ENTROPY AND INDUCED DYNAMICS ON STATE SPACES

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ABSTRACT. We consider the topological entropy of state space and quasi-state space homeomorphisms induced from C^*-algebra automorphisms. Our main result asserts that, for automorphisms of separable exact C^*-algebras, zero Voiculescu-Brown entropy implies zero topological entropy on the quasi-state space (and also more generally on the entire unit ball of the dual). As an application we obtain a simple description of the topological Pinsker algebra in terms of local Voiculescu-Brown entropy.

1. INTRODUCTION

Entropy is a numerical invariant measuring the complexity of a dynamical system. For homeomorphisms of a compact metric space, topological entropy can be defined using either open covers [1] or separated and spanning sets [10, 5]. For automorphisms of unital nuclear C^*-algebras, a notion of entropy based on approximation was introduced by Voiculescu [29], and this was subsequently extended to automorphisms of exact C^*-algebras by Brown [7]. By [29, Prop. 4.8] the topological entropy of a homeomorphism T of a compact metric space X coincides with the Voiculescu-Brown entropy of the automorphism $f \mapsto f \circ T$ of $C(X)$. In other words, the Voiculescu-Brown entropy of an automorphism of a separable unital commutative C^*-algebra coincides with the topological entropy of the induced homeomorphism on the pure state space. This paper is principally aimed at examining the relationship between the Voiculescu-Brown entropy of an automorphism of a general unital exact C^*-algebra and the topological entropy of the induced homeomorphism on the state space (or quasi-state space if we drop the requirement that the algebra be unital) with the hope of being able to obtain information about one from information about the other.

A basic problem in dynamics is to determine whether or not a given system has positive entropy, i.e., whether or not it is "chaotic". In [14] Glasner and Weiss proved that if a homeomorphism of a compact metric space has zero topological entropy, then the induced homeomorphism on the space of probability measures also has zero topological entropy. We thus have in this case that zero Voiculescu-Brown entropy implies zero topological entropy on the state space. Our main result (Theorem 3.3) shows that this implication holds for automorphisms of any separable unital exact C^*-algebra, and moreover asserts that, in the general separable exact setting, zero Voiculescu-Brown entropy implies zero topological entropy on the unit ball of the dual (which is in fact equivalent to zero topological entropy on the quasi-state space—see Lemma 2.2 and the paragraph preceding it). This has the particular consequence that if an automorphism α of a C^*-algebra has positive topological entropy on the quasi-state space (for example, if α arises from a

homeomorphism of a compact metric space with positive topological entropy), then any automorphism of a separable exact C^*-algebra which can be obtained from α as a dynamical extension will have positive Voiculescu-Brown entropy. Thus in many cases we can obtain some information about the behaviour of Voiculescu-Brown entropy under taking noncommutative dynamical extensions, about which little seems to be known in general (it is unknown whether Voiculescu-Brown entropy can strictly decrease, or even become zero, under taking dynamical extensions of a positive entropy system).

The main body of the paper is divided into three parts. In Section 2 we recall the definition of topological entropy and show that, for an automorphism of a unital C^*-algebra, the entropy of the induced homeomorphism on the state space is either zero or infinity. As an example we demonstrate that the shift on the full group C^*-algebra $C^*(\mathbb{F}_\infty)$ of the free group on countably many generators falls into the latter case. In Section 3 we begin by recalling the definition of Voiculescu-Brown entropy and then proceed to the proof of our main result, for which we develop a matrix version of an argument from [14] that uses results from the local theory of Banach spaces. In our case the key geometric fact concerns the relationship between n and k given an approximately isometric embedding of ℓ_1^n into the space of $k \times k$ matrices with the $p = \infty$ Schatten norm. We round out Section 3 with some applications and examples. In particular we show that the shift on the reduced group C^*-algebra $C_r^*(\mathbb{F}_\infty)$ has zero topological entropy on the state space in contrast to its full group C^*-algebra counterpart. Finally, in Section 4 we apply our main result (or rather a local version that follows from the same proof) to show that the topological Pinsker algebra from topological dynamics admits a simple description in terms of local Voiculescu-Brown entropy.

2. Topological entropy and state space dynamics

To start with we recall the definition of topological entropy for a homeomorphism T of a compact metric space (X, d) (see [9, 16, 23, 30] for general references). For an open cover $\mathcal{U}$ of X we denote by $N(\mathcal{U})$ the smallest cardinality of a subcover and set

$$h_{\text{top}}(T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-(n-1)}\mathcal{U}),$$

and we define the *topological entropy* of T by

$$h_{\text{top}}(T) = \sup_{\mathcal{U}} h_{\text{top}}(T, \mathcal{U})$$

where the supremum is taken over all finite open covers $\mathcal{U}$. We may equivalently express the topological entropy in terms of separated and spanning sets as follows. A set $E \subset X$ is said to be (n, ε)-*separated (with respect to T)* if for every $x, y \in E$ with $x \neq y$ there exists a $0 \leq k \leq n - 1$ such that $d(T^k x, T^k y) > \varepsilon$, and (n, ε)-*spanning (with respect to T)* if for every $x \in X$ there is a $y \in E$ such that $d(T^k x, T^k y) \leq \varepsilon$ for each $k = 0, \ldots, n - 1$. Denoting by $\mathrm{sep}_n(T, \varepsilon)$ the largest cardinality of an (n, ε)-separated set and by $\mathrm{spn}_n(T, \varepsilon)$ the smallest cardinality of an (n, ε)-spanning set, we then have

$$h_{\mathrm{top}}(T) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}_n(T, \varepsilon) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{spn}_n(T, \varepsilon).$$

Let A be a unital C^*-algebra. We denote by $S(A)$ the state space of A, i.e., the convex set of positive linear functionals ϕ on A with $\phi(1) = 1$. We equip $S(A)$ with the weak* topology, under which it is compact. Given a automorphism α of A we will denote by T_α the homeomorphism of $S(A)$ given by $T_\alpha(\sigma) = \sigma \circ \alpha$ for all $\sigma \in S(A)$.

The following result is a generalization of the proposition on p. 422 of [26] (which treats homeomorphisms of compact metric spaces), and in fact the same proof also works here. For convenience we will give a version of the argument in our broader context.

Proposition 2.1. Let A be a separable unital C^*-algebra and α an automorphism of A. Then either $h_{\mathrm{top}}(T_\alpha) = 0$ or $h_{\mathrm{top}}(T_\alpha) = \infty$.

Proof. We define a metric d on the dual A^* by taking a dense sequence $x_1, x_2, x_3, \ldots$ in the unit ball of A and setting

$$d(\sigma, \omega) = \sum_{i=1}^{\infty} 2^{-i} |\sigma(x_i) - \omega(x_i)|.$$

The metric d is compatible with the weak* topology on bounded subsets of A^*.

