POSITIVE VOICULESCU-BROWN ENTROPY IN NONCOMMUTATIVE TORAL AUTOMORPHISMS

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Abstract. We show that the Voiculescu-Brown entropy of a noncommutative toral automorphism arising from a matrix $S \in GL(d,\mathbb{Z})$ is at least half the value of the topological entropy of the corresponding classical toral automorphism. We also obtain some information concerning the positivity of local Voiculescu-Brown entropy with respect to single unitaries. In particular we show that if S has no roots of unity as eigenvalues then the local Voiculescu-Brown entropy with respect to every product of canonical unitaries is positive, and also that in the presence of completely positive CNT entropy the unital version of local Voiculescu-Brown entropy with respect to every non-scalar unitary is positive.

1. Introduction

Let $\Theta = (\theta_{jk})_{1\leq j,k\leq d}$ be a real skew-symmetric $d \times d$ matrix. The noncommutative d-torus A_Θ is defined as the universal C^*-algebra generated by unitaries $u_1, \dots, u_d$ subject to the relations

$$u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j$$

for all $1 \leq j,k \leq d$ (see [26] for a reference). For any matrix $S = (s_{jk})_{1\leq j,k\leq d}$ in $GL(d,\mathbb{Z})$ there is an isomorphism $\alpha : A_{S^t\Theta S} \to A_\Theta$ determined by

$$\alpha_\Theta(u_j) = u_1^{s_{1j}} u_2^{s_{2j}} \cdots u_d^{s_{dj}}$$

for each $j = 1, \dots, d$. Thus when $S^t\Theta S \equiv \Theta \pmod{M_d(\mathbb{Z})}$ we obtain an automorphism of A_Θ, which we denote by $\alpha_{S,\Theta}$ and refer to as a *noncommutative toral automorphism.* Note that whenever $\alpha_{S,\Theta}$ exists so does $\alpha_{S,-\Theta}$. These noncommutative analogues of toral automorphisms were initially introduced in [33] and [3] for $d = 2$, in which case for any given $S \in SL(2,\mathbb{Z})$ the automorphism $\alpha_{S,\Theta}$ is defined for all Θ. An indication of their significance from a noncommutative geometry perspective is the fact that, for $d =$

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2, if θ_{12} is an irrational number satisfying a generic Diophantine property, then every diffeomorphism of A_Θ equipped with the canonical differential structure is a composition of an inner automorphism by a smooth unitary in the connected component of the unit, a noncommutative toral automorphism, and an automorphism arising from the canonical action of $\mathbb{T}^2$ [9]. In an arbitrary dimension d, if Θ is rational (i.e., the entries of Θ are all rational) then the C^*-algebra A_Θ is homogeneous (see [22]) and when $\alpha_{S,\Theta}$ exists we recover the corresponding classical toral automorphism at the level of the pure state space upon restricting $\alpha_{S,\Theta}$ to the center of A_Θ, so that from the noncommutative point of view it is the case of nonrational Θ that is of primary interest. Unlike classical toral automorphisms, which for hyperbolic $S \in SL(d, \mathbb{Z})$ have served as prototypes for such important dynamical phenomena as hyperbolicity and structural stability (see [7, 11]), noncommutative toral automorphisms as a class have remained somewhat mysterious, although in many cases much has been ascertained from a measure-theoretic viewpoint [1, 16, 18, 17, 21].

In [31, Sect. 5] it was shown that the Voiculescu-Brown entropy of $\alpha_{S,\Theta}$ is bounded above by the topological entropy of the corresponding classical toral automorphism, i.e., by $\sum_{|\lambda_i|>1} \log |\lambda_i|$, where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of S counted with multiplicity (see, e.g., [7, Sect. 24] for a calculation of the topological entropy of toral automorphisms). Except in the case that the eigenvalues of S all lie on the unit circle, this does not resolve the basic question of whether the entropy is positive or zero, i.e., of whether the system is "chaotic" or "deterministic". When Θ is rational the Voiculescu-Brown entropy can be seen to be $\sum_{|\lambda_i|>1} \log |\lambda_i|$ by restricting $\alpha_{S,\Theta}$ to the centre of A_Θ and applying monotonicity and Proposition 4.8 of [31]. In [21] Neshveyev showed that if S has no roots of unity then the entropic K-property (and in particular positive CNT entropy and hence also positive Voiculescu-Brown entropy by [31, Prop. 4.6]) follows from a summability condition with respect to a 2-cocycle $\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$ in terms of which A_Θ can be described. In the case $d = 2$, when we are dealing with a rotation C^*-algebra A_θ ($= A_\Theta$ for $\Theta = (\theta_{jk})_{j,k=1,2}$ with $\theta_{12} = \theta$), if S has an eigenvalue λ with $|\lambda| > 1$ then the set of θ for which the CNT entropy with respect to the canonical tracial state is positive has zero Lebesgue measure [18] and it contains $\mathbb{Z} + 2\mathbb{Z}\lambda^2$ as a consequence of the above-mentioned result of Neshveyev (see [16, 21]).

