MCDUFF FACTORS FROM AMENABLE ACTIONS AND DYNAMICAL ALTERNATING GROUPS

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Abstract. Given a topologically free action of a countably infinite amenable group on the Cantor set, we prove that, for every subgroup G of the topological full group containing the alternating group, the group von Neumann algebra $\mathscr{L}G$ is a McDuff factor. This yields the first examples of nonamenable simple finitely generated groups G for which $\mathscr{L}G$ is McDuff. Using the same construction we show moreover that if a faithful action $G \curvearrowright X$ of a countable group on a countable set with no finite orbits is amenable then the crossed product of the associated shift action over a given II_1 factor is a McDuff factor. In particular, if H is a nontrivial countable ICC group and $G \curvearrowright X$ is a faithful amenable action of a countable ICC group on a countable set with no finite orbits, then the group von Neumann algebra of the generalized wreath product $H \wr_X G$ is a McDuff factor. Our technique can also be applied to show that if H is a nontrivial countable group and $G \curvearrowright X$ is an amenable action of a countable group on a countable set with no finite orbits then the generalized wreath product $H \wr_X G$ is Jones–Schmidt stable.

1. Introduction

In operator algebra theory central sequences have long played a significant role in addressing problems in and around amenability, having been used both as a mechanism for producing various examples beyond the amenable horizon and as a point of leverage for teasing out the finer structure of amenable operator algebras themselves. In the early 1940s Murray and von Neumann exhibited (sticking to the separable realm, as we do henceforth) the first example of a II_1 factor nonisomorphic to the hyperfinite II_1 factor R by showing that the free group factor $\mathscr{L}F_2$, unlike R, does not possess nontrivial central sequences, i.e., does not have what they called *property Gamma* [24]. In the late 1960s McDuff employed central sequences and an iterated group-theoretic construction to engineer an uncountable infinity of pairwise nonisomorphic II_1 factors [22]. Shortly thereafter she gave a characterization of II_1 factors admitting a pair of central sequences that asymptotically noncommute as those which tensorially absorb R, i.e., those that have the *McDuff property* [23]. A bit later in the 1970s Connes put property Gamma to work in the proof of his theorem that injectivity implies hyperfiniteness, a cornerstone in the classification of injective von Neumann algebras [7]. On the topological side, central sequences (both in operator and tracial norms) have proven their utility many times over in the corresponding Elliott classification program for simple separable nuclear C*-algebras, starting in the 1990s and with increasing intensity over the last decade. For instance, versions of property Gamma and the McDuff property formulated in terms of the uniform trace norm were critical ingredients in recent work on the Toms–Winter conjecture [3, 4], one outcome of which was the equivalence of finite nuclear dimension and $\mathcal{Z}$-stability (tensorial absorption of the Jiang–Su algebra) for nonelementary simple separable unital nuclear C*-algebras. This equivalence permitted one to install

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the relatively tractable property of $\mathcal{Z}$-stability as the operative regularity hypothesis in the final classification theorem [12, 10, 28] and cemented its position as the C*-algebraic analogue of being McDuff.

Many ICC groups will give rise to II_1 factors with property Gamma or the McDuff property on account of asymptotic commutativity relations within the group itself, which can be arranged by taking products and/or suitable inductive limit constructions (the ICC property—which asks that the conjugacy class of every nontrivial element be infinite—guarantees factoriality, and indeed is equivalent to it by a result of Murray and von Neumann). Coming up with examples of simple finitely generated groups that yield such II_1 factors is more difficult. The problem of identifying when an infinite group is simple and finitely generated can itself be a delicate task, but there is at least one rich source of examples coming from dynamics, namely the alternating groups $\mathsf{A}(\Gamma, X)$ of minimal subshift actions $\Gamma \curvearrowright X$ of countably infinite groups on the Cantor set [25]. These are subgroups of the topological full group (i.e., the group of homeomorphisms locally implemented by elements of the acting group) that, in the case of many acting groups Γ including $\mathbb{Z}$, are known to coincide with the commutator subgroup. Juschenko and Monod proved that the topological full group of a minimal $\mathbb{Z}$-action on the Cantor set is always amenable, which, by passing to the commutator subgroup and specializing to subshift actions, gave the first examples of amenable infinite simple finitely generated groups [17]. When Γ is not virtually cyclic, however, the alternating group of a minimal subshift action can fail to be amenable [9, 27, 20]. Nevertheless, the first author and Tucker-Drob showed that if Γ is amenable and the action is topologically free then for every subgroup of the topological full containing the alternating group the group von Neumann algebra (which is always a II_1 factor in this case) has property Gamma [20]. The goal of the present paper is to strengthen this last conclusion to the McDuff property:

Theorem A. Let $\Gamma \curvearrowright X$ be a topologically free continuous action of a countably infinite amenable discrete group on the Cantor set, and let G be a subgroup of the topological full group $[[\Gamma \curvearrowright X]]$ containing the alternating group $\mathsf{A}(\Gamma, X)$. Then the von Neumann algebra $\mathscr{L}G$ is a McDuff II_1 factor.

Applying the above result to the free minimal expansive actions constructed in [9, 27] we obtain the first examples of nonamenable simple finitely generated groups whose von Neumann algebra is a McDuff factor. The topologically free minimal expansive actions constructed in Section 8 of [20] give us moreover uncountably many pairwise nonisomorphic such groups.

We will actually show something a little more general (see Theorem 2.1). The virtue of formulating Theorem A as we have is that the groups G in question are automatically ICC.