Now suppose that $h_{\mathrm{top}}(T_\alpha) > 0$. Then for some $\varepsilon > 0$ we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}_n(T_\alpha, \varepsilon) > 0.$$

Fix an $r \in \mathbb{N}$. Set $\theta = (1 - \lambda)/(1 - \lambda^r)$, and choose a $\lambda > 0$ with $\lambda < \min(1, 2^{-1}\theta\varepsilon)$, and set

$$\varepsilon' = \theta \lambda^{r-1} (\varepsilon - 2\lambda\theta^{-1}) > 0.$$

For each $n \in \mathbb{N}$ let $E_n \subset S(A)$ be an (n, ε)-separated set of largest cardinality, and consider the subset

$$E_n^r = \left\{ \theta \sum_{i=1}^{r} \lambda^{i-1} \sigma_i : \sigma_1, \ldots, \sigma_r \in E_n \right\}$$

of $S(A)$. We will show that E_n^r is an (n, ε')-separated set of cardinality $|E_n|^r$.

Indeed suppose that $(\sigma_1, \ldots, \sigma_r)$ and $(\omega_1, \ldots, \omega_r)$ are distinct r-tuples of elements in E_n. Then for some $1 \leq j \leq r$ we have $\sigma_i = \omega_i$ for each $i = 1, \ldots, j - 1$ and $\sigma_j \neq \omega_j$. Since E_n is (n, ε)-separated there is a $0 \leq k \leq n - 1$ such that $d(T_\alpha^k \sigma_j, T_\alpha^k \omega_j) > \varepsilon$. Observe that if $j < r$, then in view of the definition of the metric d we have

$$d\left(T_\alpha^k \left(\theta \sum_{i=j}^{r} \lambda^{i-1} \sigma_i \right), T_\alpha^k \left(\theta \lambda^{j-1} \sigma_j \right) \right) = d\left(T_\alpha^k \left(\theta \sum_{i=j+1}^{r} \lambda^{i-1} \sigma_i \right), 0 \right)$$

$$\leq \theta \sum_{i=j+1}^{r} \lambda^{i-1}$$

$$= \lambda^j \frac{1 - \lambda^{r-j}}{1 - \lambda^r}$$

$$\leq \lambda^j$$

and similarly $d\big(T_\alpha^k\big(\theta \sum_{i=j}^{r} \lambda^{i-1}\omega_i\big), T_\alpha^k(\theta\lambda^{j-1}\omega_j)\big) \leq \lambda^j$. Thus when $j < r$ we have

$$d\Big(T_\alpha^k\Big(\theta\sum_{i=1}^{r} \lambda^{i-1}\sigma_i\Big), T_\alpha^k\Big(\theta\sum_{i=1}^{r} \lambda^{i-1}\omega_i\Big)\Big)$$

$$= d\Big(T_\alpha^k\Big(\theta\sum_{i=j}^{r} \lambda^{i-1}\sigma_i\Big), T_\alpha^k\Big(\theta\sum_{i=j}^{r} \omega^{i-1}\sigma_i\Big)\Big)$$

$$\geq d\big(T_\alpha^k(\theta\lambda^{j-1}\sigma_j), T_\alpha^k(\theta\lambda^{j-1}\omega_j)\big) - d\Big(T_\alpha^k\Big(\theta\sum_{i=j}^{r} \lambda^{i-1}\sigma_i\Big), T_\alpha^k(\theta\lambda^{j-1}\sigma_j)\Big)$$

$$- d\Big(T_\alpha^k\Big(\theta\sum_{i=j}^{r} \lambda^{i-1}\omega_i\Big), T_\alpha^k(\theta\lambda^{j-1}\omega_j)\Big)$$

$$\geq \theta\lambda^{j-1}d\big(T_\alpha^k\sigma_j, T_\alpha^k\omega_j\big) - 2\lambda^j$$

$$\geq \theta\lambda^{j-1}(\varepsilon - 2\lambda\theta^{-1})$$

$$> \varepsilon',$$

while in the case $j = r$ the first expression in this display is simply $\theta\lambda^{r-1}d(T_\alpha^k\sigma_r, T_\alpha^k\omega_r)$, which again is strictly greater than ε'. It follows that E_n^r is an (n, ε')-separated set of cardinality $|E_n|^r$ as we wished to show, and so

$$h_{\text{top}}(T_\alpha) \geq \limsup_{n\to\infty} \frac{1}{n} \log \text{sep}_n(T_\alpha, \varepsilon') \geq r \limsup_{n\to\infty} \frac{1}{n} \log \text{sep}_n(T_\alpha, \varepsilon)$$

Since r was arbitrary we conclude that $h_{\text{top}}(T_\alpha) = \infty$. $\square$

We point out that the above argument can also be used to obtain the same dichotomy for the values of topological entropy among continuous maps of state spaces induced by positive unital linear maps of separable operator systems or order-unit spaces, as well as among homeomorphisms of quasi-state spaces induced by automorphisms of general separable C^*-algebras (see Section 3).

As an example we will show that, for the shift on the full group C^*-algebra $C^*(\mathbb{F}_\infty)$ of the free group on countably many generators, the topological entropy on the state space is infinite (Proposition 2.4). Although this is an immediate consequence of the fact that the C^*-dynamical system arising from the topological 2-shift is a C^*-dynamical factor of the shift on $C^*(\mathbb{F}_\infty)$ (see the paragraph following Proposition 2.4), we will give here a more explicitly geometric proof in anticipation of the arguments in Section 3. In addition, the following two lemmas which we will require are of use in other situations. For example,

Lemma 2.2 implies that the values of topological entropy on the state space and quasi-state space agree (see the second paragraph of Section 3), and its proof can also be used to show that the values of topological entropy on the quasi-state space and on the unit ball of the dual agree. Lemma 2.3, on the other hand, will be of convenience in Example 3.6.

Lemma 2.2. Let A be a separable unital C^*-algebra and α an automorphism of A. Let S_α be the homeomorphism of the closed unit ball $B_1(A^*)$ of the dual of A (with the weak* topology) given by $S_\alpha(\sigma) = \sigma \circ \alpha$. Then

$$h_{\mathrm{top}}(S_\alpha) = h_{\mathrm{top}}(T_\alpha).$$

Proof. Since T_α is the restriction of S_α to $S(A)$, we have $h_{\mathrm{top}}(S_\alpha) \geq h_{\mathrm{top}}(T_\alpha)$. Thus, in view of Proposition 2.1, we need only show that $h_{\mathrm{top}}(T_\alpha) = 0$ implies $h_{\mathrm{top}}(S_\alpha) = 0$. Suppose then that $h_{\mathrm{top}}(T_\alpha) = 0$. Let $x_1, x_2, x_3, \ldots$ be a dense sequence in the unit ball of A and define on A^* the metric