The first aim of this article, which we carry out in Section 2, is to show that, in an arbitrary dimension d, if the eigenvalues $\lambda_1, \dots, \lambda_d$ do not all lie on the unit circle, then the Voiculescu-Brown entropy of $\alpha_{S,\Theta}$ is at least $\frac{1}{2}\sum_{|\lambda_i|>1}\log|\lambda_i|$. (This was effectively claimed in [18] for $d = 2$ but the tensor product argument given there is not correct.) In Section 3 we apply a result from [14] to obtain some information concerning the positivity of the local Voiculescu-Brown entropy of $\alpha_{S,\Theta}$ with respect to products of canonical unitaries. We prove in particular that if S has no roots of unity as eigenvalues then the local Voiculescu-Brown entropy of $\alpha_{S,\Theta}$ with respect to any product of canonical unitaries is positive. Finally, in Section 4 we show in the general unital setting that completely positive CNT entropy of the von Neumann algebraic dynamical system arising from a faithful invariant state implies positivity of the unital version of local Voiculescu-Brown entropy with respect to every non-scalar unitary, and also that in the unital case this latter condition is equivalent to a noncommutative extension of the topological-dynamical notion of completely positive entropy which we call "completely positive Voiculescu-Brown entropy". We then apply these results to the above-mentioned noncommutative toral automorphisms treated by Neshveyev in [21].

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2. Positive Voiculescu-Brown entropy

We begin by recalling the definition of Voiculescu-Brown entropy [4], which is based on completely positive approximation (see [24] for a reference on completely positive maps). Let A be an exact (equivalently, nuclearly embeddable [15]) C^*-algebra and α an automorphism of A. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a faithful *-representation. For a finite set $\Omega \subseteq A$ and $\delta > 0$ we denote by $\mathrm{CPA}(\pi, \Omega, \delta)$ the collection of triples (φ, ψ, B) where B is a finite-dimensional C^*-algebra and $\varphi : A \to B$ and $\psi : B \to \mathcal{B}(\mathcal{H})$ are contractive completely positive maps such that $\|(\psi \circ \varphi)(a) - \pi(a)\| < \delta$ for all $a \in \Omega$. This collection is nonempty by nuclear embeddability, and we define $\mathrm{rcp}(\Omega, \delta)$ to be the infimum of rank B over all $(\varphi, \psi, B) \in \mathrm{CPA}(\pi, \Omega, \delta)$, with rank referring to the dimension of a maximal Abelian C^*-subalgebra. As the notation indicates, $\mathrm{rcp}(\Omega, \delta)$ is independent of the faithful *-representation π, as shown in the proof of Proposition 1.3 in [4]. We then set

$$\begin{aligned} ht(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rcp}(\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta), \\ ht(\alpha, \Omega) &= \sup_{\delta>0} ht(\alpha, \Omega, \delta), \\ ht(\alpha) &= \sup_{\Omega} ht(\alpha, \Omega) \end{aligned}$$

where the last supremum is taken over all finite sets $\Omega \subseteq A$. The quantity $ht(\alpha)$ is a C^*-dynamical invariant which we call the *Voiculescu-Brown entropy* of α. We note that exactness passes to C^*-subalgebras and that Voiculescu-Brown entropy is nonincreasing when passing to dynamically invariant C^*-subalgebras (monotonicity).

We will also have occasion to use the unital version of $ht(\alpha, \Omega)$ in Section 4, which we will denote by $ht_{\mathrm{u}}(\alpha, \Omega)$. This is defined in the case of unital A by using unital completely positive maps instead of general contractive completely positive maps as above, and when A is nuclear it agrees with the corresponding quantity in Voiculescu's original definition [31, Sect. 4], as can be seen from an argument in the proof of Proposition 1.4 in [4] involving Arveson's extension theorem. When $1 \notin \Omega$ we may have $ht(\alpha, \Omega) \neq ht_{\mathrm{u}}(\alpha, \Omega)$, but these quantities do agree when $1 \in \Omega$ as the proof of Proposition 1.4 in [4] shows, so that $ht(\alpha)$ may be alternatively obtained in the unital case by taking the supremum of $ht_{\mathrm{u}}(\alpha, \Omega)$ over all finite sets $\Omega \subseteq A$.

Notation 2.1. For a C^*-algebra A we denote by A^{op} the opposite algebra (i.e., the C^*-algebra obtained from A by reversing the multiplication), and for each $a \in A$ we denote by $\tilde{a}$ the corresponding element in A^{op}.

We would like to thank George Elliott for suggesting the proof we give of the following lemma, which simplifies our original proof.

Lemma 2.2. Let A and B be C^*-algebras and $\varphi : A \to B$ an n-positive map. Let $\varphi^{\mathrm{op}} : A^{\mathrm{op}} \to B^{\mathrm{op}}$ be the induced linear map given by $\varphi^{\mathrm{op}}(\tilde{a}) = \widetilde{\varphi(a)}$ for all $a \in A$. Then φ^{op} is also n-positive. In particular, if φ is completely positive then so is φ^{op}.

