and let $\delta > 0$ be such that $\delta \leq \beta/|K|^2$. Note that by the monotonicity properties of entropy it equivalent to quantify over such ρ, F, and δ in the statement of theorem.

Let $\delta' > 0$, to be further specified. Take a finite partition $\alpha = \{A_1, \dots, A_q\}$ consisting cylinder sets over e and a finite set $E \subseteq G$ containing K such that α_E refines η and for every $i = 1, \dots, m$ there exists a set $D'_i \in \mathcal{A}(\alpha_E)$ for which $\mu(D_i \Delta D'_i) < \delta'$. By a simple perturbation argument we may assume that the sets $D'_1, \dots, D'_m$ form a partition of X. Let $\sigma : G \to \mathrm{Sym}(d)$ be a map for some $d \in \mathbb{N}$. We assume that d is sufficiently large and that σ is a sufficiently good sofic approximation for purposes to be described below. Since the sets $D'_1, \dots, D'_m$ partition X we can define a homomorphism $\theta : \mathcal{A}(\rho_F) \to \mathcal{A}(\alpha_{FE})$ by setting

$$\theta\bigg(\bigcap_{s \in F} sD_{f(s)}\bigg) = \bigcap_{s \in F} sD'_{f(s)}$$

for all $f \in \{1, \dots, m\}^F$ for which $\bigcap_{s \in F} sD_{f(s)} \neq \emptyset$ and redefining $\theta(\bigcap_{s \in F} sD_{f_0(s)})$ for some $f_0 \in \{1, \dots, m\}^F$ for which $\bigcap_{s \in F} sD_{f_0(s)} \neq \emptyset$ so that θ maps X to $\{1, \dots, d\}$. It is then straightforward to check that if σ is a sufficiently good sofic approximation and δ' is small enough then composing an element of $\mathrm{Hom}_\mu(\alpha_E, F, \delta/2, \sigma)$ with θ yields an element of $\mathrm{Hom}_\mu(\rho, F, \delta, \sigma)$. Setting $B_1' = \theta(B_1)$ and $B_2' = \theta(B_2)$, we may moreover take δ' to be small enough to ensure that $\max(\mu(B_1'\Delta B_1), \mu(B_2'\Delta B_2)) < \beta/4$, so that

$$\begin{aligned}\max(\mu(B_1' \cap C_2), \mu(B_2' \cap C_1)) &\leq \max(\mu(B_1'\Delta B_1), \mu(B_2'\Delta B_2)) \\ &\qquad + \max(\mu(B_1 \cap C_2), \mu(B_2 \cap C_1)) \\ &< \frac{\beta}{2}.\end{aligned}$$

Note that α_{FE} refines η, since $e \in F$ and $K \subseteq E$. For every $\gamma \in \{1, \dots, q\}^d$, regarding γ as a map $\{1, \dots, d\} \to \{1, \dots, q\}$ we define a homomorphism $\psi_\gamma : \mathcal{A}(\alpha_{FE}) \to \mathcal{P}_d$ by setting

$$\psi_\gamma\bigg(\bigcap_{s\in FE} sA_{f(s)}\bigg) = \bigcap_{s\in FE} \sigma_s \gamma^{-1}(f(s))$$

for all $f \in \{1, \dots, q\}^{FE}$. View $\{1, \dots, q\}^d$ as a probability space with probability measure ν^d where $\nu(\{k\}) = \mu(A_k)$ for each $k = 1, \dots, q$. By the second moment argument in Section 8 of [2] (see also Section 4 of [8], where the argument is recast using the homomorphism definition of entropy), if we assume that d is sufficiently large and that σ is a sufficiently good sofic approximation then there exists a set $L \subseteq \{1, \dots, q\}^d$ such that $\nu^d(L) \geq 3/4$ and $\psi_\gamma \in \mathrm{Hom}_\mu(\alpha, FE, \delta/(6|E||\alpha|^{|E|}), \sigma)$ for every $\gamma \in L$. By Lemma 2.4, if we assume σ to be a sufficiently good sofic approximation then we will have $\mathrm{Hom}_\mu(\alpha, FE, \delta/(6|E||\alpha|^{|E|}), \sigma) \subseteq \mathrm{Hom}_\mu(\alpha_E, F, \delta/2, \sigma)$, so that $\psi_\gamma \in \mathrm{Hom}_\mu(\alpha_E, F, \delta/2, \sigma)$ for all $\gamma \in L$.