$$d(\sigma, \omega) = \sum_{i=1}^{\infty} 2^{-i} |\sigma(x_i) - \omega(x_i)|,$$

which is compatible with the weak* topology on bounded subsets of A^*. Let $\varepsilon > 0$, and pick an integer $r > \varepsilon^{-1}$. Let $E_n \subset S(A)$ be an (n, ε)-spanning set (with respect to T_α) of smallest cardinality. Now if $\tau \in B_1(A^*)$ then we can write

$$\tau = \sigma_1 - \sigma_2 + i(\sigma_3 - \sigma_4)$$

where each σ_j is a positive linear functional of norm at most 1 [17, Thm. 4.3.6 and Cor. 4.3.7]. Since E_n is (n, ε)-spanning, for each $j = 1, 2, 3, 4$ for which $\sigma_j \neq 0$ we can find an $\omega_j \in E_n$ such that

$$d\big(T_\alpha^k \omega_j, T_\alpha^k(\sigma_j/\|\sigma_j\|)\big) \leq \varepsilon$$

for every $k = 0, \ldots, n-1$, and we can also find an $m_j \in \{0, 1, \ldots, r\}$ such that $\big|m_j/r - \|\sigma_j\|\big| \leq \varepsilon$, so that, for every $k = 0, \ldots, n-1$,

$$\begin{aligned}
d\big(S_\alpha^k((m_j/r)\omega_j), S_\alpha^k \sigma_j\big) &\leq d\big(S_\alpha^k((m_j/r)\omega_j), S_\alpha^k(\|\sigma_j\|\omega_j)\big) \\
&\quad + d\big(S_\alpha^k(\|\sigma_j\|\omega_j), S_\alpha^k \sigma_j\big) \\
&\leq \big|m_j/r - \|\sigma_j\|\big| + \|\sigma_j\| d\big(T_\alpha^k \omega_j, T_\alpha^k(\sigma_j/\|\sigma_j\|)\big) \\
&\leq \varepsilon + \|\sigma_j\|\varepsilon \\
&\leq 2\varepsilon.
\end{aligned}$$

For any $j = 1, 2, 3, 4$ for which $\sigma_j = 0$, let ω_j be any bounded linear functional on A and set $m_j = 0$. Set $\tau' = (m_1\omega_1 - m_2\omega_2 + im_3\omega_3 - im_4\omega_4)/r$. Then $\|\tau'\| \leq 4$, and if $\tilde{S}_\alpha$ denotes the homeomorphism of the closed ball $B_4(A^*) = \{\sigma \in A^* : \|\sigma\| \leq 4\}$ given by $\tilde{S}_\alpha(\sigma) = \sigma \circ \alpha$, then in view of the definition of the metric d we have

$$d\big(\tilde{S}_\alpha^k \tau', \tilde{S}_\alpha^k \tau\big) \leq \sum_{i=1}^{4} d\big(S_\alpha^k((m_i/r)\omega_i), S_\alpha^k \sigma_i\big) \leq 8\varepsilon + 8\varepsilon = 16\varepsilon$$

for all $k = 0, \ldots, n-1$. Let F_n be the subset of $B_4(A^*)$ consisting of all τ' which arise in the above way with respect to some $\tau \in B_1(A^*)$. We have thus shown that F_n is

$(n, 16\varepsilon)$-spanning for $B_1(A^*)$ with respect to $\tilde{S}_\alpha$ (in the obvious sense which relativizes the definition of an (n, ε)-spanning set to a subset of the space—see Definition 14.14 of [9]), and hence that the smallest cardinality $\mathrm{spn}_n(\tilde{S}_\alpha, 16\varepsilon, B_1(A^*))$ of an $(n, 16\varepsilon)$-spanning set for $B_1(A^*)$ with respect to $\tilde{S}_\alpha$ is bounded above by $|F_n|$, which is in turn bounded above by $|E_n|^{4(r+1)}$. Since r does not depend on n we therefore obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{spn}_n(\tilde{S}_\alpha, 16\varepsilon, B_1(A^*)) \leq 4(r+1) \limsup_{n\to\infty} \frac{1}{n} \log |E_n| = 0.$$

Now Proposition 14.15 of [9] shows that the supremum of the first expression in the above display over all $\varepsilon > 0$ is in fact equal to the topological entropy of $S_\alpha = \tilde{S}_\alpha\big|_{B_1(A^*)}$, and so we conclude that $h_{\mathrm{top}}(S_\alpha) = 0$, as desired. $\qquad\square$

Lemma 2.3. Let A be a unital C^*-algebra and α an automorphism of A. Suppose that there exists an $x \in A$ and a $K \geq 1$ such that, for each $n \in \mathbb{N}$, the linear map $\Gamma_n : \ell_1^n \to \mathrm{span}\{x, \alpha(x), \ldots, \alpha^{n-1}(x)\}$ which sends the ith standard basis element of ℓ_1^n to $\alpha^{i-1}(x)$ for each $i = 1, \ldots, n$ is an isomorphism whose inverse has norm at most K. Then $h_{\mathrm{top}}(T_\alpha) = \infty$.

Proof. Let $x \in A$, $K \geq 1$, and Γ_n for $n \in \mathbb{N}$ be as in the hypotheses. For each $n \in \mathbb{N}$ we denote by Λ_n the collection of functions from $\{0, \ldots, n-1\}$ to $\{-1, 1\}$. For each $f \in \Lambda_n$ we define the linear functional σ_f on $\mathrm{span}(x, \alpha(x), \ldots, \alpha^{n-1}(x))$ by specifying

$$\sigma_f\big(\alpha^k(x)\big) = K^{-1}f(k)$$

for each $k = 0, \ldots, n-1$. Since Γ_n^{-1} has norm at most K, it follows that σ_f has norm at most one. By the Hahn-Banach theorem we can extend σ_f to an element σ_f' in the dual A^* of norm at most one. Let $\mathcal{U}$ be the open cover of the closed unit ball $B_1(A^*)$ consisting of the two open sets

$$\{\sigma \in B_1(A^*) : |\sigma(x) - K^{-1}| < 2K^{-1}\},$$
$$\{\sigma \in B_1(A^*) : |\sigma(x) - K^{-1}| > K^{-1}\}.$$

Let $S_\alpha : B_1(A^*) \to B_1(A^*)$ be the homeomorphism given by $S_\alpha(\sigma) = \sigma \circ \alpha$. For every $n \in \mathbb{N}$, each element of $\mathcal{U} \vee S_\alpha^{-1}\mathcal{U} \vee \cdots \vee S_\alpha^{-(n-1)}\mathcal{U}$ contains precisely one linear functional of the form σ_f for $f \in \Lambda_n$, from which it follows that

$$N(\mathcal{U} \vee S_\alpha^{-1}\mathcal{U} \vee \cdots \vee S_\alpha^{-(n-1)}\mathcal{U}) = 2^n.$$

Consequently

$$h_{\mathrm{top}}(S_\alpha) \geq \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee S_\alpha^{-1}\mathcal{U} \vee \cdots \vee S_\alpha^{-(n-1)}\mathcal{U}) = \log 2,$$

and so $h_{\mathrm{top}}(T_\alpha) = \infty$ by Lemma 2.2 and Proposition 2.1. $\qquad\square$

Let $\{u_i\}_{i\in\mathbb{Z}}$ be the set of canonical unitaries associated to the generators in the full group C^*-algebra $C^*(\mathbb{F}_\infty)$. The shift α on $C^*(\mathbb{F}_\infty)$ is the automorphism defined by specifying $\alpha(u_i) = u_{i+1}$ for all $i \in \mathbb{Z}$.

