WWU MÜNSTER

Frobenius homomorphisms in higher algebra

Thomas Nikolaus

International Congress of Mathematicians 2022

living.knowledge

MM Mathematics Münster Cluster of Excellence

Motivation I: K-Theory

Goal: Compute K-group $K_*(R)$ for R ring

Motivation I: K-Theory

Goal: Compute K-group $K_*(R)$ for R ring

Theorem (Quillen '72)

$$K_*(\mathbb{Z}/p) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/(p^i - 1) & * = 2i - 1 \\ 0 & \textit{else} \end{cases}$$

Motivation I: K-Theory

Goal: Compute K-group $K_*(R)$ for R ring

Theorem (Quillen '72)

$$K_*(\mathbb{Z}/p) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/(p^i - 1) & * = 2i - 1 \\ 0 & \textit{else} \end{cases}$$

Question: What about $K_*(\mathbb{Z}/p^k)$?

Motivation I: K-Theory

Goal: Compute K-group $K_*(R)$ for R ring

Theorem (Quillen '72)

$$K_*(\mathbb{Z}/p) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/(p^i - 1) & * = 2i - 1 \\ 0 & \textit{else} \end{cases}$$

Question: What about $K_*(\mathbb{Z}/p^k)$? Only known for $* \leqslant 2p-2$ (Angeltveit '11)

Motivation I: K-Theory

Goal: Compute K-group $K_*(R)$ for R ring

Theorem (Quillen '72)

$$K_*(\mathbb{Z}/p) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/(p^i - 1) & * = 2i-1 \\ 0 & \textit{else} \end{cases}$$

Question: What about $K_*(\mathbb{Z}/p^k)$? Only known for $* \leqslant 2p-2$ (Angeltveit '11)

Theorem (Antieau–N.–Krause '22)

1. *For* $i \gg 0$ $\left(i \geqslant \frac{p^2(p^k-1)}{(p-1)^2}\right)$

$$K_{2i-2}(\mathbb{Z}/p^k) = 0$$

$$\#K_{2i-1}(\mathbb{Z}/p^k) = (p^i - 1)p^{i(k-1)}$$

Motivation I: K-Theory

Goal: Compute K-group $K_*(R)$ for R ring

Theorem (Quillen '72)

$$K_*(\mathbb{Z}/p) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/(p^i - 1) & * = 2i-1 \\ 0 & \textit{else} \end{cases}$$

Question: What about $K_*(\mathbb{Z}/p^k)$? Only known for $* \leqslant 2p-2$ (Angeltveit '11)

Theorem (Antieau–N.–Krause '22)

1. *For* $i \gg 0$ $\left(i \geqslant \frac{p^2(p^k-1)}{(p-1)^2}\right)$

$$K_{2i-2}(\mathbb{Z}/p^k) = 0$$
$$\#K_{2i-1}(\mathbb{Z}/p^k) = (p^i - 1)p^{i(k-1)}$$

2. *There is an explicit algorithm computing* $K_*(\mathbb{Z}/p^k)$

2^∞-torsion in $K_*(\mathbb{Z}/2^k)$

R :	$\mathbb{Z}/4$	$\mathbb{Z}/8$	$\mathbb{Z}/16$	$\mathbb{Z}/32$
K_1	2^1	$2^1, 2^1$	$2^1, 2^2$	$2^1, 2^3$
K_2	2^1	2^1	2^1	2^1
K_3	2^3	$2^3, 2^2$	$2^3, 2^4$	$2^3, 2^6$
K_4	0	2^1	2^2	2^3
K_5	2^3	$2^1, 2^6$	$2^1, 2^1, 2^9$	$2^1, 2^2, 2^{12}$
K_6	0	0	2^1	2^1
K_7	$2^1, 2^3$	$2^4, 2^4$	$2^1, 2^4, 2^8$	$2^1, 2^1, 2^4, 2^{11}$
K_8	0	0	2^1	2^2
K_9	$2^1, 2^1, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^4$	$2^1, 2^1, 2^2, 2^{12}$	$2^1, 2^1, 2^1, 2^2, 2^{17}$
K_{10}	0	0	0	2^1
K_{11}	$2^1, 2^5$	$2^1, 2^1, 2^1, 2^2, 2^2, 2^5$	$2^3, 2^3, 2^{12}$	$2^1, 2^3, 2^5, 2^{16}$
K_{12}	0	0	0	2^1
K_{13}	$2^1, 2^2, 2^4$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^1, 2^3, 2^{15}$	$2^1, 2^1, 2^1, 2^1, 2^3, 2^{22}$
K_{14}	0	0	0	2^1
K_{15}	$2^1, 2^1, 2^1, 2^5$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^2, 2^5, 2^{15}$	$2^1, 2^1, 2^2, 2^3, 2^5, 2^{21}$
K_{16}	0	0	0	2^1
K_{17}	$2^1, 2^1, 2^1, 2^3, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^3, 2^9$	$2^1, 2^2, 2^2, 2^2, 2^3, 2^{17}$	$2^1, 2^1, 2^2, 2^2, 2^2, 2^3, 2^{26}$
K_{18}	0	0	0	0
K_{19}	$2^2, 2^3, 2^5$	$2^1, 2^3, 2^4, 2^{12}$	$2^3, 2^3, 2^4, 2^{20}$	$2^3, 2^3, 2^3, 2^4, 2^{27}$