We next suppose that σ is a sufficiently good sofic approximation so that the set

$$R = \{v \in \{1, \dots, d\} : \sigma_s^{-1}(v) \neq \sigma_t^{-1}(v) \text{ for all distinct } s, t \in FE\}$$

has cardinality at least $(1 - 1/(2|K|^2))d$. Take a maximal $\sigma(K)^{-1}$-separated set $V \subseteq \{1, \dots, d\}$. Then $|V| \geq |\sigma(K)|^{-2}d$, since $\bigcup_{v\in V} \sigma(K)\sigma(K)^{-1}v = \{1, \dots, d\}$ by maximality. Hence $|V| \geq d/|K|^2$. Set $W = V \cap R$. Then

$$|W| \geq |V| - \frac{d}{2|K|^2} \geq \frac{|V|}{2}.$$

We will now once again apply the second moment argument from Section 8 of [2] but this time relativized to the set W. Let $f \in \{1, \dots, q\}^{FE}$. Set $P_f = \bigcap_{s\in FE} sA_{f(s)}$ and $Q_{\gamma,f} = \bigcap_{s\in FE} \sigma_s\gamma^{-1}(f(s))$ for $\gamma \in \{1, \dots, q\}^d$. For $v \in \{1, \dots, d\}$ let $Z_v = Z_{v,f}$ be the function on $\{1, \dots, q\}^d$ such that $Z_v(\gamma)$ is equal to 1 if $v \in W \cap Q_{\gamma,f}$ and 0 otherwise. Writing $\mathbb{E}(\cdot)$ for the expected value of a function on $\{1, \dots, q\}^d$, for $v \in W$ we have

$$\begin{aligned}\mathbb{E}(Z_v) &= \nu^d\big(\{\gamma \in \{1, \dots, q\}^d : \sigma_s^{-1}(v) \in \gamma^{-1}(f(s)) \text{ for every } s \in FE\}\big) \\ &= \prod_{s\in FE} \nu(\{f(s)\}) = \prod_{s\in FE} \nu(A_{f(s)}) = \mu(P_f).\end{aligned}$$

Set $Z = \sum_{v\in W} Z_v$. Let us estimate the variance of Z. For $v, w \in W$, if $\sigma_s^{-1}(v) \neq \sigma_t^{-1}(w)$ for all $s, t \in FE$ then Z_v and Z_w are independent, that is, $\mathbb{E}(Z_vZ_w) = \mathbb{E}(Z_v)\mathbb{E}(Z_w)$. It follows that the number of pairs $(v, w) \in W \times W$ for which Z_v and Z_w are not independent is at most $|W||FE|^2$.

Hence

$$\mathbb{E}(Z^2) = \sum_{v,w\in W} \mathbb{E}(Z_v Z_w) \leq \sum_{v,w\in W} \mathbb{E}(Z_v)\mathbb{E}(Z_w) + |W||FE|^2 = \mathbb{E}(Z)^2 + |W||FE|^2$$

and so $\mathrm{Var}(Z) \leq |W||FE|^2$. As $Z(\gamma) = |W \cap Q_{\gamma,f}|$ and $\mathbb{E}(Z) = |W|\mu(P_f)$, using Chebyshev's inequality we then have, for all $t > 0$,

$$\mathbb{P}\big(\big||W\cap Q_{\gamma,f}|/|W| - \mu(P_f)\big| > t\big) \leq \frac{\mathrm{Var}(Z)}{|W|^2t^2} \leq \frac{|FE|^2}{|W|t^2} \leq \frac{2|K|^2|FE|^2}{dt^2}.$$