Proposition 2.4. For the shift α on $C^*(\mathbb{F}_\infty)$ we have

$$h_{\mathrm{top}}(T_\alpha) = \infty.$$

Proof. For each $n \in \mathbb{N}$, the linear map from $\mathrm{span}(u_0, \ldots, u_{n-1})$ to ℓ_1^n which sends u_i to the ith standard basis element of ℓ_1^n is an isometry, as pointed out in [24, Sect. 8]. This follows from the observation that, for any scalars $c_0, \ldots, c_{n-1}$,

$$\left\| \sum_{i=0}^{n-1} c_i u_i \right\| = \sup \left\| \sum_{i=0}^{n-1} c_i v_i \right\|$$

where the supremum is taken over all unitaries $v_0, \ldots, v_{n-1} \in \mathcal{B}(\ell_2)$. We can thus appeal to Lemma 2.3 to obtain the result. $\square$

Proposition 2.4 can also be established by observing that if T is the left shift on $X = \{-1, 1\}^{\mathbb{Z}}$ and β is the automorphism of $C(X)$ defined by $\beta(f) = f \circ T$ for all $f \in C(X)$, then β is a C^*-dynamical factor of α. Indeed if we consider for each $n \in \mathbb{N}$ the function $g_n \in C(X)$ given by

$$g_n((a_k)_{k \in \mathbb{Z}}) = a_n$$

for all $(a_k)_{k \in \mathbb{Z}} \in X$, then we can define a *-homomorphism $\gamma : C^*(\mathbb{F}_\infty) \to C(X)$ by specifying $\gamma(u_n) = g_n$ for each $n \in \mathbb{N}$. By the Stone-Weierstrass theorem γ is surjective, and evidently $\gamma \circ \alpha = \beta \circ \gamma$. It follows that T_α contains a subsystem conjugate to T_β and hence has infinite entropy by Proposition 2.1.

In contrast we will show in Proposition 3.7 that, for the corresponding shift on the reduced group C^*-algebra $C_r^*(\mathbb{F}_\infty)$, the topological entropy on the state space is zero.

3. Zero Voiculescu-Brown entropy implies zero topological entropy on the unit ball of the dual

We begin this section by recalling the definition of Voiculescu-Brown entropy, which is an extension to exact C^*-algebras [7] of the approximation-based entropy for automorphisms of unital nuclear C^*-algebras introduced in [29]. Let A be an exact C^*-algebra, and let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a faithful representation. For a finite set $\Omega \subset A$ and $\delta > 0$ we denote by $\mathrm{CPA}(\pi, \Omega, \delta)$ the collection of triples (ϕ, ψ, B) where B is a finite-dimensional C^*-algebra and $\phi : A \to B$ and $\psi : B \to \mathcal{B}(\mathcal{H})$ are contractive completely positive linear maps such that $\|(\psi \circ \phi)(x) - \pi(x)\| < \delta$ for all $x \in \Omega$. This collection is non-empty by nuclear embeddability [19]. We define $\mathrm{rcp}(\Omega, \delta)$ to be the infimum of rank B over all $(\phi, \psi, B) \in \mathrm{CPA}(\pi, \Omega, \delta)$, where rank refers to the dimension of a maximal Abelian C^*-subalgebra. As the notation indicates, this infimum is independent of the particular faithful representation π, as demonstrated in the proof of Proposition 1.3 in [7]. For an automorphism α of A we set

$$ht(\alpha, \Omega, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{rcp}(\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta),$$
$$ht(\alpha, \Omega) = \sup_{\delta > 0} ht(\alpha, \Omega, \delta),$$
$$ht(\alpha) = \sup_{\Omega} ht(\alpha, \Omega)$$

with the last supremum taken over all finite sets $\Omega \subset A$. We refer to $ht(\alpha)$ as the *Voiculescu-Brown entropy* of α.

For any C^*-algebra A, the quasi-state space $Q(A)$ of A is defined as the convex set of positive linear functionals ϕ on A with $\|\phi\| \leq 1$. Equipped with the weak*-topology, $Q(A)$ is compact. Given an automorphism α of A we denote by $\tilde{T}_\alpha$ the homeomorphism $\sigma \mapsto \sigma \circ \alpha$ of $Q(A)$. As in the previous section, when A is unital we denote by T_α the homeomorphism $\sigma \mapsto \sigma \circ \alpha$ of the state space $S(A)$, (i.e., the restriction of $\tilde{T}_\alpha$ to $S(A)$), and for general A we denote by S_α the homeomorphism $\sigma \mapsto \sigma \circ \alpha$ of the closed unit ball $B_1(A^*)$ of the dual of A with the weak* topology. Note that, since $Q(A)$ is a subset of the unit ball of the dual A^*, by Lemma 2.2 we have $h_{\mathrm{top}}(\tilde{T}_\alpha) = h_{\mathrm{top}}(T_\alpha)$ in the separable unital case. The argument in the proof of Lemma 2.2 can also be used to show that $h_{\mathrm{top}}(S_\alpha) = h_{\mathrm{top}}(\tilde{T}_\alpha)$ in the general separable case.

Before coming to the statement of our main result we establish two lemmas. These involve problems of a typical nature in the local theory of Banach spaces and will be proved using methods from this theory. For the basic background we refer the reader to [20] and [27]. Here we may take our Banach spaces to be over either the real or complex numbers, but it is the complex case which is relevant for our applications. For $1 \leq p \leq \infty$ we denote by C_p^k the Schatten p-class, i.e., the space of $k \times k$ matrices with norm $\|x\|_p = \mathrm{Tr}(|x|^p)^{1/p}$ in the case $1 \leq p < \infty$ (where Tr is the trace taking value 1 on minimal projections), or the operator norm (with the matrices operating on ℓ_2^k) in the case $p = \infty$. Given isomorphic Banach spaces X and Y and $K \geq 1$, we say that X and Y are K-*isomorphic* if the Banach-Mazur distance

$$d(X, Y) = \inf\{\|\Gamma\|\|\Gamma^{-1}\| : \Gamma : X \to Y \text{ is an isomorphism}\}$$

is no greater than K.

The following lemma and its proof were communicated to me by Nicole Tomczak-Jaegermann.

Lemma 3.1. Let X be an n-dimensional subspace of C_∞^k which is K-isomorphic to ℓ_1^n. Then

$$n \leq aK^2 \log k$$

where $a > 0$ is a universal constant.