The argument in [20] for deriving property Gamma makes use of finite permutational wreath products inside of $\mathsf{A}(\Gamma, X)$ that can be expressed spectrally as permutational Bernoulli actions $S_F \curvearrowright \{0,1\}^F$ indexed by Følner sets F of Γ. A set of measure one half is constructed in each of the Bernoulli spaces $\{0,1\}^F$ via a summation condition on the coordinates that takes into account the Følner boundary effect. This set is shown to be approximately invariant using the central limit theorem, as was done by Kechris and Tsankov in [18] for the different purpose of obtaining a characterization of amenability for actions in terms of the existence of approximately invariant sets of measure one half for the associated generalized Bernoulli actions. The corresponding projection e in the group von Neumann algebra is then approximately central to within a prescribed tolerance, yielding property Gamma. A natural strategy for boosting this to the McDuff property would be to take the tensor factors in the group algebra of the wreath product to be something noncommutative instead of the algebra $\mathbb{C}^2$ sitting over the original Bernoulli base $\{0,1\}$, reinterpret the original binary alternative as a choice of a seed projection p in these (common) tensor factors, and then choose a second seed projection q that is far from commuting with p and using it in the same way as p to construct another almost central projection f. We have been unable to determine, however, if such seed projections p and q can be found so that the corresponding e and f asymptotically noncommute as the Følner sets F become more and more invariant. In fact we suspect, on the basis of numerical computations carried out for us by Giles Gardam, that such e and f will always asymptotically commute, even when p and q are approximately freely related.

What we have discovered is that one can dispense with the above probabilistic approach altogether and instead start with a projection p in the noncommutative Bernoulli base which has trace extremely close to 1, close enough so that if we copy it out into a single elementary tensor over the Følner core of the set F then we will obtain a projection $\tilde{p}$ with trace approximately one half. This requires that the Bernoulli base be a finite-dimensional $*$-subalgebra B of $\mathscr{L}G$ of very large dimension. Some basic representation theory for finite alternating groups (guaranteeing that B can be chosen with enough noncommutativity) then enables us to construct a partial isometry v in B such that if we copy it into an elementary tensor $\tilde{v}$ over the Følner core of F, just like we did to p in order to produce $\tilde{p}$, then we will have $\tilde{v}^*\tilde{v} = \tilde{p}$ and the projections $\tilde{v}\tilde{v}^*$ and $\tilde{p}$ will commute and be approximately independent, which implies that the commutator $[\tilde{v}, \tilde{v}^*]$ is bounded away from zero in trace norm. Finally one observes that both $\tilde{v}$ and $\tilde{v}^*$ are approximately central, with the tolerance being controlled by the approximate invariance of the set F. From this we conclude the McDuff property.

Our construction can also be applied to establish a connection to amenability for actions in the spirit of Kechris and Tsankov, only now via the McDuff property for the crossed product of shift actions over a II_1 factor. In this setting there is also an obstruction related to inner amenability of the group that prevents one from obtaining a full characterization of amenability for the action. Recall that a group action $G \curvearrowright X$ on a set is *amenable* if there is a finitely additive G-invariant probability measure on X, or equivalently if there is a state on $\ell^\infty(X)$ that is invariant under the induced G-action. Every action of an amenable group is amenable, but many nonamenable groups admit nontrivial and even faithful amenable actions (see [32] for the case of free groups and [29] for further discussion and references). A group G is *inner amenable* if there exists a finitely additive atomless probability measure on $G \setminus \{e\}$ which is invariant under the conjugation action of G, which in the case that G is ICC simply means that the conjugation action $G \curvearrowright G \setminus \{e\}$ is amenable. Inner amenability fails for free groups on two or more generators but does hold for many nonamenable groups. It is implied by property Gamma [8] (so that the groups G in Theorem A all satisfy it, as was already shown in [20]) but is strictly weaker [30].

If the action $G \curvearrowright X$ has a finite orbit then it is amenable for obvious reasons and the factoriality condition on the crossed product below fails, and so this case is naturally omitted from the theorem statement.

Theorem B. Let M be a II_1 factor with trace τ. Let $G \curvearrowright X$ be an amenable action of a countable group on a countable set, and suppose that the action has no finite orbits. Suppose furthermore that the crossed product $M^{\overline{\otimes} X} \rtimes G$ of the associated shift action $G \curvearrowright M^{\overline{\otimes} X}$ is a II_1 factor (which will be the case, for example, when the action $G \curvearrowright X$ is faithful). Then $M^{\overline{\otimes} X} \rtimes G$ is McDuff.

In the special case when the action is that of an amenable group on itself by left translation, the conclusion of the theorem follows from a general result of Bédos on actions of amenable groups on McDuff II_1 factors [1].

When G is non-inner-amenable, the amenability of the action $G \curvearrowright X$ is actually equivalent to both the McDuff property and property Gamma for $M^{\overline{\otimes} X} \rtimes G$, with the implication from property Gamma to amenability following from [8] and Lemma 2.7 of [31], as observed in Proposition 2.8 of [26] (the non-inner-amenability assumption cannot be dropped here, as we illustrate in Example 3.1). Using deformation/rigidity techniques, it was proved by Isono and Marrakchi that if G is nonamenable then the crossed product $M^{\overline{\otimes} G} \rtimes G$ of the shift action is prime [14] and by Patchell that if G is non-inner-amenable and ICC, the action $G \curvearrowright X$ is transitive and nonamenable, and the stabilizer of some nonempty finite subset of X is amenable (a kind of mixing condition) then $M^{\overline{\otimes} X} \rtimes G$ is prime [26].

As a special case of Theorem B, we obtain the following result for II_1 factors arising as group von Neumann algebras of generalized wreath products that conform, as crossed products, to the framework of the theorem statement. Recall that the generalized (restricted) wreath product $H \wr_X G$ of two groups relative to an action $G \curvearrowright X$ on a set is defined as the semidirect product $H^{\oplus X} \rtimes G$ where G acts on the restricted direct sum $H^{\oplus X}$ by $g \cdot (h_x)_{x \in X} = (h_{g^{-1}x})_{x \in X}$. In this case there is a natural isomorphism $\mathscr{L}(H \wr_X G) \cong \mathscr{L}(H)^{\overline{\otimes} X} \rtimes G$, and under this identification we get the two factoriality conditions in Theorem B precisely when both H and $H \wr_X G$ are ICC.