Taking $t = \beta/(2q^{|FE|})$ we thus obtain, for d sufficiently large,

$$\mathbb{P}\big(\big||W\cap Q_{\gamma,f}|/|W| - \mu(P_f)\big| > \beta/(2q^{|FE|})\big) \leq \frac{1}{4q^{|FE|}}.$$

Since B_1', B_2', C_1, and C_2 are unions of sets of the form P_f for $f \in \{1,\dots,q\}^{FE}$, it follows that if d is sufficiently large then there is a set $L' \subseteq \{1,\dots,q\}^d$ with $\nu^d(L') \geq 3/4$ such that for every $\gamma \in L'$ we have $|\pi_W \circ \psi_\gamma(B_1'\Delta C_2)|/|W| < \mu(B_2'\Delta C_1) + \beta/2$ and $|\pi_W \circ \psi_\gamma(B_2'\Delta C_1)|/|W| < \mu(B_1'\Delta C_2) + \beta/2$ so that

$$\max(|\pi_W \circ \psi_\gamma(B_1'\Delta C_2)|, |\pi_W \circ \psi_\gamma(B_2'\Delta C_1)|) < \beta|W|,$$

which will allow us below to invoke the conclusion of Lemma 2.3 in accord with our choice of β.

Since W is $\sigma(K)^{-1}$-separated, $W \subseteq R$, and C_1 and C_2 are unions of sets in α_K, for every $v \in W$ and $i = 1,2$ the probability that $v \in \psi_\gamma(C_i)$ is $\mu(C_i)$, while for all distinct $v,w \in W$ and all $i,j \in \{1,2\}$ the events $\{\gamma \in \{1,\dots,q\}^d : v \in \psi_\gamma(C_i)\}$ and $\{\gamma \in \{1,\dots,q\}^d : w \in \psi_\gamma(C_j)\}$ are independent. Define the function I on $\{1,\dots,q\}^d$ by

$$I(\gamma) = -\sum_{v\in W} \log \mu(C_{i_{\gamma,v}})$$

where $i_{\gamma,v}$ is such that $v \in \psi_\gamma(C_{i_{\gamma,v}})$. Then, by the law of large numbers,

$$\lim_{|W|\to\infty} \mathbb{P}\bigg(\bigg|\frac{1}{|W|}I(\gamma) - H(\eta)\bigg| > \frac{\varepsilon}{2}\bigg) = 0.$$

Since for each $\gamma \in \{1,\dots,q\}^d$ the quantity $e^{-I(\gamma)} = \prod_{v\in W}\mu(C_{i_{\gamma,v}})$ is equal to the ν^d-measure of the set of all $\gamma' \in \{1,\dots,q\}^d$ for which $\pi_W \circ \psi_{\gamma'}|_\eta = \pi_W \circ \psi_\gamma|_\eta$, we can thus find, assuming d is sufficiently large and σ is a sufficiently good sofic approximation, an $L'' \subseteq \{1,\dots,n\}^d$ for which $\nu^d(L'') \geq 3/4$ and, for all $\gamma \in L''$,

$$\nu^d(\{\gamma' \in \{1,\dots,q\}^d : \pi_W \circ \psi_{\gamma'}|_\eta = \pi_W \circ \psi_\gamma|_\eta\}) \leq e^{-(H(\eta)-\varepsilon/2)|W|}.$$

Write Ψ for the set of all ψ_γ such that $\gamma \in L \cap L' \cap L''$. We then have

$$|\{\pi_W \circ \psi|_\eta : \psi \in \Psi\}| \geq \nu^d(L\cap L'\cap L'')e^{(H(\eta)-\varepsilon/2)|W|} \geq \frac{1}{4}e^{(H(\eta)-\varepsilon/2)|W|}.$$

If d is large enough so that $|W|$ is large enough, the last expression above will be at least $e^{(H(\eta)-\varepsilon)|W|}$.