Proof. Suppose that φ is n-positive, i.e., the map $\varphi \otimes id : A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$ is positive. Since for every C^*-algebra D the map $D \to D^{\mathrm{op}}$ given by $a \mapsto \tilde{a}$ for all $a \in D$ is an order isomorphism, it follows that the map $(\varphi\otimes id)^{\mathrm{op}} : (A\otimes M_n(\mathbb{C}))^{\mathrm{op}} \to (B\otimes M_n(\mathbb{C}))^{\mathrm{op}}$ is positive. But this is just the map $\varphi^{\mathrm{op}}\otimes id^{\mathrm{op}} : A^{\mathrm{op}}\otimes(M_n(\mathbb{C}))^{\mathrm{op}} \to B^{\mathrm{op}}\otimes(M_n(\mathbb{C}))^{\mathrm{op}}$. Take an isomorphism $\beta : (M_n(\mathbb{C}))^{\mathrm{op}} \to M_n(\mathbb{C})$. Then $(id_{B^{\mathrm{op}}}\otimes\beta)\circ(\varphi^{\mathrm{op}}\otimes id^{\mathrm{op}})\circ(id_{A^{\mathrm{op}}}\otimes\beta)^{-1} = \varphi^{\mathrm{op}}\otimes(\beta\circ id^{\mathrm{op}}\circ\beta^{-1}) = \varphi^{\mathrm{op}}\otimes id : A^{\mathrm{op}}\otimes M_n(\mathbb{C}) \to B^{\mathrm{op}}\otimes M_n(\mathbb{C})$ is positive, i.e., φ^{op} is n-positive. □

Proposition 2.3. Let α be an automorphism of an exact C^*-algebra A and let α^{op} be the induced automorphism of A^{op}. Then $ht(\alpha) = ht(\alpha^{\mathrm{op}})$.

Proof. Given a faithful *-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ consider the induced injective *-homomorphism $\pi^{\mathrm{op}} : A^{\mathrm{op}} \to \mathcal{B}(\mathcal{H})^{\mathrm{op}}$. Let $\tilde{\mathcal{H}} = \{\tilde{x} : x \in \mathcal{H}\}$ be the Hilbert space conjugate to $\mathcal{H}$ with scalar multiplication $\bar{\lambda}\tilde{x} = \widetilde{\lambda x}$ and inner product $\langle\tilde{x},\tilde{y}\rangle = \langle y, x\rangle$. Then we have a natural identification of $\mathcal{B}(\mathcal{H})^{\mathrm{op}}$ with $\mathcal{B}(\tilde{\mathcal{H}})$ under which we can consider π^{op} as a *-representation. It follows then by Lemma 2.2 that for any finite set $\Omega \subseteq A$ and $\delta > 0$ we have $\mathrm{rcp}(\tilde{\Omega},\delta) = \mathrm{rcp}(\Omega,\delta)$ where $\tilde{\Omega} = \{\tilde{a} : a \in \Omega\}$, and so we conclude that $ht(\alpha) = ht(\alpha^{\mathrm{op}})$. □

Lemma 2.4. Let $d \geq 2$ and let Θ be a real skew-symmetric $d\times d$ matrix. Let β be the canonical action of $\mathbb{T}^d$ on the noncommutative torus A_Θ, and let α be an automorphism of A_Θ such that $\alpha\beta(\mathbb{T}^n)\alpha^{-1} = \beta(\mathbb{T}^n)$ in $\mathrm{Aut}(A_\Theta)$. Then $ht(\beta_x\alpha) = ht(\alpha\beta_x) = ht(\alpha)$ for all $x \in \mathbb{T}^n$.

Proof. It suffices to show $ht(\beta_x\alpha) \leq ht(\alpha)$. Let $\pi : A_\Theta \to \mathcal{B}(\mathcal{H})$ be a faithful *-representation of A_Θ. We can choose, for each $p \in \mathbb{Z}^d$, a unitary u_p in A_Θ in such a way that $\mathrm{span}\{u_p : p \in \mathbb{Z}^d\}$ is dense in A_Θ and $\beta_x(u_p) = \langle p, x\rangle\, u_p$, where $\langle\cdot,\cdot\rangle : \mathbb{Z}^d\times\mathbb{T}^d \to \mathbb{T}$ is the canonical pairing. For every $\omega \subseteq \mathbb{Z}^d$ we set $U_\omega = \{u_p : p \in \omega\}$. Since $U_{\mathbb{Z}^d}$ is total in A_Θ, by Proposition 2.6 of [4] $ht(\alpha)$ and $ht(\beta_x\alpha)$ are equal to the supremum of $ht(\alpha, U_\omega)$ and $ht(\beta_x\alpha, U_\omega)$ over all finite sets $\omega \subseteq \mathbb{Z}^d$ respectively. Thus it suffices to show that $ht(\alpha, U_\omega) \leq ht(\beta_x\alpha, U_\omega)$ for every finite set $\omega \subseteq \mathbb{Z}^d$, and this will follow once we show that

$$\begin{aligned}\mathrm{rcp}(U_\omega \cup (\beta_x\alpha)(U_\omega) \cup \cdots \cup (\beta_x\alpha)^{m-1}(U_\omega), \delta)&\\ \leq \mathrm{rcp}(U_\omega \cup \alpha(U_\omega) \cup \cdots \cup \alpha^{m-1}(U_\omega), \delta)&\end{aligned}$$

for any given finite set $\omega \subseteq \mathbb{Z}^d$, $m \in \mathbb{N}$, and $\delta > 0$. Suppose then that (φ,ψ,B) is a triple in $\mathrm{CPA}(\pi, U_\omega \cup \alpha(U_\omega) \cup \cdots \cup \alpha^{m-1}(U_\omega), \delta)$ such that $\mathrm{rank}(B) = \mathrm{rcp}(U_\omega \cup \alpha(U_\omega) \cup \cdots \cup$