2^∞-torsion in $K_*(\mathbb{Z}/2^k)$

R :	$\mathbb{Z}/4$	$\mathbb{Z}/8$	$\mathbb{Z}/16$	$\mathbb{Z}/32$
K_1	$\mathbf{2^1}$	$\mathbf{2^1, 2^1}$	$\mathbf{2^1, 2^2}$	$\mathbf{2^1, 2^3}$
K_2	$\mathbf{2^1}$	$\mathbf{2^1}$	$\mathbf{2^1}$	$\mathbf{2^1}$
K_3	2^3	$2^3, 2^2$	$2^3, 2^4$	$2^3, 2^6$
K_4	0	2^1	2^2	2^3
K_5	2^3	$2^1, 2^6$	$2^1, 2^1, 2^9$	$2^1, 2^2, 2^{12}$
K_6	0	0	2^1	2^1
K_7	$2^1, 2^3$	$2^4, 2^4$	$2^1, 2^4, 2^8$	$2^1, 2^1, 2^4, 2^{11}$
K_8	0	0	2^1	2^2
K_9	$2^1, 2^1, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^4$	$2^1, 2^1, 2^2, 2^{12}$	$2^1, 2^1, 2^1, 2^2, 2^{17}$
K_{10}	0	0	0	2^1
K_{11}	$2^1, 2^5$	$2^1, 2^1, 2^1, 2^2, 2^2, 2^5$	$2^3, 2^3, 2^{12}$	$2^1, 2^3, 2^5, 2^{16}$
K_{12}	0	0	0	2^1
K_{13}	$2^1, 2^2, 2^4$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^1, 2^3, 2^{15}$	$2^1, 2^1, 2^1, 2^1, 2^3, 2^{22}$
K_{14}	0	0	0	2^1
K_{15}	$2^1, 2^1, 2^1, 2^5$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^2, 2^5, 2^{15}$	$2^1, 2^1, 2^2, 2^3, 2^5, 2^{21}$
K_{16}	0	0	0	2^1
K_{17}	$2^1, 2^1, 2^1, 2^3, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^3, 2^9$	$2^1, 2^2, 2^2, 2^2, 2^3, 2^{17}$	$2^1, 2^1, 2^2, 2^2, 2^2, 2^3, 2^{26}$
K_{18}	0	0	0	0
K_{19}	$2^2, 2^3, 2^5$	$2^1, 2^3, 2^4, 2^{12}$	$2^3, 2^3, 2^4, 2^{20}$	$2^3, 2^3, 2^3, 2^4, 2^{27}$

2^∞-torsion in $K_*(\mathbb{Z}/2^k)$

R :	$\mathbb{Z}/4$	$\mathbb{Z}/8$	$\mathbb{Z}/16$	$\mathbb{Z}/32$
K_1	2^1	$2^1, 2^1$	$2^1, 2^2$	$2^1, 2^3$
K_2	2^1	2^1	2^1	2^1
K_3	2^3	$2^3, 2^2$	$2^3, 2^4$	$2^3, 2^6$
K_4	**0**	2^1	2^2	2^3
K_5	2^3	$2^1, 2^6$	$2^1, 2^1, 2^9$	$2^1, 2^2, 2^{12}$
K_6	**0**	**0**	2^1	2^1
K_7	$2^1, 2^3$	$2^4, 2^4$	$2^1, 2^4, 2^8$	$2^1, 2^1, 2^4, 2^{11}$
K_8	**0**	**0**	2^1	2^2
K_9	$2^1, 2^1, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^4$	$2^1, 2^1, 2^2, 2^{12}$	$2^1, 2^1, 2^1, 2^2, 2^{17}$
K_{10}	**0**	**0**	**0**	2^1
K_{11}	$2^1, 2^5$	$2^1, 2^1, 2^1, 2^2, 2^2, 2^5$	$2^3, 2^3, 2^{12}$	$2^1, 2^3, 2^5, 2^{16}$
K_{12}	**0**	**0**	**0**	2^1
K_{13}	$2^1, 2^2, 2^4$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^1, 2^3, 2^{15}$	$2^1, 2^1, 2^1, 2^1, 2^3, 2^{22}$
K_{14}	**0**	**0**	**0**	2^1
K_{15}	$2^1, 2^1, 2^1, 2^5$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^2, 2^5, 2^{15}$	$2^1, 2^1, 2^2, 2^3, 2^5, 2^{21}$
K_{16}	**0**	**0**	**0**	2^1
K_{17}	$2^1, 2^1, 2^1, 2^3, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^3, 2^9$	$2^1, 2^2, 2^2, 2^2, 2^3, 2^{17}$	$2^1, 2^1, 2^2, 2^2, 2^2, 2^3, 2^{26}$
K_{18}	**0**	**0**	**0**	**0**
K_{19}	$2^2, 2^3, 2^5$	$2^1, 2^3, 2^4, 2^{12}$	$2^3, 2^3, 2^4, 2^{20}$	$2^3, 2^3, 2^3, 2^4, 2^{27}$