Theorem C. Let H be a nontrivial countable ICC group and $G \curvearrowright X$ be an amenable action of a countable group on a countable set with no finite orbits such that the generalized wreath product $H \wr_X G$ is ICC (which will be the case, for example, if G is ICC). Then $\mathscr{L}(H \wr_X G)$ is a McDuff II_1 factor.

As before, when G is non-inner-amenable the amenability of the action $G \curvearrowright X$ is equivalent to both the McDuff property and property Gamma for $\mathscr{L}(H \wr_X G)$.

It was shown in [13, 2] that many generalized wreath products are W*-superrigid, i.e., uniquely determined as groups by their group von Neumann algebra. The base groups in [13, 2] are Abelian, in contrast to the above ICC hypothesis on H, which is there to guarantee the factoriality of $\mathscr{L}H$ and hence the applicability of Theorem B. The recent papers [6, 5] however treat wreath-like products that include ones with ICC bases.

In response to a question of Robin Tucker-Drob, we show that our technique can also be used to establish the following result on JS-stability for generalized wreath products. A countable discrete p.m.p. (probability-measure-preserving) equivalence relation is said to be *JS-stable* if it satisfies the McDuff-like property of being isomorphic to its product with the unique ergodic hyperfinite p.m.p. equivalence relation [16]. A countable group is *JS-stable* if it admits a free ergodic p.m.p. action whose orbit equivalence relation is JS-stable. We thank Robin Tucker-Drob for a suggestion that permitted us to remove the non-Abelianness assumption on H in our original version of the theorem.

Theorem D. Let H be a nontrivial countable group and $G \curvearrowright X$ an amenable action of a countable group on a countable set with no finite orbits. Then the generalized wreath product $H \wr_X G$ is JS-stable.

The details of the proof of Theorem A, along with a review of definitions and basic background material, are contained in Section 2. The proofs of Theorem B and Theorem D are contained in Section 3 and 4.

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2. Proof of Theorem A

Let M be a II_1 factor and τ its faithful normal tracial state. Our factors are always assumed to be separable for the trace norm $\|a\|_2 = \tau(a^*a)^{1/2}$. A bounded sequence (a_n) in M is said to be a *central sequence* if $\|[a_n, b]\|_2 \to 0$ for every $b \in M$. The factor M has the *McDuff property* if $M\overline{\otimes}R \cong M$ where R is the hyperfinite II_1 factor. By a theorem of McDuff [23], M has the McDuff property if and only if there exist central sequences (a_n) and (b_n) in M such that $\|[a_n, b_n]\|_2 \not\to 0$. It is this central sequence criterion that we will use to establish the McDuff property in Theorem A. Later in Section 3 we will also invoke *property Gamma*, which asks for the existence of a central sequence of unitaries with trace zero, a condition that is readily seen to be weaker than the McDuff property (which itself can also be characterized by the existence of a sequence of unital 2×2 matrix subalgebras that is central in the obvious sense).

Let Γ be a countable discrete group and $\Gamma \curvearrowright X$ a continuous action on the Cantor set. The *topological full group* $[[\Gamma \curvearrowright X]]$ of the action is the group of all homeomorphisms h from X to itself for which there exist a clopen partition $\{A_1, \dots, A_n\}$ of X and $s_1, \dots, s_n \in \Gamma$ such that $h(x) = s_i x$ for all $i = 1, \dots, n$ and $x \in A_i$. This group is countable because Γ is countable and X admits only countably many clopen partitions.

Next we recall from [25] the definition of the alternating group $\mathsf{A}(\Gamma, X)$. Let $d \in \mathbb{N}$ and write S_d for the symmetric group on $\{1, \dots, d\}$. Consider the homomorphisms $\psi : \mathsf{S}_d \to [[\Gamma \curvearrowright X]]$ for which there exist pairwise disjoint sets $A_1, \dots A_d \subseteq X$ such that the image of a permutation σ under ψ acts as the identity on the complement of $A_1 \sqcup \dots \sqcup A_d$ and for each $i = 1, \dots, d$ maps A_i to $A_{\sigma(i)}$ via some element of Γ. The image of these homomorphisms generate a subgroup $\mathsf{S}_d(\Gamma, X)$ of $[[\Gamma \curvearrowright X]]$, and we can also consider the subgroup $\mathsf{A}_d(\Gamma, X)$ of $\mathsf{S}_d(\Gamma, X)$ generated by the images of the restrictions of the homomorphisms to the alternating group $\mathsf{A}_d \subseteq \mathsf{S}_d$. The group $\mathsf{A}_3(\Gamma, X)$ is called the *alternating group* of the action $\Gamma \curvearrowright X$ and written $\mathsf{A}(\Gamma, X)$. When the actions has no finite orbits one has $\mathsf{A}(\Gamma, X) = \mathsf{A}_3(\Gamma, X)$ for every $d \geq 2$. It is shown in [25] that if the action of Γ is minimal then $\mathsf{A}_d(\Gamma, X)$ is simple, while if Γ is finitely generated and the action is expansive (equivalently, is a subshift action with finitely many symbols) and has no orbits of cardinality less than 5 then $\mathsf{A}_d(\Gamma, X)$ is finitely generated.

We invariably denote by τ the unique normal tracial state on a II_1 factor, with the particular algebra being understood from the context. The identity element of a group will always be written e.