Proof. The idea, which is standard in the local theory of Banach spaces, is to compare (Rademacher) type 2 constants. It can be seen from the proof of Theorem 3.1(ii) in [28] that the type 2 constant of the Schatten p-class for $2 \leq p < \infty$ satisfies

$$T_2(C_p^k) \leq C\sqrt{p}$$

where C is a universal constant. Thus, since the Banach-Mazur distance $d(C_\infty^k, C_p^k)$ is equal to $k^{1/p}$ for $2 \leq p < \infty$ [27, Thm. 45.2], we have

$$T_2(C_\infty^k) \leq d(C_\infty^k, C_p^k)T_2(C_p^k) \leq Ck^{1/p}\sqrt{p}$$

for every $2 \leq p < \infty$. Setting $p = \log k$ we obtain for sufficiently large k the bound

$$T_2(C_\infty^k) \leq Ce\sqrt{\log k},$$

and since the type 2 constant for ℓ_1^n satisfies $T_2(\ell_1^n) \geq \sqrt{n}$ (see §4 in [27]) it follows that

$$\sqrt{n} \leq T_2(\ell_1^n) \leq KT_2(X) \leq KT_2(C_\infty^k) \leq KCe\sqrt{\log k},$$

yielding the assertion of the lemma.   □

The next lemma is a matrix analogue of Proposition 2.1 of [14]. I thank one of the referees for suggesting its formulation as a general Banach space result. The main part of its proof is concerned with symmetric convex subsets of the unit cube which contain many ε-separated points, and was first established by Elton [12] in the real case and Pajor [21, 22] in the complex case. Our general line of argument follows [14], with the part involving almost Hilbertian sections of unit balls being replaced in our case with an appeal to Lemma 3.1. As is our convention, for a Banach space X we write $B_r(X)$ to refer to the closed ball $\{x \in X : \|x\| \leq r\}$.

Lemma 3.2. Given $\varepsilon > 0$ and $\lambda > 0$ there exists a $\mu > 0$ such that, for all $n \geq 1$, if $\phi : C_1^{r_n} \to \ell_\infty^n$ is a contractive linear map such that $\phi(B_1(C_1^{r_n}))$ contains an ε-separated set of cardinality at least $e^{\lambda n}$, then $r_n \geq e^{\mu n}$.

Proof. It is well known in Banach space theory that for every $\varepsilon > 0$ and $\lambda > 0$ there exist $d > 0$ and $\delta > 0$ such that the following holds for all $n \geq 1$: if $S \subset B_1(\ell_\infty^n)$ is a symmetric convex set which contains an ε-separated set F of cardinality at least $e^{\lambda n}$ then there is a subset $I_n \subset \{1, 2, \ldots, n\}$ with cardinality at least dn such that

$$B_\delta(\ell_\infty^{I_n}) \subset \pi_n(S)$$

where $\pi_n : \ell_\infty^n \to \ell_\infty^{I_n}$ is the canonical projection. In the real case this is implicit in the argument on p. 117 in [12], while in the complex case it follows immediately from Théorème 5 of [22] (take $t = \varepsilon/2$; then, with $K = B_1(\ell_\infty^n)$, the cubes $x + tK$ for $x \in F$ have pairwise disjoint interiors, whence $\mathrm{Vol}(S + tK) \geq t^n 2^n e^{\lambda n}$, yielding the conclusion with, e.g., $\delta = (e^\lambda - 1)\varepsilon/4$, assuming $\varepsilon \leq 1/2$). Thus taking $S = \phi(B_1(C_1^{r_n}))$ we obtain

$$B_\delta(\ell_\infty^{I_n}) \subset \pi_n(\phi(B_1(C_1^{r_n}))).$$

Hence the dual map $(\pi_n \circ \phi)^*$ from $(\ell_\infty^{I_n})^* \cong \ell_1^{I_n}$ to $(C_1^{r_n})^* \cong C_\infty^{r_n}$ is an embedding of norm at most 1 whose inverse has norm at most $1/\delta$. Since these bounds do not depend on n, by Lemma 3.1 there is a $c > 0$ such that, for all $n \geq 1$,

$$|I_n| \leq c \log r_n$$

and hence $dn \leq c \log r_n$. Setting $\mu = d/c$ we obtain the assertion of the lemma.   □

Theorem 3.3. Let A be a separable exact C^*-algebra and α an automorphism of A. Then $ht(\alpha) = 0$ implies $h_{\mathrm{top}}(S_\alpha) = 0$, and hence also $h_{\mathrm{top}}(\tilde{T}_\alpha) = 0$ and (when A is unital) $h_{\mathrm{top}}(T_\alpha) = 0$.

Proof. Let K be any compact subset of the unit ball of A whose linear span is dense in A (for example, we may take $K = \{k^{-1}x_k\}_{k \in \mathbb{N}}$ where $\{x_k\}_{k \in \mathbb{N}}$ is a dense sequence in the unit ball of A). The metric d on $B_1(A^*)$ defined by

$$d(\sigma, \omega) = \sup_{x \in K} |\sigma(x) - \omega(x)|$$

for all $\sigma, \omega \in B_1(A^*)$ is readily seen to give rise to the weak* topology. Now suppose $h_{\mathrm{top}}(S_\alpha) > 0$. Then there exist an $\varepsilon > 0$, a $\lambda > 0$, and an infinite set $J \subset \mathbb{N}$ such that

for all $n \in J$ there is an $(n, 4\varepsilon)$-separated set $E_n \subset B_1(A^*)$ of cardinality at least $e^{\lambda n}$. By compactness there is a finite set $\Omega \subset K$ such that, for all $\sigma, \omega \in B_1(A^*)$,

$$d(\sigma, \omega) \leq \sup_{x \in \Omega} |\sigma(x) - \omega(x)| + \varepsilon.$$

We will show that $ht(\alpha, \Omega, \varepsilon) > 0$.

Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be any faithful representation. For each $n \in J$, let (ϕ_n, ψ_n, B_n) be an element in $\mathrm{CPA}(\pi, \Omega \cup \cdots \cup \alpha^{n-1}\Omega, \varepsilon)$ with B_n of smallest possible rank, and set $r_n = \mathrm{rank}\, B_n$. Writing $\Omega = \{x_1, \ldots, x_m\}$ we define a map Γ_n from the Schatten class $C_1^{r_n}$ to $(\ell_\infty^n)^m \cong \ell_\infty^{nm}$ by

$$\Gamma_n(h) = ((\mathrm{Tr}(h\phi_n(\alpha^k(x_i))))_{k=1}^{n-1})_{i=1}^m$$

for all $h \in C_1^{r_n}$, where Tr is the trace on $M_{r_n}(\mathbb{C})$ taking value 1 on minimal projections and B_n is considered as a C^*-subalgebra of $M_{r_n}(\mathbb{C})$ under some fixed embedding. Note that Γ_n is contractive, since $\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega$ lies in the unit ball of A, ϕ_n is contractive, and $|\mathrm{Tr}(hx)| \leq \mathrm{Tr}(|h|)\|x\|$ for all $h \in C_1^{r_n}$ and $x \in B_n$.