2^∞-torsion in $K_*(\mathbb{Z}/2^k)$

R:	$\mathbb{Z}/4$	$\mathbb{Z}/8$	$\mathbb{Z}/16$	$\mathbb{Z}/32$
K_1	2^1	$2^1, 2^1$	$2^1, 2^2$	$2^1, 2^3$
K_2	2^1	2^1	2^1	2^1
K_3	2^3	$2^3, 2^2$	$2^3, 2^4$	$2^3, 2^6$
K_4	$\mathbf{0}$	2^1	2^2	2^3
K_5	2^3	$2^1, 2^6$	$2^1, 2^1, 2^9$	$2^1, 2^2, 2^{12}$
K_6	$\mathbf{0}$	$\mathbf{0}$	2^1	2^1
K_7	$2^1, 2^3$	$2^4, 2^4$	$2^1, 2^4, 2^8$	$2^1, 2^1, 2^4, 2^{11}$
K_8	$\mathbf{0}$	$\mathbf{0}$	2^1	2^2
K_9	$2^1, 2^1, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^4$	$2^1, 2^1, 2^2, 2^{12}$	$2^1, 2^1, 2^1, 2^2, 2^{17}$
K_{10}	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	2^1
K_{11}	$2^1, 2^5$	$2^1, 2^1, 2^1, 2^2, 2^2, 2^5$	$2^3, 2^3, 2^{12}$	$2^1, 2^3, 2^5, 2^{16}$
K_{12}	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	2^1
K_{13}	$2^1, 2^2, 2^4$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^1, 2^3, 2^{15}$	$2^1, 2^1, 2^1, 2^1, 2^3, 2^{22}$
K_{14}	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	2^1
K_{15}	$2^1, 2^1, 2^1, 2^5$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^2, 2^5, 2^{15}$	$2^1, 2^1, 2^2, 2^3, 2^5, 2^{21}$
K_{16}	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	2^1
K_{17}	$2^1, 2^1, 2^1, 2^3, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^3, 2^9$	$2^1, 2^2, 2^2, 2^2, 2^3, 2^{17}$	$2^1, 2^1, 2^2, 2^2, 2^2, 2^3, 2^{26}$
K_{18}	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
K_{19}	$2^2, 2^3, 2^5$	$2^1, 2^3, 2^4, 2^{12}$	$2^3, 2^3, 2^4, 2^{20}$	$2^3, 2^3, 2^3, 2^4, 2^{27}$

Idea

Use trace methods:

$$K_*(R) \leftsquigarrow TC_*(R) \leftsquigarrow THH_*(R)$$

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\simeq H_*(S^m, \mathbb{Z})$ as groups

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\simeq H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \simeq S^2 \vee S^4$?

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\simeq H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$, since $H^*(\mathbb{C}P^2, \mathbb{Z}) \not\simeq H^*(S^2 \vee S^4, \mathbb{Z})$ as rings

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\cong H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$, since $H^*(\mathbb{C}P^2, \mathbb{Z}) \not\cong H^*(S^2 \vee S^4, \mathbb{Z})$ as rings
- $\Sigma\mathbb{C}P^2 \simeq S^3 \vee S^5$?

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\cong H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$, since $H^*(\mathbb{C}P^2, \mathbb{Z}) \not\cong H^*(S^2 \vee S^4, \mathbb{Z})$ as rings
- $\Sigma\mathbb{C}P^2 \not\simeq S^3 \vee S^5$, since $H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2) \not\cong H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2)$ as rings with Steenrod action

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\cong H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$, since $H^*(\mathbb{C}P^2, \mathbb{Z}) \not\cong H^*(S^2 \vee S^4, \mathbb{Z})$ as rings
- $\Sigma\mathbb{C}P^2 \not\simeq S^3 \vee S^5$, since $H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2) \not\cong H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2)$ as rings with Steenrod action

Ultimate invariant: $C^*(M, \mathbb{Z})$ as an $\mathbb{E}_\infty$-ring

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\simeq H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$, since $H^*(\mathbb{C}P^2, \mathbb{Z}) \not\simeq H^*(S^2 \vee S^4, \mathbb{Z})$ as rings
- $\Sigma\mathbb{C}P^2 \not\simeq S^3 \vee S^5$, since $H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2) \not\simeq H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2)$ as rings with Steenrod action