$\alpha^{m-1}(U_\omega), \delta)$. For each $j \in \mathbb{Z}_{\geq 0}$ there exists some $x(j) \in \mathbb{T}^d$ such that $(\beta_x\alpha)^j = \alpha^j\beta_{x(j)}$. Then $(\beta_x\alpha)^j(u_p) = \langle p, x(j)\rangle\, \alpha^j(u_p)$ for every $p \in \mathbb{Z}^d$, and so

$$\|(\psi \circ \varphi)((\beta_x\alpha)^j(u_p)) - \pi((\beta_x\alpha)^j(u_p))\| = \|(\psi \circ \varphi)(\alpha^j(u_p)) - \pi(\alpha^j(u_p))\| < \delta$$

for all $j = 0, \dots, m-1$ and $p \in \omega$. Thus the triple (φ, ψ, B) is also contained in $\mathrm{CPA}(\pi, U_\omega \cup (\beta_x\alpha)(U_\omega) \cup \cdots \cup (\beta_x\alpha)^{m-1}(U_\omega), \delta)$, finishing the proof. □

Remark 2.5. (1) Let A be any exact C^*-algebra with a sequence of finite dimensional subspaces $V_1 \subseteq V_2 \subseteq \cdots$ such that $\bigcup_{j\in\mathbb{N}} V_j$ is dense in A, and let G be a subgroup of $\mathrm{Aut}(A)$ preserving every V_j. If $\alpha \in \mathrm{Aut}(A)$ satisfies $\alpha G \alpha^{-1} = G$ then $ht(\beta\alpha) = ht(\alpha\beta) = ht(\alpha)$ for every $\beta \in G$. The proof of Lemma 2.4 applies with minor modifications.

(2) It is easy to show that an automorphism α of A_Θ satisfies the hypothesis of Lemma 2.4 if and only if it is of the form $\alpha_{S,\Theta}\beta_x$ for some noncommutative toral automorphism $\alpha_{S,\Theta}$ and $x \in \mathbb{T}^d$.

Lemma 2.6. Let $\alpha_{S,\Theta}$ be any noncommutative toral automorphism. Then

$$ht(\alpha_{S,\Theta}) + ht(\alpha_{S,-\Theta}) \geq \sum_{|\lambda_i|>1} \log|\lambda_i|$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of S counted with multiplicity.

Proof. Since the Voiculescu-Brown entropy of an automorphism of a separable commutative C^*-algebra agrees with the topological entropy of the induced homeomorphism on the pure state space by [31, Prop. 4.8], in the case $\Theta = 0$ we have $ht(\alpha_{S,0}) = \sum_{|\lambda_i|>1} \log|\lambda_i|$. Consider now the tensor product $A_\Theta \otimes A_{-\Theta}$. Denoting by $u_1, \dots, u_d$ and $v_1, \dots, v_d$ the canonical unitaries of A_Θ and $A_{-\Theta}$, respectively, we have that the unitaries $u_j \otimes v_j$ for $j = 1, \dots, d$ form canonical generators for a copy C of $C(\mathbb{T}^d)$. This can been seen from the fact that they operate as shifts in different coordinate directions on the Hilbert subspace

$$\overline{\mathrm{span}}\{\pi_+(u_1^{k_1} \cdots u_d^{k_d})\xi_+ \otimes \pi_-(v_1^{k_1} \cdots v_d^{k_d})\xi_- : (k_1, \dots, k_d) \in \mathbb{Z}^d\}$$

(identified with $\ell^2(\mathbb{Z}^d)$) with respect to the tensor product of the canonical tracial state GNS representations $\pi_\pm$ of $A_{\pm\Theta}$ with canonical cyclic vectors $\xi_\pm$. We furthermore see that this identification of the $\alpha_{S,\Theta} \otimes \alpha_{S,-\Theta}$-invariant C^*-subalgebra C with $C(\mathbb{T}^d)$ establishes a conjugacy between $\alpha_{S,\Theta} \otimes \alpha_{S,-\Theta}|_C$ and $\alpha_{S,0}$. The monotonicity and tensor product subadditivity of Voiculescu-Brown entropy then yields

$$\sum_{|\lambda_i|>1} \log|\lambda_i| = ht(\alpha_{S,0}) = ht(\alpha_{S,\Theta} \otimes \alpha_{S,-\Theta}\big|_C) \leq ht(\alpha_{S,\Theta}) + ht(\alpha_{S,-\Theta}).$$

□

Theorem 2.7. Let $\alpha_{S,\Theta}$ be any noncommutative toral automorphism. Then

$$ht(\alpha_{S,\Theta}) = ht(\alpha_{S,-\Theta}) \geq \frac{1}{2} \sum_{|\lambda_i|>1} \log|\lambda_i|$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of S counted with multiplicity.