By Proposition 5.1 of [20], if $\Gamma \curvearrowright X$ is a topologically free continuous action of group on the Cantor set then every subgroup of $[[\Gamma \curvearrowright X]]$ containing $\mathsf{A}(\Gamma, X)$ is ICC. Theorem A is thus a consequence of the following result. The amenability of the group Γ will be applied in the form of the Følner property, which requires, for every finite set $e \in K \subseteq \Gamma$ and $\delta > 0$, that there exist a nonempty finite set $T \subseteq \Gamma$ such that $|\bigcap_{s\in K} s^{-1}T| \geq (1-\delta)|T|$.

Theorem 2.1. *Let $\Gamma \curvearrowright X$ be an action of a countably infinite amenable group on the Cantor set with at least one free orbit. Then the group von Neumann algebra of any ICC subgroup of $[[\Gamma \curvearrowright X]]$ containing $\mathsf{A}(\Gamma, X)$ is a McDuff II_1 factor.*

Proof. Factoriality follows from the ICC condition.

Let Ω be a finite symmetric subset of $[[\Gamma \curvearrowright X]]$ and let $\varepsilon > 0$. By the definition of the topological full group, we can find a clopen partition $\mathcal{P} = \{P_1, \dots, P_N\}$ of X such that for each $h \in \Omega$ there exist $s_{h,1}, \dots, s_{h,N} \in \Gamma$ for which $hx = s_{h,i}x$ for every $i = 1, \dots, N$ and $x \in P_i$. Let K be the collection of all of these $s_{h,i}$ together with the identity element of Γ.

Take a $\delta > 0$ such that

$$4^{-\delta/(1-\delta)} \geq 1 - \frac{\varepsilon}{32}. \tag{2.1}$$

By taking a logarithm and applying l'Hôpital's rule, one can verify, for all $\theta \in \mathbb{R}$, that

$$\lim_{r\to\infty} (2 \cdot 2^{-r^{-1}} - 1)^{\theta r} = \frac{1}{4^\theta}. \tag{2.2}$$

It follows that we can find an $r_0 > 0$ so that, for all $r \geq r_0$ and $\theta \in \{1, \delta/(1-\delta)\}$,

$$\left|(2 \cdot 2^{-r^{-1}} - 1)^{\theta r} - \frac{1}{4^\theta}\right| \leq \frac{\varepsilon}{32}. \tag{2.3}$$

By hypothesis there exists an $x_0 \in X$ such that the action of Γ on the orbit Γx_0 is free. By amenability, there exists a nonempty finite set $T \subseteq \Gamma$ such that the set $T' = \bigcap_{s\in K} s^{-1}T$ (which is a subset of T since $e \in K$) satisfies $|T'| \geq \lceil (1-\delta)|T|\rceil$. Since Γ is infinite, we can choose T so that its cardinality is larger that r_0, and also large enough so that

$$(2^{-((1-\delta)|T|)^{-1}})^{\lceil (1-\delta)|T|\rceil} \geq \frac{1}{2} - \frac{\varepsilon}{8}. \tag{2.4}$$

Since the action of Γ on the orbit of x_0 is free and Γ is infinite, we can find a sequence (d_k) in Γ such that the points td_kx_0 for $k \in \mathbb{N}$ and $t \in T$ are pairwise distinct. By the pigeonhole principle we can find a subsequence (d_{k_j}) such that for every $t \in T$ the points $td_{k_j}x_0$ for $j \in \mathbb{N}$ belong to a common member of $\mathcal{P}$. In particular, using continuity we can find a finite set $D \subset \Gamma$ of cardinality as large as we wish (to be specified below) and a clopen neighbourhood B of x_0 such that the sets tdB for $d \in D$ and $t \in T$ are pairwise disjoint and for every $t \in T$ the sets tdB for $d \in D$ are contained in a common member of $\mathcal{P}$. This choice of D guarantees the existence of a function $\theta \in K^{\Omega\times T}$ defined by $htdx = \theta(h,t)tdx$ for $h \in \Omega, t \in T, d \in D$, and $x \in B$.

For each $s \in K$ we have $sT' \subseteq T$ and so for every $h \in \Omega$ we can find a $\sigma_h \in \mathrm{Sym}(T)$ such that $\sigma_h t = \theta(h,t)t$ for all $t \in T'$. Consider the alternating group $\mathsf{A}(D)$. We regard the product $\mathsf{A}(D)^T$ as a subgroup of $\mathsf{A}(\Gamma, X)$ with an element $(\omega_t)_{t\in T}$ in $\mathsf{A}(D)^T$ acting by $dtx \mapsto \omega_t(d)tx$ for all $x \in B$, $t \in T$, and $d \in D$ and by $x \mapsto x$ for all $x \in X \setminus TDB$.

By the representation theory of alternating groups [11, 15], the Artin–Wedderburn decomposition of $\mathscr{L}\mathsf{A}(D)$ takes the form $\mathbb{C} \oplus \left(\bigoplus_{l\in L} \mathbb{M}_{k_l}\right)$ where $k_l \geq |D| - 1$ for every l in the finite set L. Write tr for the tracial state on $\mathscr{L}\mathsf{A}(D)$ associated to the left regular representation of $\mathsf{A}(D)$ on $\ell^2(\mathsf{A}(D))$, i.e., the vector state $a \mapsto \langle a\delta_e, \delta_e\rangle$ where $\{\delta_g : g \in \mathsf{A}(D)\}$ is the canonical orthonormal basis for $\ell^2(\mathsf{A}(D))$. The summand $\mathbb{C}$ in the Artin–Wedderburn decomposition corresponds to the trivial representation of $\mathsf{A}(D)$ and thus must act on $\ell^2(\mathsf{A}(D))$ as the orthogonal projection onto the one-dimensional subspace of $\mathsf{A}(D)$-invariant vectors, which is spanned by the unit vector $\xi = |\mathsf{A}(D)|^{-1/2}\sum_{g\in\mathsf{A}(D)}\delta_g$. It follows that the projection $f := (1,0) \in \mathbb{C}\oplus\big(\bigoplus_{l\in L}\mathbb{M}_{k_l}\big)$ satisfies

$$\mathrm{tr}(f) = \langle f\delta_e, \delta_e\rangle = \langle\langle\delta_e,\xi\rangle\xi,\delta_e\rangle = \frac{1}{|\mathsf{A}(D)|}. \tag{2.5}$$