Ultimate invariant: $C^*(M, \mathbb{Z})$ as an $\mathbb{E}_\infty$-ring

Theorem (Mandell '06)

$C^*(M, \mathbb{Z}) \simeq C^*(N, \mathbb{Z})$ *as* $\mathbb{E}_\infty$*-rings over* $\mathbb{Z}$ *(*M, N *simply-connected, finite type)*

$\Rightarrow M \simeq N$

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\simeq H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$, since $H^*(\mathbb{C}P^2, \mathbb{Z}) \not\simeq H^*(S^2 \vee S^4, \mathbb{Z})$ as rings
- $\Sigma\mathbb{C}P^2 \not\simeq S^3 \vee S^5$, since $H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2) \not\simeq H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2)$ as rings with Steenrod action

Ultimate invariant: $C^*(M, \mathbb{Z})$ as an $\mathbb{E}_\infty$-ring

Theorem (Mandell '06)

$C^*(M, \mathbb{Z}) \simeq C^*(N, \mathbb{Z})$ *as* $\mathbb{E}_\infty$*-rings over* $\mathbb{Z}$ *(M, N simply-connected, finite type)*

$\Rightarrow M \simeq N$

Question: Is $C^*(-, \mathbb{Z})$ an equivalence between spaces and $\mathbb{E}_\infty$-rings?

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\simeq H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$, since $H^*(\mathbb{C}P^2, \mathbb{Z}) \not\simeq H^*(S^2 \vee S^4, \mathbb{Z})$ as rings
- $\Sigma\mathbb{C}P^2 \not\simeq S^3 \vee S^5$, since $H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2) \not\simeq H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2)$ as rings with Steenrod action

Ultimate invariant: $C^*(M, \mathbb{Z})$ as an $\mathbb{E}_\infty$-ring

Theorem (Mandell '06)

$C^*(M, \mathbb{Z}) \simeq C^*(N, \mathbb{Z})$ *as* $\mathbb{E}_\infty$*-rings over* $\mathbb{Z}$ (*M, N simply-connected, finite type*)

$\Rightarrow M \simeq N$

Question: Is $C^*(-, \mathbb{Z})$ an equivalence between spaces and $\mathbb{E}_\infty$-rings with some extra structure?

Motivation II: Spaces

Algebraic Topology: study invariants of spaces, e.g. $H_*(M, \mathbb{Z})$

Example

- $S^n \not\simeq S^m$ for $n \neq m$, since $H_*(S^n, \mathbb{Z}) \not\simeq H_*(S^m, \mathbb{Z})$ as groups
- $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$, since $H^*(\mathbb{C}P^2, \mathbb{Z}) \not\simeq H^*(S^2 \vee S^4, \mathbb{Z})$ as rings
- $\Sigma\mathbb{C}P^2 \not\simeq S^3 \vee S^5$, since $H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2) \not\simeq H^*(\Sigma\mathbb{C}P^2, \mathbb{F}_2)$ as rings with Steenrod action

Ultimate invariant: $C^*(M, \mathbb{Z})$ as an $\mathbb{E}_\infty$-ring

Theorem (Mandell '06)

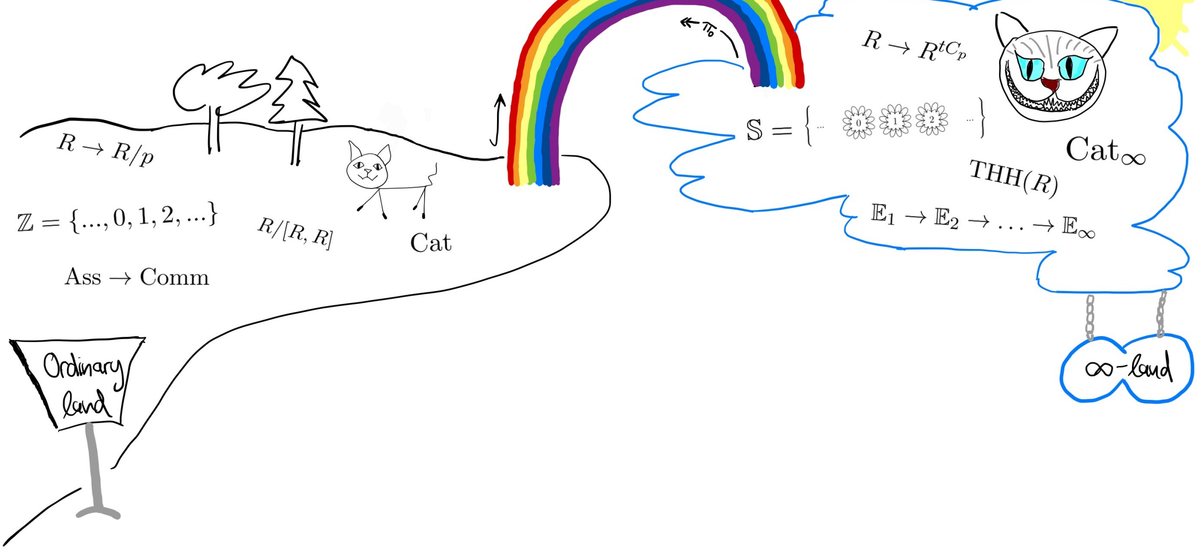
$C^*(M, \mathbb{Z}) \simeq C^*(N, \mathbb{Z})$ *as* $\mathbb{E}_\infty$*-rings over* $\mathbb{Z}$ *(M, N simply-connected, finite type)*