Proof. Denoting by $u_1, \dots, u_d$ and $v_1, \dots, v_d$ the canonical unitaries of A_Θ and $A_{-\Theta}$, respectively, we have an isomorphism $A_\Theta \to A_{-\Theta}^{\mathrm{op}}$ given by $u_j \mapsto \tilde{v}_j$. Identify A_Θ and $A_{-\Theta}^{\mathrm{op}}$ via this isomorphism. By computing $\alpha_{S,-\Theta}^{\mathrm{op}} \in \mathrm{Aut}(A_\Theta)$ on the canonical unitaries we see that it has the form $\alpha_{S,\Theta}\beta_x$ for some $x \in \mathbb{T}^d$, where β is the canonical action of $\mathbb{T}^d$ on A_Θ. Proposition 2.3 and Lemma 2.4 then yield $ht(\alpha_{S,-\Theta}) = ht(\alpha_{S,-\Theta}^{\mathrm{op}}) = ht(\alpha_{S,\Theta})$. By Lemma 2.6 we also have $ht(\alpha_{S,\Theta}) + ht(\alpha_{S,-\Theta}) \geq \sum_{|\lambda_i|>1} \log|\lambda_i|$. The assertion of the theorem now follows. □

3. Positive local Voiculescu-Brown entropy with respect to products of canonical unitaries

Our goal in this section is to obtain some information concerning positivity of local Voiculescu-Brown entropy with respect to products of canonical unitaries. We will proceed by first relating a noncommutative toral automorphism to the corresponding classical toral automorphism as in the proof of Lemma 2.6 but at a local level, and then appealing to a result from [14] involving local Voiculescu-Brown entropy in the separable unital commutative setting. Throughout this section we will denote the canonical unitaries of the commutative d-torus $A_0 \cong C(\mathbb{T}^d)$ by $f_1, \dots, f_d$. We will continue to denote the canonical unitaries of a general noncommutative d-torus by $u_1, \dots, u_d$.

Lemma 3.1. Let $\alpha_{S,\Theta}$ be a noncommutative toral automorphism, $k_1, \dots, k_d \in \mathbb{Z}$, and $\lambda \in \mathbb{C}$. Then

$$\frac{1}{2} ht(\alpha_{S,0}, \{\lambda f_1^{k_1} \cdots f_d^{k_d}\}) \leq ht(\alpha_{S,\Theta}, \{\lambda u_1^{k_1} \cdots u_d^{k_d}\}).$$

Proof. We may assume that $\lambda = 1$. Let C be the $\alpha_{S,\Theta} \otimes \alpha_{S,-\Theta}$-invariant commutative C^*-algebra of $A_\Theta \otimes A_{-\Theta}$ identified in the proof of Lemma 2.6. Denoting by $v_1, \cdots, v_d$ the canonical unitaries of $A_{-\Theta}$, we have

$$\begin{aligned} ht(\alpha_{S,0}, \{f_1^{k_1} \cdots f_1^{k_d}\}) &= ht(\alpha_{S,\Theta} \otimes \alpha_{S,-\Theta}|_C, \{u_1^{k_1} \cdots u_d^{k_d} \otimes v_1^{k_1} \cdots v_d^{k_d}\}) \\ &\leq ht(\alpha_{S,\Theta}, \{u_1^{k_1} \cdots u_d^{k_d}\}) + ht(\alpha_{S,-\Theta}, \{v_1^{k_1} \cdots v_d^{k_d}\}), \end{aligned} \tag{1}$$

where the last inequality follows from an argument similar to that in the proof of Proposition 3.10 in [31]. As in the proof of Theorem 2.7 we identify A_Θ with $A_{-\Theta}^{\mathrm{op}}$ via $u_j \mapsto \tilde{v}_j$ and observe that $\alpha_{-\Theta}^{\mathrm{op}} \in \mathrm{Aut}(A_\Theta)$ has the form $\alpha_{S,\Theta}\beta_x$ for some $x \in \mathbb{T}^d$, where β is the canonical action of $\mathbb{T}^d$ on A_Θ. Following Notation 2.1, we then have $\widetilde{v_1^{k_1} \cdots v_d^{k_d}} = \eta \tilde{v}_1^{k_1} \cdots \tilde{v}_d^{k_d}$ for some $\eta \in \mathbb{C}$ of unit modulus, and so

$$ht(\alpha_{S,-\Theta}, \{v_1^{k_1} \cdots v_d^{k_d}\}) = ht(\alpha_{S,-\Theta}^{\mathrm{op}}, \{\widetilde{v_1^{k_1} \cdots v_d^{k_d}}\}) = ht(\alpha_{S,\Theta}, \{u_1^{k_1} \cdots u_d^{k_d}\}) \tag{2}$$

in view of the proofs of Proposition 2.3 and Lemma 2.4. The assertion of the lemma now follows from (1) and (2). □

Theorem 3.2. Let $\alpha_{S,\Theta}$ be a noncommutative toral automorphism and suppose that S has no roots of unity as eigenvalues. Then

$$ht(\alpha_{S,\Theta}, \{\lambda u_1^{k_1} \cdots u_d^{k_d}\}) > 0$$

for any non-zero $(k_1, \dots, k_d) \in \mathbb{Z}^d$ and non-zero $\lambda \in \mathbb{C}$.