Consequently there exist $\lambda_l > 0$ for $l \in L$ with $|\mathsf{A}(D)|^{-1} + \sum_{l\in L}\lambda_l = 1$ (in fact $\lambda_l = k_l^2/(1+\sum_{l\in L}k_l^2)$ by standard theory) such that for all $a = (b,(c_l)_{l\in L}) \in \mathbb{C}\oplus\big(\bigoplus_{l\in L}\mathbb{M}_{k_l}\big)$ we have, denoting by tr_l the unique tracial state on M_{k_l},

$$\mathrm{tr}(a) = \frac{b}{|\mathsf{A}(D)|} + \sum_{l\in L}\lambda_l\,\mathrm{tr}_l(c_l). \tag{2.6}$$

For $l \in L$ write $\{e^{(l)}_{i,j}\}_{1\le i,j\le k_l}$ for the matrix units of the summand M_{k_l}. Set $d_l = \lfloor 2^{-((1-\delta)|T|)^{-1}}k_l\rfloor$ and define

$$\begin{aligned} p &= \sum_{l\in L}\sum_{i=1}^{d_l} e^{(l)}_{i,i},\\ q &= \sum_{l\in L}\Bigg(\sum_{i=1}^{2d_l-k_l} e^{(l)}_{i,i} + \sum_{i=d_l+1}^{k_l} e^{(l)}_{i,i}\Bigg),\\ v &= \sum_{l\in L}\Bigg(\sum_{i=1}^{2d_l-k_l} e^{(l)}_{i,i} + \sum_{i=d_l+1}^{k_l} e^{(l)}_{i,i-k_l+d_l}\Bigg). \end{aligned}$$

We have $v^*v = p$, $vv^* = q$, and $vpq = pq$. Moreover, writing $e_l = \sum_{i=1}^{d_l} e_{i,i}$ we have, by (2.6),

$$\mathrm{tr}(p) = \mathrm{tr}(q) = \sum_{l\in L}\mathrm{tr}(e_l) = \sum_{l\in L}\lambda_l\frac{d_l}{k_l}.$$

It follows, by virtue of the equation $\sum_{l\in L}\lambda_l = 1 - |\mathsf{A}(D)|^{-1}$, our choice of d_l, and the fact that $k_l \ge |D| - 1$ for every $l \in L$, that we can make the quantity $\mathrm{tr}(p)$ as close to $2^{-((1-\delta)|T|)^{-1}}$ as we wish by taking $|D|$ sufficiently large. Thus given an $\varepsilon' > 0$ we can take D to have large enough cardinality so that

$$\mathrm{tr}(p) \ge 2^{-((1-\delta)|T|)^{-1}} - \varepsilon'.$$

Since $\mathrm{tr}(pq) = \sum_{l\in L}\lambda_l(2d_l - k_l)/k_l$, we may similarly assume that $|D|$ is large enough so that

$$2\cdot 2^{-((1-\delta)|T|)^{-1}} - 1 - \varepsilon' \le \mathrm{tr}(pq) \le 2\cdot 2^{-((1-\delta)|T|)^{-1}} - 1.$$

Therefore by taking ε' small enough we can guarantee, in view of (2.4), that

$$\mathrm{tr}(p)^{\lceil(1-\delta)|T|\rceil} \ge \big(2^{-((1-\delta)|T|)^{-1}} - \varepsilon'\big)^{\lceil(1-\delta)|T|\rceil} \ge \frac{1}{2} - \frac{\varepsilon}{4} \tag{2.7}$$

and, in view of (2.3), that

$$\mathrm{tr}(pq)^{\delta|T|} \ge \big(2\cdot 2^{-((1-\delta)|T|)^{-1}} - 1 - \varepsilon'\big)^{\delta|T|} \ge 1 - \frac{\varepsilon}{8}. \tag{2.8}$$

Note also by (2.3) that

$$\mathrm{tr}(pq)^{(1-\delta)|T|} \leq (2 \cdot 2^{-((1-\delta)|T|)^{-1}} - 1)^{(1-\delta)|T|} \leq \frac{1}{4} + \frac{\varepsilon}{4}. \tag{2.9}$$

For every $a \in \mathscr{L}\mathsf{A}(D)$ and $R \subseteq T$ write $\tilde{a}_R$ for the element $\otimes_{t\in T} a_t \in \mathscr{L}\mathsf{A}(D)^{\otimes T} = \mathscr{L}(\mathsf{A}(D)^T) \subseteq \mathscr{L}\mathsf{A}(\Gamma, X)$ where $a_t = a$ if $t \in R$ and $a_t = 1$ otherwise. Note that the restriction of the trace τ on $\mathscr{L}\mathsf{A}(\Gamma, X)$ to $\mathscr{L}(\mathsf{A}(D)^T)$, under the identification of the latter with $\mathscr{L}\mathsf{A}(D)^{\otimes T}$, is equal to the tensor product trace $\mathrm{tr}^{\otimes T}$. Note also that for $h \in \Omega$, $a \in \mathscr{L}\mathsf{A}(D)$, and $S \subseteq T'$ we have $u_h \tilde{a}_S u_h^{-1} = \tilde{a}_{\sigma_h S}$. Since T' has cardinality at least $\lceil (1-\delta)|T| \rceil$, we can choose a $S \subseteq T'$ with exactly this cardinality. It follows using (2.7) and (2.9) that

$$\begin{aligned}
\|[\tilde{v}_S, \tilde{v}_S^*]\|_2^2 &= \|\tilde{p}_S - \tilde{q}_S\|_2^2 \\
&= \tau(\tilde{p}_S + \tilde{q}_S - 2\widetilde{pq}_S) \\
&= 2\big(\mathrm{tr}(p)^{|S|} - \mathrm{tr}(pq)^{|S|}\big) \\
&\geq \frac{1}{2} - \varepsilon.
\end{aligned}$$