$\Rightarrow M \simeq N$

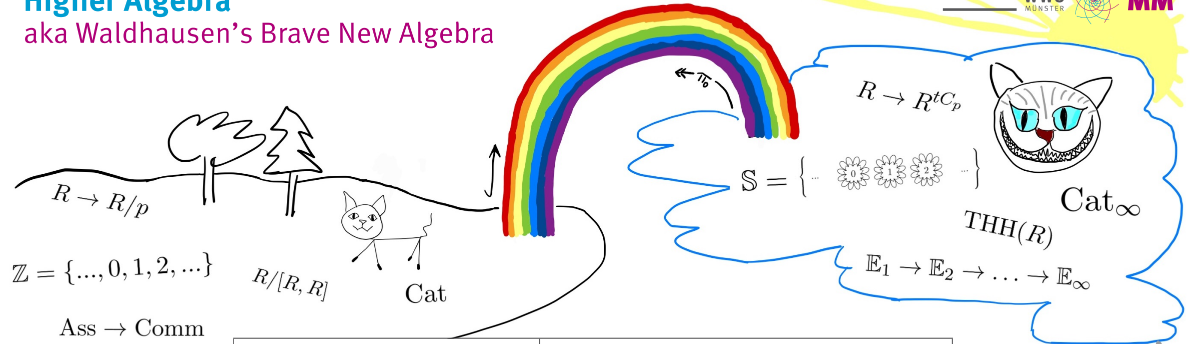
Question: Is $C^*(-, \mathbb{Z})$ an equivalence between spaces and $\mathbb{E}_\infty$-rings with some extra structure?

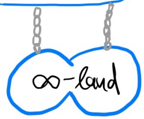
Answer: Yes, $\mathbb{E}_\infty$-rings with trivialized Frobenius (Mandell '01,...,Yuan '21)

Higher Algebra
aka Waldhausen's Brave New Algebra



Higher Algebra
aka Waldhausen’s Brave New Algebra

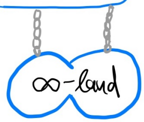




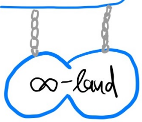
ordinary algebra	higher algebra

Higher Algebra
aka Waldhausen's Brave New Algebra

ordinary algebra	higher algebra
sets	spaces (homotopy types, anima)



Higher Algebra
aka Waldhausen's Brave New Algebra

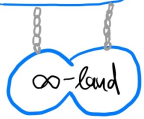


ordinary algebra	higher algebra
sets	spaces (homotopy types, anima)
abelian groups	spectra

Higher Algebra

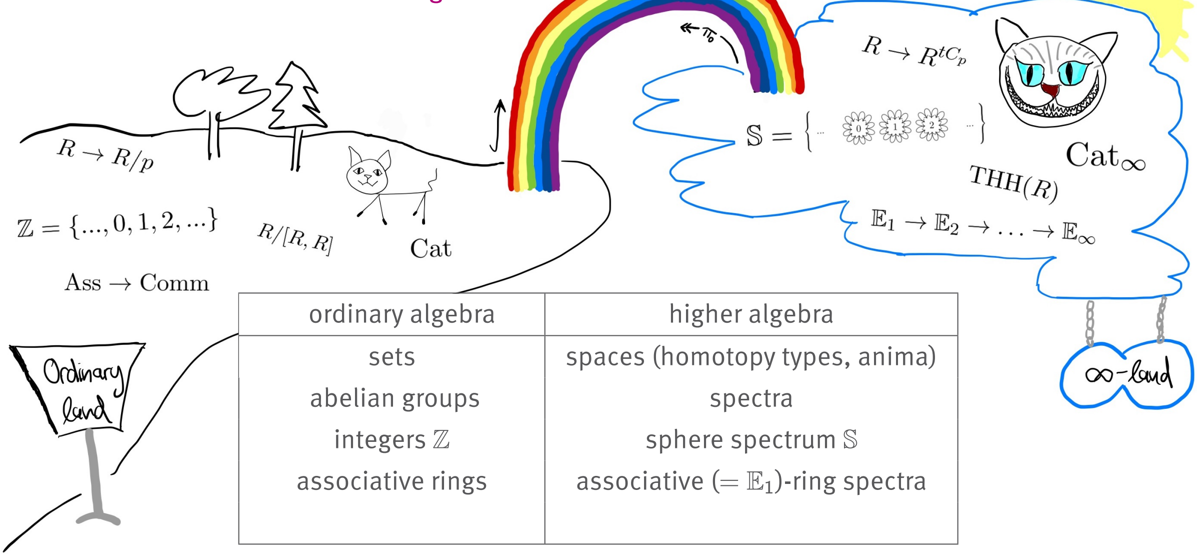
aka Waldhausen's Brave New Algebra

ordinary algebra	higher algebra
sets	spaces (homotopy types, anima)
abelian groups	spectra
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$