Proof. Since the measure-theoretic toral automorphism associated to S via Lebesgue measure is ergodic (see [32]) and hence has completely positive (Kolmogorov-Sinai) entropy [27], the topological toral automorphism associated to S (i.e., the case $\Theta = 0$ at the level of the pure state space) has completely positive (topological) entropy, i.e., each of its nontrivial factors has positive topological entropy (see [2]). Thus $ht(\alpha_{S,0}, \{x\}) > 0$ for every non-scalar $x \in A_0 \cong C(\mathbb{T}^d)$ by Corollary 4.4 of [14]. Lemma 3.1 then yields the result. □

For a general noncommutative toral automorphism $\alpha_{S,\Theta}$ it follows from Lemma 3.1 that to conclude that $ht(\alpha_{S,\Theta}, \{\lambda u_1^{k_1} \cdots u_d^{k_d}\}) > 0$ we need only show that $ht(\alpha_{S,0}, \{\lambda f_1^{k_1} \cdots f_d^{k_d}\}) > 0$. If we are simply dealing with a canonical unitary u_j then this occurs, for example, if the jth coordinate axis in $\mathbb{R}^d$ is not orthogonal to some one-dimensional subspace of a eigenspace in $\mathbb{R}^d$ corresponding to a real eigenvalue of S not equal to ± 1. To see this, suppose L is such a one-dimensional subspace and let λ be the associated real eigenvalue.

We may assume $|\lambda| > 1$ since $ht(\alpha_{S,\Theta}^{-1}, \{u_j\}) = ht(\alpha_{S,\Theta}, \{u_j\})$ (see the proof of Proposition 2.5 in [4]) and $\alpha_{S,\Theta}^{-1} = \alpha_{S^{-1},\Theta}\beta_x$ for some $x \in \mathbb{T}^d$, where β is the canonical action of $\mathbb{T}^d$ on A_Θ. Define the pseudo-metric d_j on $\mathbb{T}^d$ by

$$d_j(x, y) = |f_j(x) - f_j(y)|$$

for all $x, y \in \mathbb{T}$, with the unitary f_j being considered here in the canonical way as a function on the pure state space $\mathbb{T}^d$. Since the jth coordinate subspace of $\mathbb{R}^d$ is not orthogonal to L there exists a $\delta > 0$ such that, for any $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in L, the jth coordinate distance $|x_j - y_j|$ is at least δ times the Euclidean distance between x and y. Now since the action of T on L is simply multiplication by λ it can be seen via a covering space argument (see the proof of Theorem 24.5 in [7]) that there exists a $C > 0$ and an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ the image of L under the quotient map onto $\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$ contains an (n, ε)-separated set of cardinality at least $C|\lambda|^n$, from which it follows that the entropy $h_{d_j}(\bar{S})$ is strictly positive, where $\bar{S}$ is the automorphism of $\mathbb{T}^d$ corresponding to S. Here we are using standard notation and terminology from topological dynamics (see [7]) except that we are allowing the metric in the definition of entropy to be merely a pseudo-metric. Now by Theorem 4.3 of [14] we conclude that $ht(\alpha_{S,0}, \{u_j\}) > 0$, as desired.

By a similar argument which allows for the possibility of non-trivial Jordan cells we have the following more general statement.

Theorem 3.3. Let $\alpha_{S,\Theta}$ be a noncommutative toral automorphism. Let $j \in \{1, \dots, d\}$ and suppose that the jth coordinate axis in $\mathbb{R}^d$ is not orthogonal to the span of the generalized eigenspaces associated to the set of real eigenvalues of S not equal to ± 1. Then $ht(\alpha_{S,\Theta}, \{u_j\}) > 0$.

We could evidently further generalize this theorem to handle products of canonical unitaries, and also formulate a similar result involving pairs of canonical unitaries and complex eigenvalues not on the unit circle.

4. Completely positive Voiculescu-Brown entropy

In [21] Neshveyev showed that, for the von Neumann algebraic dynamical system arising from a noncommutative toral automorphism $\alpha_{S,\Theta}$ via the canonical tracial state on A_Θ, the property of being an entropic K-system (and in particular of having completely positive

CNT entropy), in the case of S having no roots of unity as eigenvalues (which occurs if and only if S is aperiodic (see [25] or [32])), is a consequence of a summability condition which for $d = 2$ is satisfied for a certain countable set of rotation parameters. We will show in the general unital setting that completely positive CNT entropy of the von Neumann algebraic dynamical system arising from a faithful invariant state implies "completely positive Voiculescu-Brown entropy" (i.e., every restriction of the automorphism to a non-trivial invariant C^*-subalgebra has positive Voiculescu-Brown entropy), and that in the unital case the latter property is equivalent to the positivity of the unital version of local Voiculescu-Brown entropy with respect to every non-scalar unitary.