Furthermore, for all $h \in \Omega$ we have $|\sigma_h S \setminus S| \leq \delta|T|$ and hence, using (2.8),

$$\begin{aligned}
\|u_h \tilde{v}_S u_h^{-1} - \tilde{v}_S\|_2^2 &= \|\tilde{v}_{\sigma_h S} - \tilde{v}_S\|_2^2 \\
&= \|\tilde{v}_{\sigma_h S \cap S}(\tilde{v}_{\sigma_h S\setminus S} - \tilde{v}_{S\setminus \sigma_h S})\|_2^2 \\
&\leq \|\tilde{v}_{\sigma_h S\cap S}\|^2 \|\tilde{v}_{\sigma_h S\setminus S} - \tilde{v}_{S\setminus\sigma_h S}\|_2^2 \\
&\leq (\|\tilde{v}_{\sigma_h S\setminus S} - 1\|_2 + \|1 - \tilde{v}_{S\setminus\sigma_h S}\|_2)^2 \\
&= 4\|\tilde{v}_{\sigma_h S\setminus S} - 1\|_2^2 \\
&= 4\tau(\tilde{v}^*_{\sigma_h S\setminus S}\tilde{v}_{\sigma_h S\setminus S} - \tilde{v}^*_{\sigma_h S\setminus S} - \tilde{v}_{\sigma_h S\setminus S} + 1) \\
&\leq 8\tau(1 - \tilde{v}_{\sigma_h S\setminus S}) \\
&= 8(1 - \mathrm{tr}(pq)^{|\sigma_h S\setminus S|}) \\
&\leq 8(1 - \mathrm{tr}(pq)^{\delta|T|}) \\
&\leq \varepsilon.
\end{aligned} \tag{2.10}$$

Thus if we take such $\tilde{v}_S$ and $\tilde{v}_S^*$ over an increasing sequence of sets Ω with union $[[\Gamma \curvearrowright X]]$ and a sequence of tolerances ε converging to zero, we obtain noncommuting approximately central sequences for $\mathscr{L}[[\Gamma \curvearrowright X]]$ inside of $\mathscr{L}\mathsf{A}(\Gamma, X)$. This yields the McDuff conclusion in the theorem. □

In the above proof we could have avoided the application of l'Hospital's rule by alternatively taking the seed projections p and q to be approximately independent with respect to the trace. In fact this is how we will proceed in the proof of Theorem B, where the whole picture simplifies due to the diffuseness of the seed space.

3. Proof of Theorem B

It is a standard fact (provable in the same way as for groups acting on themselves by translation) that amenability for a group action $G \curvearrowright X$ on a set is equivalent to the following Følner property: for every finite set $e \in K \subseteq G$ and $\delta > 0$ there exists a nonempty finite set $T \subseteq X$ such that

$|\bigcap_{s\in K} s^{-1}T| \geq (1-\delta)|T|$, in which case we say that T is (K,δ)-*invariant*. Such sets T are informally referred to as *Følner sets* with the understanding that a certain degree of approximate invariance is at play. It can also be shown, in the say way as for groups themselves with respect to the left regular representation, that amenability for an action $G \curvearrowright X$ is equivalent to the existence of approximately invariant unit vectors for the induced unitary representation π on $\ell^2(X)$, i.e., to the existence, for every finite set $F \subseteq G$ and $\delta > 0$, of a unit vector $\xi \in \ell^2(X)$ satisfying $\|\pi(g)\xi - \xi\| < \delta$ for every $g \in F$. If G itself is amenable (i.e., the action of G on itself by left translation is amenable) then all of its actions are amenable, as is easy to verify. See Section 4.1 of [19] for more information.

Proof of Theorem B. We wish to show, given a finite subset Ω of $M^{\overline{\otimes} X} \rtimes G$ and an $\varepsilon > 0$, that there exists a pair of elements in $M^{\overline{\otimes} X} \rtimes G$ whose commutators with elements in Ω have trace norm less than ε and whose commutator with each other has trace norm bounded away from zero independently of ε. It evidently suffices to check this for Ω drawn from a subset of the crossed product which generates a trace-norm dense subalgebra. We may thus assume that $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 consists of elementary tensors in $M^{\overline{\otimes} X}$ of finite support and Ω_2 is the set $\{u_g\}_{g\in F}$ of canonical unitaries corresponding to elements in a given finite subset F of G containing e. Write Y for the union of the supports of elements in Ω_1.

Choose a $\delta > 0$ small enough so that $(1 - 2^{-2\delta/(1-\delta)}) \leq \varepsilon/8$. Since by assumption the action on X has no finite orbits, the cardinality of the Følner sets for the action will tend to infinity as we demand more and more invariance. We can thus find an (F,δ)-invariant subset T of X that is disjoint from Y by first shrinking the tolerance δ a little and then finding a Følner set for this tightened tolerance that has sufficiently large cardinality so that its intersection with the complement of Y will do the job. Set $S = \bigcap_{s\in F} s^{-1}T$, which by (F,δ)-invariance satisfies $|S| \geq (1-\delta)|T|$. Since M is a II_1 factor it contains commuting projections p and q of trace $2^{-|S|^{-1}}$ which are independent, i.e., $\tau(pq) = \tau(p)\tau(q)$ (for example, choose a masa in M, write it in the form $A \overline{\otimes} A$ in such a way that the trace τ on M restricts to $\tau|_{A\otimes 1} \otimes \tau|_{1\otimes A}$ under the canonical identification of the two copies of A with $A \otimes 1$ and $1 \otimes A$, and take $f \otimes 1$ and $1 \otimes f$ for a suitable projection f). Since the projections $p - pq$ and $q - pq$ have the same trace they are Murray–von Neumann equivalent, and so we can construct a partial isometry $v \in M$ such that $v^*v = p$, $vv^* = q$, and $vpq = pq$. Note that