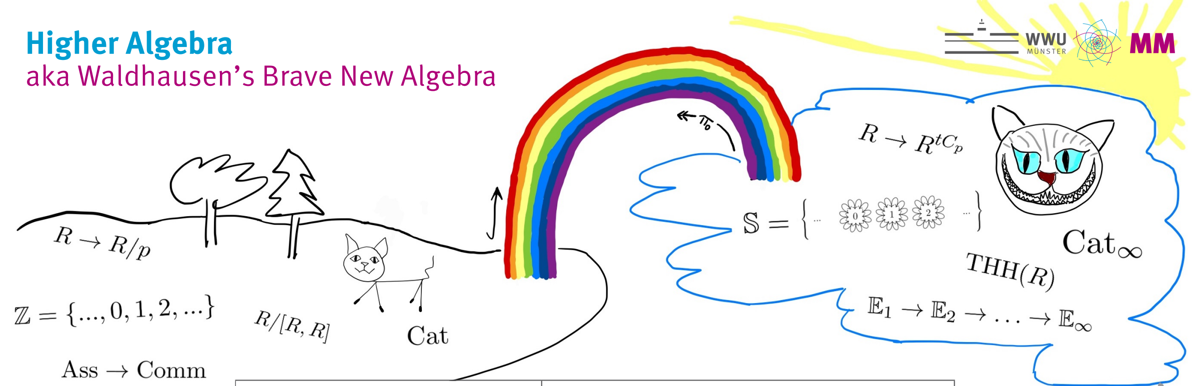
Higher Algebra

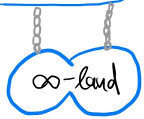
aka Waldhausen's Brave New Algebra



ordinary algebra	higher algebra
sets	spaces (homotopy types, anima)
abelian groups	spectra
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$
associative rings	associative ($= \mathbb{E}_1$)-ring spectra

Higher Algebra
aka Waldhausen's Brave New Algebra





ordinary algebra	higher algebra
sets	spaces (homotopy types, anima)
abelian groups	spectra
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$
associative rings	associative ($= \mathbb{E}_1$)-ring spectra
commutative rings	commutative ($= \mathbb{E}_\infty$)-ring spectra

The Frobenius homomorphism...
...in ordinary algebra

Commutative Frobenius

R commutative ring

$$\varphi_p : R \to R/p \qquad r \mapsto [r^p]$$

Map of commutative rings

The Frobenius homomorphism...

...in ordinary algebra

Commutative Frobenius

R commutative ring

$$\varphi_p : R \to R/p \qquad r \mapsto [r^p]$$

Map of commutative rings

Associative Frobenius

R associative ring

The Frobenius homomorphism...

...in ordinary algebra

Commutative Frobenius

R commutative ring

$$\varphi_p : R \to R/p \qquad r \mapsto [r^p]$$

Map of commutative rings

Associative Frobenius

R associative ring

$$\varphi_p : R/[R,R] \to (R/[R,R])/pR \qquad [r] \mapsto [r^p]$$

Map of abelian groups

$[R,R] \subseteq R$: subgroup generated by $rs - sr$

The Frobenius homomorphism...
...in ordinary & higher algebra

Commutative Frobenius

R commutative ring

$$\varphi_p : R \to R/p \qquad r \mapsto [r^p]$$

Map of commutative rings

Tate valued Frobenius

R commutative ring spectrum

$$\varphi_p : R \to R^{tC_p}$$

Map of commutative ring spectra

Associative Frobenius

R associative ring

$$\varphi_p : R/[R, R] \to (R/[R, R])/pR \qquad [r] \mapsto [r^p]$$

Map of abelian groups

The Frobenius homomorphism...
...in ordinary & higher algebra

Commutative Frobenius

R commutative ring

$$\varphi_p : R \to R/p \qquad r \mapsto [r^p]$$

Map of commutative rings

Tate valued Frobenius

R commutative ring spectrum

$$\varphi_p : R \to R^{tC_p}$$

Map of commutative ring spectra

Associative Frobenius

R associative ring

$$\varphi_p : R/[R,R] \to (R/[R,R])/pR \qquad [r] \mapsto [r^p]$$

Map of abelian groups

Frobenius on THH

R associative ring spectrum

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

Map of spectra

The Tate construction

G finite group, M abelian group with G-action.

Norm map

$$\mathrm{Nm} : M_G \to M^G$$

with $x \mapsto \sum_{g \in G} gx$

G finite group, M abelian group with G-action.