For each $\sigma \in B_1(A^*)$ we can extend $\sigma \circ \pi^{-1}$ on $\pi(A)$ to a contractive linear functional σ' on $\mathcal{B}(\mathcal{H})$ by the Hahn-Banach theorem. Now if σ and ω are distinct elements of E_n then there is a k with $0 \leq k \leq n-1$ such that $d(T_\alpha^k \sigma, T_\alpha^k \omega) > 4\varepsilon$. Then

$$\sup_{x \in \Omega} \left| (\sigma \circ \alpha^k)(x) - (\omega \circ \alpha^k)(x) \right| > 3\varepsilon,$$

and since for every $x \in \Omega$ we have

$$\|(\psi_n \circ \phi_n)(\alpha^k(x)) - \pi(\alpha^k(x))\| < \varepsilon$$

it follows by the triangle inequality that

$$\sup_{x \in \Omega} \left| (\sigma' \circ \psi_n)(\phi_n(\alpha^k(x))) - (\omega' \circ \psi_n)(\phi_n(\alpha^k(x))) \right| > \varepsilon.$$

Taking a conditional expectation $P : M_{r_n}(\mathbb{C}) \to B_n$ and isometrically identifying a linear functional on $M_{r_n}(\mathbb{C})$ with its density matrix in $C_1^{r_n}$, we thus have that the image of the subset $\{\sigma' \circ \psi_n \circ P : \sigma \in E_n\}$ of $B_1(C_1^{r_n})$ under Γ_n is an ε-separated set with cardinality at least $e^{\lambda n}$. Since m does not depend on n, by Lemma 3.2 there is a $\mu > 0$ such that $r_n \geq e^{\mu n}$ for all $n \in J$, and so $ht(\alpha, \Omega, \varepsilon) \geq \mu > 0$, yielding the result. $\qquad\square$

Remark 3.4. Notice that in the proof of Theorem 3.3 we made no use of the order structure. In fact Pop and Smith have shown in [25] that Voiculescu-Brown entropy can be alternatively defined using completely contractive linear maps.

The following corollary is an immediate consequence of Theorem 3.3 and the fact that topological entropy does not increase under taking factors or restrictions to closed invariant subsets.

Corollary 3.5. Let A and B be separable exact C^*-algebras and $\alpha : A \to A$ and $\beta : B \to B$ automorphisms with $h_{\mathrm{top}}(S_\alpha) > 0$. Suppose that there exists a surjective contractive linear map $\gamma : B \to A$ such that $\alpha \circ \gamma = \gamma \circ \beta$, or an injective contractive linear map $\rho : A \to B$ such that $\beta \circ \rho = \rho \circ \alpha$. Then $ht(\beta) > 0$. This conclusion also holds more generally if α can be obtained from β via a finite chain of intermediary automorphisms intertwined in succession by maps of the same form as γ or ρ.

Example 3.6. Using Theorem 3.3 we can exhibit positive Voiculescu-Brown entropy in a large class of systems constructed in a operator-theoretic fashion as demonstrated by the following examples. Let $f \in \{-1, 0, 1\}^{\mathbb{Z}}$ be a sequence in which every finite string of -1's and 1's is represented. For each $i = -1, 0, 1$ we set $E_i = \{k \in \mathbb{Z} : f(k) = i\}$. Let $x \in \mathcal{B}(\ell_2(E_{-1} \cup E_1))$ be the operator obtained by specifying by

$$x\xi_k = f(k)\xi_k$$

on the set $\{\xi_k : k \in E_{-1} \cup E_1\}$ of standard basis elements. Let y be any self-adjoint operator in $\mathcal{B}(\ell_2(E_0))$ of norm at most 1 and set $a = (x, y) \in \mathcal{B}(\ell_2(E_{-1} \cup E_1)) \oplus \mathcal{B}(\ell_2(E_0)) \subset \mathcal{B}(\ell_2(\mathbb{Z}))$. Let u be the shift $u\xi_k = \xi_{k+1}$ on $\mathcal{B}(\ell_2(\mathbb{Z}))$ with respect to the canonical basis $\{\xi_k : k \in \mathbb{Z}\}$, and let $A \subset \mathcal{B}(\ell_2(\mathbb{Z}))$ be the C^*-algebra generated by $\{u^n a u^{-n}\}_{n \in \mathbb{Z}}$. By restricting $\mathrm{Ad}\, u$ to A we obtain an automorphism α of A. By our assumption on f, for every $g \in \{-1, 1\}^{\{0,\ldots,n-1\}}$ we can find a $j \in \mathbb{Z}$ such that for each $k = 0, \ldots, n-1$ we have $a\xi_{j-k} = g(k)\xi_{j-k}$ and hence

$$\alpha^k(a)\xi_j = g(k)\xi_j.$$

As a consequence, for each $n \in \mathbb{N}$ the real linear map which sends the kth standard basis element of ℓ_1^n over the real scalars to $\alpha^k(a)$ for each $k = 0, \ldots, n-1$ is an isometry, and the complexification of this map is an isomorphism of norm at most 2 with inverse of norm at most 2. Lemma 2.3 then yields $h_{\mathrm{top}}(T_\alpha) = \infty$, and so it follows from Theorem 3.3 that $ht(\alpha) > 0$ whenever A is exact (this can in fact also be deduced directly from Lemma 3.1—see Remark 3.10). In the case that a is a diagonal operator with respect to the canonical basis of $\ell_2(\mathbb{Z})$, the C^*-algebra A is commutative and the topological entropy of the induced homeomorphism of the pure state space coincides with $ht(\alpha)$ by Proposition 4.8 of [29] and hence is positive (as can also be seen from Theorem A of [14]). The main point of these examples is to demonstrate that positive Voiculescu-Brown entropy can be established in many systems without having either to relate the given system to a topological dynamical system which is known a priori to have positive topological entropy or to rely on measure-theoretic dynamical invariants like CNT or Sauvageot-Thouvenot entropy. It is sufficient, for example, that the eigenspaces of the iterates of an operator of norm 1 corresponding to the respective eigenvalues ± 1 are sufficiently mixed along their intersection.