Set $\xi' = \{B_1', B_2'\}$. Take a set Ψ' of representatives in Ψ for the relation of equality under restriction to ξ'. For each $\psi \in \Psi'$ write Φ_ψ for the set of all $\varphi \in \Psi$ which agree with ψ on ξ'. Since every $\psi \in \Psi$ is of the form φ_γ for some $\gamma \in L'$, by our choice of β we have

$|\{\pi_W \circ \varphi|_\eta : \varphi \in \Phi_\psi\}| \leq e^{\varepsilon|W|}$ for every $\psi \in \Psi'$, assuming that d is large enough so that $|W| \geq M$. Therefore, granted that d is large enough,

$$\begin{aligned} e^{(H(\eta)-\varepsilon)|W|} &\leq |\{\pi_W \circ \psi|_\eta : \psi \in \Psi\}| \\ &\leq \sum_{\psi \in \Psi'} |\{\pi_W \circ \varphi|_\eta : \varphi \in \Phi_\psi\}| \\ &\leq |\Psi'| \cdot e^{\varepsilon|W|} \\ &\leq |\mathrm{Hom}_\mu(\alpha_E, F, \delta/2, \sigma)|_{\xi'} \cdot e^{\varepsilon|W|}. \end{aligned}$$

Using the fact that $|W| \geq |V| - d/(2|K|^2) \geq d/(2|K|^2)$, we thus obtain

$$\frac{1}{d} \log |\mathrm{Hom}_\mu(\alpha_E, F, \delta/2, \sigma)|_{\xi'} \geq (H(\eta) - 2\varepsilon)\frac{|W|}{d} \geq \frac{H(\eta)}{4|K|^2}.$$

Considering now the map $\varphi \mapsto \varphi \circ \theta$ from $\mathrm{Hom}_\mu(\alpha_E, F, \delta/2, \sigma)$ to $\mathrm{Hom}_\mu(\rho, F, \delta, \sigma)$, the fact that for every $\psi \in \mathrm{Hom}_\mu(\alpha_E, F, \delta/2, \sigma)$ the restriction of $\psi \circ \theta$ to ξ determines the restriction of ψ to ξ' yields

$$|\mathrm{Hom}_\mu(\rho, F, \delta, \sigma)|_\xi \geq |\mathrm{Hom}_\mu(\alpha_E, F, \delta/2, \sigma)|_{\xi'}.$$

We thus obtain the theorem with $\lambda = H(\eta)/(4|K|^2)$. □

The following is an immediate consequence of Lemma 2.5 and the definition of sofic measure entropy.

Theorem 2.6. *Every Bernoulli action of a sofic group has completely positive entropy with respect to every sofic approximation net.*

3. The f-invariant

Throughout this section F_r is a free group on a fixed but arbitrary nonempty finite set of r generators. Our goal is to show the analogue of Theorem 2.6 for the f-invariant, namely that every nontrivial factor of a Bernoulli action of F_r with a finite generating partition has positive f-invariant.

The f-invariant of a measure-preserving action $F_r \curvearrowright (X, \mu)$ is defined as follows [4]. Write S for the standard set of r generators of F_r. For a partition ξ with finite Shannon entropy we set

$$F(\xi) = (1 - 2|S|)H(\xi) + \sum_{s \in S} H(\xi \vee s\xi),$$

$$f(\xi) = \inf_{n \in \mathbb{N}} F\Bigg(\bigvee_{s \in B_n} s\xi\Bigg)$$

where B_n denotes the set of all words in $S \cup S^{-1}$ of length at most n. Then $f(\xi)$ takes a common value over all generating partitions ξ with finite Shannon entropy [4] and we define $f(X, F_r)$ to be this value in the case that such a generating partition exists.

The f-invariant can be alternatively expressed by averaging the local quantities in the definition sofic entropy over permutation models for F_r. More precisely, by Theorem 1.3 of [3] we have, for a finite generating partition ξ,

$$(*) \qquad f(X, F_r) = \inf_F \inf_{\delta>0} \limsup_{d\to\infty} \frac{1}{d} \log \left(\frac{1}{d!^r} \sum_{\sigma\in\mathrm{Hom}(F_r,\mathrm{Sym}(d))} |\mathrm{Hom}_\mu(\xi, F, \delta, \sigma)|_\xi \right)$$

where F ranges over the finite subsets of G and $\mathrm{Hom}(F_r, \mathrm{Sym}(d))$ denotes the set of all group homomorphisms from F_r to $\mathrm{Sym}(d)$. Note that for each d there are elements of $\mathrm{Hom}(F_r, \mathrm{Sym}(d))$ which fail to be good sofic approximations due to a lack of sufficient freeness, but the following lemma shows that this lack of freeness occurs with asymptotically vanishing probability as $d \to \infty$.