Norm map

$$\mathrm{Nm} : M_G \to M^G$$

with $x \mapsto \sum_{g \in G} gx$

Definition

Consider $M^G/_{\mathrm{Nm}} = \mathrm{coker}(M_G \to M^G)$

The Tate construction

G finite group, M abelian group with G-action.
Norm map

$$\mathrm{Nm} : M_G \to M^G$$

with $x \mapsto \sum_{g \in G} gx$

Definition

Consider $M^G/_{\mathrm{Nm}} = \mathrm{coker}(M_G \to M^G)$

Example

If $G = C_p$ acts trivially on M, then

$$M^{C_p}/_{\mathrm{Nm}} \cong M/p$$

The Tate construction

G finite group, M abelian group with G-action.
Norm map

$$\mathrm{Nm} : M_G \to M^G$$

with $x \mapsto \sum_{g \in G} gx$

Definition

Consider $M^G/_{\mathrm{Nm}} = \mathrm{coker}(M_G \to M^G)$

Example

If $G = C_p$ acts trivially on M, then

$$M^{C_p}/_{\mathrm{Nm}} \cong M/p$$

G finite group, X spectrum with G-action.
Norm map

$$\mathrm{Nm} : X_{hG} \to X^{hG}$$

from (homotopy) coinvariants to invariants.

G finite group, M abelian group with G-action.
Norm map
$$\mathrm{Nm}: M_G \to M^G$$
with $x \mapsto \sum_{g \in G} gx$

Definition

Consider $M^G/_{\mathrm{Nm}} = \mathrm{coker}(M_G \to M^G)$

Example

If $G = C_p$ acts trivially on M, then
$$M^{C_p}/_{\mathrm{Nm}} \cong M/p$$

G finite group, X spectrum with G-action.
Norm map
$$\mathrm{Nm}: X_{hG} \to X^{hG}$$
from (homotopy) coinvariants to invariants.

Definition

The Tate spectrum X^{tG} is the cofibre of the norm.

The Tate construction

G finite group, M abelian group with G-action.
Norm map

$$\mathrm{Nm} : M_G \to M^G$$

with $x \mapsto \sum_{g \in G} gx$

Definition

Consider $M^G/_{\mathrm{Nm}} = \mathrm{coker}(M_G \to M^G)$

Example

If $G = C_p$ acts trivially on M, then

$$M^{C_p}/_{\mathrm{Nm}} \cong M/p$$

G finite group, X spectrum with G-action.
Norm map

$$\mathrm{Nm} : X_{hG} \to X^{hG}$$

from (homotopy) coinvariants to invariants.

Definition

The Tate spectrum X^{tG} is the cofibre of the norm.

Theorem (Lin '80, Gunawardena '80)

If X is finite spectrum, then

$$X^{tC_p} \simeq X_p^\wedge .$$

The Tate diagonal

Example

1. A abelian group. The map

$$\Delta_p : A \to (A \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} A)^{C_p}$$
$$a \mapsto (a \otimes \dots \otimes a)$$

is not additive.

The Tate diagonal

Example

1. A abelian group. The map

$$\begin{aligned} \Delta_p : A &\to (A \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} A)^{C_p}/\mathrm{Nm} \\ a &\mapsto (a \otimes \ldots \otimes a) \end{aligned}$$

is additive!

The Tate diagonal

Example

1. A abelian group. The map

$$\begin{aligned}\Delta_p : A &\to (A \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} A)^{C_p}/\mathrm{Nm} \\ a &\mapsto (a \otimes \dots \otimes a)\end{aligned}$$

   is additive!

2. If A is p-torsion then this map is an isomorphism.

The Tate diagonal

Example

1. A abelian group. The map

$$\begin{aligned}\Delta_p : A &\to (A \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} A)^{C_p}/\mathrm{Nm} \\ a &\mapsto (a \otimes \ldots \otimes a)\end{aligned}$$

   is additive!

2. If A is p-torsion then this map is an isomorphism.

Theorem (Rognes–Nielsen '10, N.–Scholze '17)

1. *X a spectrum. There is a (unique) natural map*

$$\Delta_p : X \to (X \otimes_{\mathbb{S}} \ldots \otimes_{\mathbb{S}} X)^{tC_p}$$

   which is lax symmetric monoidal.

The Tate diagonal

Example

1. A abelian group. The map

$$\Delta_p : A \to (A \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} A)^{C_p}/\mathrm{Nm}$$
$$a \mapsto (a \otimes \dots \otimes a)$$

   is additive!

2. If A is p-torsion then this map is an isomorphism.

Theorem (Rognes–Nielsen '10, N.–Scholze '17)

1. X *a spectrum. There is a (unique) natural map*

$$\Delta_p : X \to (X \otimes_{\mathbb{S}} \dots \otimes_{\mathbb{S}} X)^{tC_p}$$

   which is lax symmetric monoidal.