We will use the standard notation for CNT (Connes-Narnhofer-Thirring) entropy [6]. Let α be an automorphism of a unital C^*-algebra A and ω a faithful α-invariant state on A. Denoting by π_ω the GNS representation corresponding to ω, we obtain extensions $\bar{\alpha}$ and $\bar{\omega}$ of α and ω, respectively, to $\pi_\omega(A)''$. By definition the automorphism $\bar{\alpha}$ has completely positive CNT entropy if $h_{\bar{\omega},\bar{\alpha}}(N) > 0$ for all unital finite-dimensional *-subalgebras N of $\pi_\omega(A)''$ which are different from the scalars. We recall that $h_{\bar{\omega},\bar{\alpha}}(N) = \lim_{n\to\infty} n^{-1} H_{\bar{\omega}}(N, \bar{\alpha}N, \dots, \bar{\alpha}^{n-1}N)$, where for unital finite-dimensional C^*-subalgebras $N_1, \dots, N_m \subseteq \pi_\omega(A)''$ the quantity $H_{\bar{\omega}}(N_1, \dots, N_m)$ refers to the supremum of the entropies of the Abelian models for $(\pi_\omega(A)'', \bar{\omega}, \{N_1, \dots, N_m\})$ (see Section III of [6], and note that we are using the convention that a unital finite-dimensional *-subalgebra N of $\pi_\omega(A)''$ stands for the (completely positive) inclusion $N \hookrightarrow \pi_\omega(A)''$). Entropic K-systems have completely positive CNT entropy, and the two notions coincide in the commutative case. For definitions and discussions see [19, 10]. Note that in [10] what we are referring to for clarity as "completely positive CNT entropy" is simply called "completely positive entropy".

Definition 4.1. Let α be an automorphism of a non-trivial exact C^*-algebra A. We say that α has *completely positive Voiculescu-Brown entropy* if $ht(\alpha|_B) > 0$ for every non-zero α-invariant C^*-subalgebra $B \subseteq A$ which, if A is unital, is not equal to the scalars.

Remark 4.2. If A is unital then in the above definition we may take the C^*-subalgebras B to be unital, for if $B \subseteq A$ is an α-invariant C^*-subalgebra not containing the unit of A then the Voiculescu-Brown entropies of the restrictions $\alpha|_B$ and $\alpha|_{B+\mathbb{C}1}$ agree by Lemma

1.7 of [4]. Thus in the separable unital commutative situation we recover the topological-dynamical notion of completely positive entropy, which refers to the absence of non-trivial factors with zero topological entropy [2].

Proposition 4.3. Let α be an automorphism of a unital exact C^*-algebra A and ω a faithful α-invariant state on A, and suppose that the extension $\bar{\alpha}$ of α to $\pi_\omega(A)''$ has completely positive CNT entropy. Then α has completely positive Voiculescu-Brown entropy.

Proof. Let $B \subseteq A$ be a unital α-invariant C^*-subalgebra different from the scalars. Since u.c.p. maps $M_d \to \pi_\omega(B)''$ can be approximated by u.c.p. maps $M_d \to \pi_\omega(B)$ in the strong topology (see the proof of Lemma 2.2 in [20], which we thank Sergey Neshveyev for pointing out) we have $h_{\omega|_B}(\alpha\big|_B) = h_{\bar{\omega}|\pi_\omega(B)''}(\bar{\alpha}\big|_{\pi_\omega(B)''})$ by Corollary VI.4 of [6]. Since B is different from the scalars, $\pi_\omega(B)''$ contains finite-dimensional C^*-subalgebras different from the scalars (e.g., span$\{p, 1-p\}$ where p is a projection in $\pi_\omega(B)''$ different from 0 and 1). Applying Proposition 9 of [8] we thus have

$$ht(\alpha\big|_B) \geq h_{\omega|_B}(\alpha\big|_B) = h_{\bar{\omega}|\pi_\omega(B)''}(\bar{\alpha}\big|_{\pi_\omega(B)''}) \geq \sup_N h_{\bar{\omega},\bar{\alpha}}(N) > 0,$$

where the supremum is taken over all finite-dimensional C^*-subalgebras $N \subseteq \pi_\omega(B)''$. We remark that the second last inequality, which follows from the definition of CNT entropy, is in fact an equality when $\pi_\omega(B)''$ is hyperfinite [6, Thm. VII.4] (this is automatic if A is separable and nuclear and $\pi_\omega(A)''$ is finite, or more generally if $\pi_\omega(A)''$ is finite and injective and has separable predual, for in this case every von Neumann subalgebra of $\pi_\omega(A)''$ is injective [30, Prop. V.2.36] and hence hyperfinite [5]). □

Recall from the beginning of Section 4 that $ht_{\mathrm{u}}(\alpha, \Omega)$ denotes the unital version of the local Voiculescu-Brown entropy $ht(\alpha, \Omega)$. We will next show that, in the unital case, completely positive Voiculescu-Brown entropy is equivalent to positivity of the unital version of local Voiculescu-Brown entropy with respect to every non-scalar unitary. For this we will need the following Kolmogorov-Sinai-type property, which is similar to that of Lemma A.2 in [23].