Using Theorem 3.3 we can also show that, for the shift on the reduced crossed product $C_r^*(\mathbb{F}_\infty)$ of the free group on countably many generators, the topological entropy on the state space is zero (cf. Proposition 2.4):

Proposition 3.7. With α the shift on $C_r^*(\mathbb{F}_\infty)$ we have

$$h_{\mathrm{top}}(T_\alpha) = 0.$$

Proof. By [11] or [8] the Voiculescu-Brown entropy of α is zero, and so we can apply Theorem 3.3. $\qquad\square$

We don't know whether or not the converse of Theorem 3.3 holds. As a test case for this problem we might consider the class of automorphisms of rotation C^*-algebras arising from a matrix in $SL(2, \mathbb{Z})$ with eigenvalues off the unit circle [31, 6]. By [18] such a noncommutative 2-toral automorphism has positive Voiculescu-Brown entropy, but the arguments in [18] using opposite maps and tensor products give no clue about the value of topological entropy on the state space, which we have been unable to determine

in the case of irrational rotation parameters. Note that for rational rotation parameters the topological entropy on the state space is infinite since we obtain the corresponding commutative 2-toral automorphism as a subsystem by restricting to the centre of the C^*-algebra.

We conclude this section with three remarks.

Remark 3.8. To show in the proof of Proposition 4.8 in [29] that the Voiculescu(-Brown) entropy dominates the topological entropy on the pure state space in the separable commutative setting, Voiculescu applies the classical variational principle along with several properties of the Connes-Narnhofer-Thirring entropy. It has been a problem to find a proof of this inequality that does not involve measure-theoretic entropies. In this regard Theorem 3.3 at least gives a geometric picture of why positive topological entropy on the pure state space yields positive Voiculescu-Brown entropy at the C^*-algebra level.

Remark 3.9. As a corollary to Theorem 3.3 we recover the result of Glasner and Weiss asserting that if a homeomorphism of a compact metric space has zero topological entropy then the induced homeomorphism on the space of probability measures also has zero topological entropy [14]. To obtain this corollary we merely need the fact that the topological entropy of a homeomorphism dominates the Voiculescu-Brown entropy of the induced C^*-algebra automorphism, and this can be established by a straightforward partition of unity argument (see the proof of Proposition 4.8 in [29]). Now in the geometric approach of [14] the construction of the key Banach space map is most easily managed in the zero-dimensional situation, and indeed an auxiliary reduction result is invoked to handle the general case. Thus by adopting a C^*-algebraic viewpoint we have obtained a functional-analytically more streamlined geometric proof of Glasner and Weiss's result.

Remark 3.10. Lemma 3.1 also yields a means for obtaining lower bounds for Voiculescu-Brown entropy directly at the completely positive approximation level, and is particularly useful when dealing with dynamical extensions. To illustrate, let $\Omega = \{x_1, \dots, x_n\}$ be a subset of the unit ball of an exact C^*-algebra A, and suppose that the linear map $\Gamma : \ell_1^n \to \operatorname{span}\Omega$ which sends the ith standard basis element of ℓ_1^n to x_i for each $i = 1, \dots, n$ is an isomorphism whose inverse is bounded in norm by some $K \geq 1$. Since Γ is necessarily contractive, it is a K-isomorphism, i.e., $\|\Gamma\|\|\Gamma^{-1}\| \leq K$. Now if $(\phi, \psi, B) \in \operatorname{CPA}(\pi, \Omega, \delta)$ for some faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and $0 < \delta < K^{-1}$, then for any linear combination $\sum c_i x_i$ of the elements of Ω we have

$$\left\| \sum c_i x_i \right\| \leq \left\| \pi\left(\sum c_i x_i \right) - (\psi \circ \phi)\left(\sum c_i x_i \right) \right\| + \left\| (\psi \circ \phi)\left(\sum c_i x_i \right) \right\|$$

$$\leq \delta \sum |c_i| + \left\| \phi\left(\sum c_i x_i \right) \right\|$$

$$\leq K\delta \left\| \sum c_i x_i \right\| + \left\| \phi\left(\sum c_i x_i \right) \right\|$$

so that $\left\| \phi\left(\sum c_i x_i \right) \right\| \geq (1 - K\delta)\left\| \sum c_i x_i \right\|$, and since ϕ is contractive it follows that $\phi|_{\operatorname{span}\Omega}$ is a $(1 - K\delta)^{-1}$-isomorphism onto its image. Hence the Banach-Mazur distance between ℓ_1^n and the image of $\operatorname{span}\Omega$ under ϕ is at most $K(1 - K\delta)^{-1}$, and so by Lemma 3.1 we conclude that

$$\log \operatorname{rcp}(\Omega, \delta) \geq na^{-1}K^{-2}(1 - K\delta)^2$$

where $a > 0$ is a universal constant. Thus if α is an automorphism of A and x is an element of A such that there exists a $K \geq 1$ such that for every $n \in \mathbb{N}$ the linear map which sends the ith standard basis element of ℓ_1^n to $\alpha^{i-1}(x)$ for each $i = 1, \ldots, n$ is an isomorphism whose inverse has norm at most K, then

$$ht(\alpha) \geq \sup_{\delta > 0} ht(\alpha, \{x\}, \delta) \geq \sup_{\delta \in (0, K^{-1})} a^{-1} K^{-2} (1 - K\delta)^2 = a^{-1} K^{-2}.$$

This lower bound for entropy also applies to any automorphism β of an exact C^*-algebra D such that there exists a surjective *-homomorphism $\gamma : D \to A$ with $\gamma \circ \beta = \alpha \circ \gamma$, for in such a case we can lift x under γ to a element $y \in D$ of the same norm, and the linear map sending the ith standard basis element of ℓ_1^n to $\beta^i(y)$ for each $i = 1, \ldots, n$ is an isomorphism with inverse of norm at most K, as is easily checked. More generally, we also obtain $ht(\beta) > 0$ for any automorphism β of a separable exact C^*-algebra such that α can obtained from β via a finite chain of intermediary automorphisms intertwined in succession by contractive linear surjections or linear isometries in the reverse direction (cf. Corollary 3.5).

4. A description of the topological Pinsker algebra in terms of local Voiculescu-Brown entropy

Let $T : X \to X$ be a homeomorphism of a compact metric space and α_T the automorphism of $C(X)$ given by $\alpha_T(f) = f \circ T$ for all $f \in C(X)$. We recall from [2] that a pair $(x, y) \in X \times X \setminus \Delta$ (with Δ denoting the diagonal) is called an *entropy pair* if $h_{\mathrm{top}}(T, \mathcal{U}) > 0$ for every two-element open cover $\mathcal{U} = \{U, V\}$ with $x \in \mathrm{int}(X \setminus U)$ and $x \in \mathrm{int}(X \setminus V)$. We denote by E_X the set of entropy pairs in $X \times X$. The *topological Pinsker factor* is defined as the quotient system arising from the closed T-invariant equivalence relation on X generated by the collection of entropy pairs [4] (here we have adopted the terminology of [13]). This translates at the C^*-algebra level as the α_T-invariant C^*-subalgebra $P_{X,T}$ of $C(X)$ consisting of all $f \in C(X)$ satisfying $f(x) = f(y)$ for every entropy pair (x, y). Note that $P_{X,T}$ is indeed α_T-invariant because E_X is invariant under $T \times T$ by Proposition 3 of [2]. We refer to $P_{X,T}$ as the *topological Pinsker algebra*. It is an analogue of the Pinsker σ-algebra in ergodic theory (see [30]).