$$\tau(v) = \tau(qvp) = \tau(vpq) = \tau(pq). \tag{3.1}$$

For $a \in M$ and $R \subseteq T$ write $\tilde{a}_R$ for the element $\otimes_{t\in T} a_t \in M^{\overline{\otimes} X}$ where $a_t = a$ if $t \in R$ and $a_t = 1$ otherwise. Then $\tau(\tilde{p}_S) = \tau(\tilde{q}_S) = \tau(p)^{|S|} = 1/2$ and $\tau(\widetilde{pq}_S) = \tau(p)^{2|S|} = 1/4$. By our choice of T, both $\tilde{v}$ and $\tilde{v}^*$ commute with the elements in Ω_1. Moreover

$$\|[\tilde{v}_S, \tilde{v}_S^*]\|_2^2 = \|\tilde{p}_S - \tilde{q}_S\|_2^2 = \tau(\tilde{p}_S) + \tau(\tilde{q}_S) - 2\tau(\widetilde{pq}_S) = \frac{1}{2},$$

and, using (3.1) and estimating as in (2.10) in the proof of Theorem 2.1, we have, for every $g \in F$,

$$\begin{aligned}
\|u_g^{-1}\tilde{v}_S u_g - \tilde{v}_S\|_2^2 &= \|\tilde{v}_{gS} - \tilde{v}_S\|_2^2 \\
&\leq 4\|\tilde{v}_{gS\setminus S} - 1\|_2^2 \\
&= 4\tau(\tilde{v}_{gS\setminus S}^*\tilde{v}_{gS\setminus S} - \tilde{v}_{gS\setminus S}^* - \tilde{v}_{gS\setminus S} + 1) \\
&\leq 8\tau(1 - \tilde{v}_{gS\setminus S})
\end{aligned}$$

$$
\begin{aligned}
&= 8(1 - \mathrm{tr}(pq)^{|gS\setminus S|}) \\
&\leq 8(1 - (2^{-2|S|^{-1}})^{\delta|T|}) \\
&\leq 8(1 - 2^{-2\delta/(1-\delta)}) \\
&\leq \varepsilon.
\end{aligned}
$$

Taking such $\tilde{v}_S$ and $\tilde{v}_S^*$ over an increasing sequence of finite sets Ω with trace-norm dense union in $M^{\overline{\otimes} X} \rtimes G$ and a sequence of tolerances ε converging to zero, we obtain central sequences witnessing the McDuff property. □

As mentioned in the introduction, it follows by [8] and Lemma 2.7 of [31] that if G in the context of Theorem B is assumed to be non-inner-amenable then the action $G \curvearrowright X$ is amenable whenever $M^{\overline{\otimes} X} \rtimes G$ has property Gamma, and so in this case amenability of the action $G \curvearrowright X$ is equivalent to both the McDuff property and property Gamma for $M^{\overline{\otimes} X} \rtimes G$. The non-inner-amenability cannot be dropped here, as the following example illustrates.

Example 3.1. Let $G := F_2 \times F_2^{\oplus \mathbb{N}}$ act on the set $X := F_2 \times \mathbb{N}$ by $(s, (t_k)_{k\in\mathbb{N}}) \cdot (r, n) = (srt_n^{-1}, n)$. Since F_2 is nonamenable, there are a finite set $L \subseteq F_2$ and a $\delta > 0$ such that $\sum_{g\in L} |gD\Delta D| \geq \delta|D|$ for every finite set $D \subseteq F_2$. Given any nonempty finite subset E of X there is a nonempty finite set $J \subseteq \mathbb{N}$ such that we can write $E = \bigsqcup_{n\in J}(E_n \times \{n\})$ where E_n is nonempty for each $n \in J$, in which case, identifying F_2 with the subgroup $F_2 \times \{e\}$ of G, we have

$$\sum_{g\in L} |gE\Delta E| = \sum_{g\in L}\sum_{n\in J} |gE_n\Delta E_n| \geq \sum_{n\in J} \delta|E_n| = \delta|E|.$$

This shows that the action $G \curvearrowright X$ fails the Følner condition and hence is nonamenable.

Now let M be any II_1 factor, let $\Omega_1 \subseteq M^{\overline{\otimes} X}$ be a finite set of elementary tensors with finite support, and let F be a finite subset of G. Write Ω_2 for the set $\{u_g\}_{g\in F}$ of unitaries in the crossed product $M^{\overline{\otimes} X} \rtimes G$ corresponding to F. For an element $(s, (t_k)_{k\in\mathbb{N}})$ in G we will call the (finite) set of indices k in $\mathbb{N}$ for which $t_k \neq e$ its *support*. Furthermore, for $s \in F_2$ and $n \in \mathbb{N}$ we denote by $g_{s,n}$ the element $(e, (t_k)_{k\in\mathbb{N}})$ supported on $\{n\}$ with $t_n = s$. Write K_1 for the set of all $n \in \mathbb{N}$ such that $F_2 \times \{n\}$ intersects the support of some element of Ω_1 and write K_2 for the union of the supports of the elements in F. Then $K := K_1 \cup K_2$ is a finite subset of $\mathbb{N}$. Write a, b for the generators of F_2. Then for every $n \in \mathbb{N} \setminus K$ and $s \in \{a, b\}$ we have $gg_{s,n}g^{-1} = g_{s,n}$ for all $g \in F$ and $u_{g_{s,n}} y u_{g_{s,n}}^* = y$ for all $y \in \Omega_1$. Furthermore, $\|[u_{g_{a,n}}, u_{g_{b,n}}]\|_2 = \sqrt{2}$, and so one can construct a noncommuting sequence of such pairs of unitaries which are asymptotically central, showing that the II_1 factor $M^{\overline{\otimes} X} \rtimes G$ is McDuff.