Lemma 4.4. Let α be an automorphism of a unital exact C^*-algebra A. If $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \cdots$ is a nested sequence of finite sets of unitaries in A such that $\bigcup_{k\in\mathbb{N},n\in\mathbb{Z}} \alpha^n\Omega_k$

generates A as a C^*-algebra, then

$$ht(\alpha) = \sup_{k\in\mathbb{N}} ht_{\mathrm{u}}(\alpha, \Omega_k).$$

Proof. Given a unital completely positive map φ from A into any unital C^*-algebra B, Lemma 3.1 of [12] yields

$$\|\varphi(x^*y) - \varphi(x)^*\varphi(y)\| \leq \|\varphi(x^*x) - \varphi(x)^*\varphi(x)\|^{1/2}\|\varphi(y^*y) - \varphi(y)^*\varphi(y)\|^{1/2}$$

for all $x, y \in A$, and so in particular for any unitaries $u, v \in A$ we have

$$\begin{aligned} \|\varphi(uv) - \varphi(u)\varphi(v)\| &\leq \|1 - \varphi(u)^*\varphi(u)\|^{1/2}\|1 - \varphi(v)^*\varphi(v)\|^{1/2} \\ &\leq (\|u^*u - \varphi(u)^*u\| + \|\varphi(u)^*u - \varphi(u)^*\varphi(u)\|)^{1/2} \\ &\qquad \times (\|v^*v - \varphi(v)^*v\| + \|\varphi(v)^*v - \varphi(v)^*\varphi(v)\|)^{1/2} \\ &\leq 2\|\varphi(u) - u\|^{1/2}\|\varphi(v) - v\|^{1/2} \end{aligned}$$

whence

$$\begin{aligned} \|\varphi(uv) - uv\| &\leq \|\varphi(uv) - \varphi(u)\varphi(v)\| + \|\varphi(u)\varphi(v) - u\varphi(v)\| + \|u\varphi(v) - uv\| \\ &\leq (\|\varphi(u) - u\|^{1/2} + \|\varphi(v) - v\|^{1/2})^2. \end{aligned}$$

We can now proceed along the lines of the proofs of Propositions 1.4 and 3.4 of [31] to obtain the result. □

Proposition 4.5. Let α be an automorphism of a unital exact C^*-algebra A. Then α has completely positive Voiculescu-Brown entropy if and only if $ht_{\mathrm{u}}(\alpha, \{u\}) > 0$ for every non-scalar unitary $u \in A$.

Proof. For the "only if" direction, we can consider for any non-scalar unitary $u \in A$ the unital α-invariant C^*-subalgebra it generates and appeal to Lemma 4.4. For the "if" direction, we need simply observe that every unital C^*-algebra different from the scalars contains a non-scalar unitary, as can be obtained by applying the functional calculus to the real part (or imaginary part if the real part is a scalar) of any non-scalar element. □

For a real skew-symmetric $d \times d$ matrix Θ the noncommutative d-torus A_Θ may alternatively be described as the universal unital C^*-algebra generated by unitaries $\{u_g\}_{g\in\mathbb{Z}^d}$ subject to the relations

$$u_g u_h = \beta(g, h) u_{g+h},$$

where $\beta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$ is a bicharacter satisfying

$$\beta(g,h)\beta(h,g)^{-1} = e^{2\pi i g\cdot\Theta h}.$$

Theorem 4.6. Let $\alpha_{S,\Theta}$ be a noncommutative toral automorphism, and suppose that S is β-preserving and has no roots of unity as eigenvalues, and that

$$\sum_{n\in\mathbb{Z}} |1-\beta(g,S^n h)| < \infty$$

for all $g,h \in \mathbb{Z}^d$. Then $\alpha_{S,\Theta}$ has completely positive Voiculescu-Brown entropy and $ht_{\mathrm{u}}(\alpha_{S,\Theta},\{u\}) > 0$ for every non-scalar unitary $u \in A$.

Proof. By Theorem 2 of [21] the hypotheses of the theorem statement imply that the von Neumann algebraic system obtained from $\alpha_{S,\Theta}$ via the canonical tracial state on A_Θ is an entropic K-system, and hence has completely positive CNT entropy. Propositions 4.3 and 4.5 then yield the desired conclusion. □

Theorem 4.6 applies in particular in the case $d = 2$ when the matrix S has eigenvalues off the unit circle and the rotation parameter θ of the rotation C^*-algebra A_θ ($= A_\Theta$ for $\Theta = (\theta_{jk})_{j,k=1,2}$ with $\theta_{12} = \theta$) lies in $\mathbb{Z} + 2\mathbb{Z}\lambda^2$ where λ is the (necessarily real) eigenvalue of S of largest absolute value (see [21]).

Finally, we would like to point out that the argument in [21] is not quite complete. Indeed in the proof of the lemma in [21] it is incorrectly taken to be the case that a matrix $T \in GL(n,\mathbb{Z})$ is aperiodic if and only if it has no eigenvalues on the unit circle. (Aperiodicity is defined as the non-existence of non-trivial finite orbits of T acting on $\mathbb{Z}^n$ and is equivalent to T having no roots of unity as eigenvalues and also equivalent to the ergodicity of the measure-theoretic automorphism of $\mathbb{T}^n$ associated to T via Lebesgue measure (see [25] or [32], and also Section 24 of [7]).) However, if we let $\mathbb{R}^n = V_1 \oplus V_2$ be the decomposition of $\mathbb{R}^n$ corresponding to the eigenvalues of T of modulus at least one and strictly less than one, respectively, and denote by P_1 and P_2 the associated projections, then the proof of the lemma in [21] demonstrates that, for large n, if $y_1 + T^n y_2 + \cdots + T^{n(k-1)} y_k = 0$ then $P_2(y_1) = \cdots = P_2(y_k) = 0$. The argument in the first paragraph of p. 191 of [13] then shows that P_2 is injective on $\mathbb{Z}^n$, whence $y_1 = \cdots = y_k = 0$, as desired.

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