The main goal of this section is to apply the argument of the proof of Theorem 3.3 to show that $P_{X,T}$ is equal to the set of all $f \in C(X)$ such that the local Voiculescu-Brown entropy of α_T with respect to the singleton $\{f\}$ is zero. Thus by viewing the dynamics at the function level we are able to obtain a simple description of the topological Pinsker factor/algebra that avoids entropy pairs and the awkward fact that the set $E_X \cup \Delta$ does not always form an equivalence relation (see [15]). As shown in [18], our functional-analytic description of $P_{X,T}$ can be applied to obtain some information concerning the positivity of local Voiculescu-Brown entropy with respect to products of canonical unitaries for certain noncommutative toral automorphisms.

For economy, in this section we will simply write $ht(T, f)$ instead of $ht(\alpha_T, \{f\})$ (as it appears in the definition of Voiculescu-Brown entropy) for any function $f \in C(X)$. Also, given a function $f \in C(X)$ we define the pseudo-metric d_f on X by

$$d_f(x, y) = |f(x) - f(y)|$$

for all $x, y \in X$. We furthermore need to extend the metric space formulation of topological entropy to pseudo-metrics. Thus for any pseudo-metric d on X we set

$$h_d(T) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{sep}_n(T, \varepsilon)$$

where $\operatorname{sep}_n(T, \varepsilon)$ is the largest cardinality of an (n, ε)-separated set, with the latter defined in the same way as for metrics (see Section 2).

The following lemma is a local version of Theorem 3.3 at the level of a single element in the C^*-algebra.

Lemma 4.1. Let $f \in C(X)$. Then $h_{d_f}(T) > 0$ implies $ht(T, f) > 0$.

Proof. Notice that, in the proof of Theorem 3.3, if the compact set K is taken to be finite, then the argument still shows that if we define the pseudo-metric

$$d(\sigma, \omega) = \max_{x \in K} |\sigma(x) - \omega(x)|$$

on the unit ball of the dual and take $h_d(S_\alpha) > 0$ as our hypothesis, then $ht(\alpha, K) > 0$. Thus in our present context we can take K in the proof of Theorem 3.3 to be the singleton $\{f\}$ to obtain the desired conclusion. $\square$

Lemma 4.2. If (x, y) is an entropy pair then $ht(T, f) > 0$ for every $f \in C(X)$ with $f(x) \neq f(y)$.

Proof. Suppose (x, y) is an entropy pair and f is a function in $C(X)$ with $f(x) \neq f(y)$. Set $\delta = |f(x) - f(y)|/3$ and define the two open sets

$$U = \{z \in X : |f(x) - f(z)| > \delta\},$$
$$V = \{z \in X : |f(x) - f(z)| < 2\delta\}.$$

Then $\mathcal{U} = \{U, V\}$ is an open cover with $x \in \operatorname{int}(X \setminus U)$ and $y \in \operatorname{int}(X \setminus V)$, and so $h_{\text{top}}(T, \mathcal{U}) > 0$ by virtue of the fact that (x, y) is an entropy pair. Now if $n \in \mathbb{N}$ and $\mathcal{V}$ is a subcover of $\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-(n-1)}\mathcal{U}$ of smallest cardinality then in each element of $\mathcal{V}$ we can choose a point which is not contained in any other element of $\mathcal{V}$, for otherwise $\mathcal{V}$ would not be a minimal subcover. The set E obtained by collecting these points together is $(n, \delta/2)$-separated relative to the pseudo-metric d_f and has the same cardinality as $\mathcal{V}$. Hence

$$h_{d_f}(T) \geq h_{\text{top}}(T, \mathcal{U}) > 0$$

and so $ht(T, f) > 0$ by Lemma 4.1, yielding the result. $\square$

The converse of Lemma 4.2 is false. This is a consequence of the fact that set $E_X \cup \Delta$ is not necessarily transitive as a relation (see [15]). Indeed if (x, y) and (y, z) are elements of $E_X \cup \Delta$ such that $(x, z) \notin E_X \cup \Delta$, and if $f \in C(X)$ satisfies $f(x) \neq f(z)$, then we cannot have both $f(x) = f(y)$ and $f(y) = f(z)$, whence $ht(T, f) > 0$ by Lemma 4.2.

Theorem 4.3. The topological Pinsker algebra $P_{X,T}$ is equal to the set of all $f \in C(X)$ such that $ht(T, f) = 0$.

Proof. What we need to prove is that for any $f \in C(X)$ we have $ht(T, f) = 0$ if and only if $f(x) = f(y)$ for all entropy pairs (x, y). The "only if" direction follows immediately from Lemma 4.2. For the "if" direction, let $f \in C(X)$ and suppose $ht(T, f) > 0$. Let B be the unital α_T-invariant C^*-subalgebra of $C(X)$ generated by $\{\alpha_T^n(f)\}_{n \in \mathbb{Z}}$, i.e., the closure in $C(X)$ of the set of polynomials in $\{\alpha_T^n(f)\}_{n \in \mathbb{Z}}$. For convenience, in the rest of the proof we will identify points in a compact metric space with pure states on the corresponding unital C^*-algebra. Now if T_B denotes the homeomorphism induced by $\alpha_T|_B$ on the pure state space of B, then since $ht(\alpha_T|_B) \geq ht(T, f) > 0$ we have $h_{\text{top}}(T_B) > 0$ by [29, Prop. 4.8], and so by [2, Props. 1 and 2] there are pure states σ, ω on B such that (σ, ω) is an entropy pair with respect to T_B. We must then have $\sigma(\alpha_T^n(f)) \neq \omega(\alpha_T^n(f))$ for some $n \in \mathbb{Z}$ since σ and ω are distinct, and since the set of entropy pairs is $T \times T$-invariant [2, Prop. 3] we may assume that $n = 0$, i.e., $\sigma(f) \neq \omega(f)$. By [2, Prop. 4] there are pure states σ' and ω' on $C(X)$ extending σ and ω, respectively, such that (σ', ω') forms an entropy pair, and we have $\sigma'(f) \neq \omega'(f)$, completing the proof. $\square$

The system (X, T) is said to have *completely positive entropy* if each of its non-trivial factors has positive topological entropy [3]. Since positive entropy systems always have an entropy pair [2, Props. 1 and 2], $P_{X,T}$ is equal to the scalars (resp. $C(X)$) precisely when the system (X, T) has completely positive entropy (resp. zero entropy), and so we obtain the following corollaries to Theorem 4.3.

Corollary 4.4. The system (X, T) has completely positive entropy if and only if $ht(T, f) > 0$ for all non-constant functions $f \in C(X)$.

Corollary 4.5. We have $h_{\text{top}}(T) > 0$ if and only if $ht(T, f) > 0$ for some $f \in C(X)$.

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