Lemma 3.1. *Let r and m be positive integers with $r \le m$. Let $\alpha_1 \ne \alpha_2 \ne \cdots \ne \alpha_m$ be numbers in $\{1, \dots, r\}$ which exhaust this set. Let $n_1, \dots, n_m \in \mathbb{Z} \setminus \{0\}$. Set*

$$\Omega_{d,\varepsilon} = \big\{(U_1, \dots, U_r) \in S_d^r : \mathrm{tr}_d(U_{\alpha_1}^{n_1} \cdots U_{\alpha_m}^{n_m}) < \varepsilon\big\}.$$

Then $\lim_{d\to\infty} |\Omega_{d,\varepsilon}|/d!^r = 1$.

Proof. By a theorem of Nica [12] we have

$$\lim_{d\to\infty} \frac{1}{d!^r} \sum_{(U_1,\dots,U_r)\in S_d^r} \mathrm{tr}_d(U_{\alpha_1}^{n_1} \cdots U_{\alpha_m}^{n_m}) = 0.$$

Since each of the above trace values is nonnegative, it follows that for all sufficiently large d we have $|\Omega_{d,\varepsilon/2}|/d!^r \ge 1/2$. Expressing the normalized Hamming metric ρ_d on S_d as

$$\rho_d(U, V) = \frac{1}{2}\mathrm{tr}_d(|U - V|^2) = \frac{1}{2}\|U - V\|_2^2$$

and applying the Cauchy-Schwarz and triangle inequalities, we deduce the existence of a $\delta > 0$ such that $N_\delta(\Omega_{d,\varepsilon/2}) \subseteq \Omega_{d,\varepsilon}$ where $N_\delta(\cdot)$ denotes the δ-neighbourhood with respect to the metric $\rho_{d,r}((U_1, \dots, U_r), (V_1, \dots, V_r)) = \max_{i=1,\dots,r} \rho_d(U_i, V_i)$. Gromov and Milman remark in Section 3.6 of [7] that a result of Maurey [11] shows that for $d \in \mathbb{N}$ the symmetric groups S_d equipped with the uniform probability measures and normalized Hamming metrics form a Lévy family (see Section 1 of [7]), and they observe in Section 2.2 of [7] that the Lévy property is preserved under finite products. Thus for $d \in \mathbb{N}$ the products S_d^r with the product measures and metrics $\rho_{d,r}$ form a Lévy family, in which case $\lim_{d\to\infty} |N_\delta(\Omega_{d,\varepsilon/2})|/d!^r = 1$, yielding the lemma. □

Theorem 3.2. *Let $F_r \curvearrowright (Y, \nu)$ be a nontrivial factor of a Bernoulli action of F_r and suppose that this factor has a finite generating partition. Then $f(Y, F_r) > 0$.*

Proof. For every finite set $F \subseteq G$, $\varepsilon > 0$, and $d \in \mathbb{N}$ write $\Omega_{d,F,\varepsilon}$ for the set of all group homomorphisms $\sigma : F_r \to \mathrm{Sym}(d)$ such that

$$\big|\{v \in \{1, \dots, d\} : \sigma_s(v) \ne \sigma_t(v)\}\big| \ge 1 - \varepsilon$$

for all distinct $s, t \in F$. Given the one-to-one correspondence between group homomorphisms from F_r to $\mathrm{Sym}(d)$ and r-tuples of $d \times d$ permutation matrices that is set up via the standard generators of F_r, we have $\lim_{d\to\infty} |\Omega_{d,F,\varepsilon}|/d!^r = 1$ by Lemma 3.1. The result now follows by appealing to Lemma 2.5 and $(*)$. □

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