4. Generalized wreath products and JS-stability

Recall that the full group of a p.m.p. action $G \curvearrowright (Y, \nu)$ is the set of all measurable maps $T : Y \to G$ with the property that the transformation T^0 of Y given by $T^0(y) := T(y)y$ is a measure automorphism. By Kida's general version of a criterion due to Jones and Schmidt in the free ergodic case [16, 21], to verify that a countable group G is JS-stable it suffices to show that it admits a p.m.p. action $G \curvearrowright (Y, \nu)$ possessing a *stability sequence*, i.e., a sequence of pairs (T_n, A_n) where T_n is a member of the full group and A_n is a measurable subset of X such that

(i) $\nu(\{y \in Y : T_n(gy) = gT_n(y)g^{-1}\}) \to 1$ for every $g \in G$,

(ii) $\nu(T_n^0(B)\Delta B) \to 0$ for every measurable $B \subseteq Y$,
(iii) $\nu(gA_n\Delta A_n) \to 0$ for every $g \in G$,
(iv) $\nu(T_n^0(A_n)\Delta A_n) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$.

The following proof uses the same kind of idea as in Sections 2 and 3, but there is an additional technical twist here in the construction of the full group elements in the definition of stability sequence, one that has no analogue in von Neumann algebra framework of the previous two sections.

Proof of Theorem D. Denoting by λ the Lebesgue measure on $[0,1]$, we consider the p.m.p. action $H \wr_X G \overset{\alpha}{\curvearrowright} (Y,\nu) := (([0,1]^H)^X, (\lambda^H)^X) = ([0,1]^{H\times X}, \lambda^{H\times X})$ induced by the given action $G \curvearrowright X$ and the Bernoulli action $H \overset{\gamma}{\curvearrowright} ([0,1]^H, \lambda^H)$, as determined by $\alpha_{ag}((y_{s,x})_{s\in H, x\in X}) = (y_{a_x^{-1}s, g^{-1}x})_{s\in H, x\in X}$ for $a = (a_x)_{x\in X} \in H^{\oplus X}$ and $g \in G$. By the discussion above, it suffices to show that α admits a stability sequence.

To that end, let $F = \{\tilde{h}_i g_i : i \in I\}$ be a finite subset of $H \wr_X G$ with $\tilde{h}_i \in H^{\oplus X}$ and $g_i \in G$ for every $i \in I$. Set $W = \bigcup_{i\in I} \operatorname{supp} \tilde{h}_i$ and $K = \{g_i : i \in I\}$, and take an $\varepsilon > 0$ such that $2^{-3\varepsilon} > 1 - |F|^{-1}$. As in the proof of Theorem B, we can find a finite subset E of X that is (K,ε)-invariant and disjoint from $W \cup \bigcup_{i\in I} g_i^{-1}W$.

For $s \in H$ write $\pi_s : [0,1]^H \to [0,1]$ for the projection map onto the coordinate at s, and set $U_0 = [0, 2^{-|E|^{-1}}]$ and $U_1 = [0,1] \setminus U_0$. Let $h \in H \setminus \{e\}$ and consider the map $\omega : [0,1]^H \to H$ that is equal to h on $Z := \pi_e^{-1}(U_0) \cap \pi_h^{-1}(U_1)$, to h^{-1} on the image of Z under the shift γ_h (which is disjoint from Z), and to e otherwise. Define $T : Y \to H^{\oplus X} \subseteq H \wr_X G$ by declaring, for all $y \in Y$ and $x \in X$, that

$$T(y)(x) = \begin{cases} \omega(y_x), & x \in E, \\ e, & \text{otherwise.} \end{cases}$$

By construction, T is an element of the full group of α. Set

$$A = \big\{(y_x)_{x\in X} \in Y : y_x \in \pi_e^{-1}(U_0) \text{ for all } x \in E\big\}.$$

Because E is disjoint from W, both T and its image are invariant under the action (via α and by conjugation, respectively) of any element of $H^{\oplus X}$ supported on W. Let $g \in K$, denote by Y_g the set of all $y \in Y$ such that $T(\alpha_g(y)) = gT(y)g^{-1}$, and set

$$C = \big\{(y_x)_{x\in X} \in Y : y_x \in \pi_e^{-1}(U_0) \cap \pi_h^{-1}(U_0) \cap \pi_{h^2}^{-1}(U_0) \text{ for all } x \in E\Delta g^{-1}E\big\}.$$

One can easily check that $C \subseteq Y_g$, and therefore

$$\nu(Y_g) \geq \nu(C) \geq (2^{-|E|^{-1}})^{3\varepsilon|E|} \geq 1 - \frac{1}{|F|}.$$

Moreover, by construction we have $T_0(B) = B$ for all measurable rectangles B such that $\mathrm{ev}_x(B) = [0,1]^H$ for all $x \in E$, where ev_x denotes the evaluation map at x.

Finally, we have

$$\nu(T_0(A)\Delta A) = 2(\nu(A) - \nu(T_0(A)\cap A)) = 2\Big(\frac{1}{2} - \frac{1}{4}\Big) = \frac{1}{2}$$

and, for $t \in F$,

$$\nu(\alpha_t(A)\Delta A) = 2(\nu(A) - \nu(\alpha_t(A)\cap A)) \leq 2\Big(\frac{1}{2} - \Big(\frac{1}{2}\Big)^{(1+\varepsilon)}\Big) \leq \frac{1}{|F|}.$$

By constructing such A and T with respect to an increasing sequence of finite sets F_n such that $\bigcup_n F_n = H \wr_X G$, we obtain a stability sequence for α. $\square$

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