2. X p*-complete, bounded below then this map is an equivalence.*

The Tate diagonal

Example

1. A abelian group. The map

$$\Delta_p : A \to (A \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} A)^{C_p}/\mathrm{Nm}$$
$$a \mapsto (a \otimes \dots \otimes a)$$

   is additive!

2. If A is p-torsion then this map is an isomorphism.

Theorem (Rognes–Nielsen '10, N.–Scholze '17)

1. X *a spectrum. There is a (unique) natural map*

$$\Delta_p : X \to (X \otimes_{\mathbb{S}} \dots \otimes_{\mathbb{S}} X)^{tC_p}$$

   which is lax symmetric monoidal.

2. X p*-complete, bounded below then this map is an equivalence.*

Conjecture

G *finite* p*-group,* X p*-complete, bounded below spectrum. Then the analogous map*

$$\Delta_G : X \to \left(X^{\otimes_{\mathbb{S}} G}\right)^{\varphi G}$$

is an equivalence.

The Tate valued Frobenius

Example

R commutative ring. The Frobenius is the composite

$$R \xrightarrow{\Delta_p} (R \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} R)^{C_p}/_{\mathrm{Nm}} \xrightarrow{m} R^{C_p}/_{\mathrm{Nm}} = R/p$$

The Tate valued Frobenius

Example

R commutative ring. The Frobenius is the composite

$$R \xrightarrow{\Delta_p} (R \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} R)^{C_p}/\mathrm{Nm} \xrightarrow{m} R^{C_p}/\mathrm{Nm} = R/p$$

Definition

R an $\mathbb{E}_\infty$-ring. The Tate-valued Frobenius is the map of $\mathbb{E}_\infty$-ring spectra

$$\varphi_p : R \xrightarrow{\Delta_p} (R \otimes_{\mathbb{S}} \ldots \otimes_{\mathbb{S}} R)^{tC_p} \xrightarrow{m^{tC_p}} R^{tC_p}$$

The Tate valued Frobenius

Example

R commutative ring. The Frobenius is the composite

$$R \xrightarrow{\Delta_p} (R \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} R)^{C_p}/\mathrm{Nm} \xrightarrow{m} R^{C_p}/\mathrm{Nm} = R/p$$

Definition

R an $\mathbb{E}_\infty$-ring. The Tate-valued Frobenius is the map of $\mathbb{E}_\infty$-ring spectra

$$\varphi_p : R \xrightarrow{\Delta_p} (R \otimes_{\mathbb{S}} \dots \otimes_{\mathbb{S}} R)^{tC_p} \xrightarrow{m^{tC_p}} R^{tC_p}$$

Example

If $R = C^*(M, \mathbb{F}_2)$ then φ_2 induces on π_* the map

$$H^*(M, \mathbb{F}_2) \to H^*(M, \mathbb{F}_2)((t)) \qquad x \mapsto \sum Sq^i(x) t^{-i}$$

Spaces and the Frobenius

Example

- If R is p-complete, finite spectrum, then $R^{tC_p} \simeq R \quad \Rightarrow \quad \varphi_p : R \to R$

Spaces and the Frobenius

Example

- If R is p-complete, finite spectrum, then $R^{tC_p} \simeq R \quad \Rightarrow \quad \varphi_p : R \to R$
- $R = C^*(M, \mathbb{S}_p^\wedge)$ for M a finite space. Then $\varphi_p \simeq \mathrm{id}_R$.

Example

- If R is p-complete, finite spectrum, then $R^{tC_p} \simeq R \quad \Rightarrow \quad \varphi_p : R \to R$
- $R = C^*(M, \mathbb{S}_p^\wedge)$ for M a finite space. Then $\varphi_p \simeq \mathrm{id}_R$.

Theorem (Yuan '21)

$$C^*(-, \mathbb{S}) : \quad \left\{ \begin{array}{l} \textit{finite, simply} \\ \textit{conn. spaces} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{l} \textit{finite } \mathbb{E}_\infty\textit{-algebras } R \textit{ with coherent trivializations} \\ \varphi_p \simeq \mathrm{id}_{R_p^\wedge} \textit{ and } \widetilde{H}^i(R, \mathbb{Z}) = 0 \textit{ for } i > -1. \end{array} \right\}$$

Spaces and the Frobenius

Example

- If R is p-complete, finite spectrum, then $R^{tC_p} \simeq R \quad \Rightarrow \quad \varphi_p : R \to R$
- $R = C^*(M, \mathbb{S}_p^\wedge)$ for M a finite space. Then $\varphi_p \simeq \mathrm{id}_R$.

Theorem (Yuan '21)

$$C^*(-,\mathbb{S}) : \left\{ \begin{array}{l} \textit{finite, simply} \\ \textit{conn. spaces} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{l} \textit{finite } \mathbb{E}_\infty\textit{-algebras } R \textit{ with coherent trivializations} \\ \varphi_p \simeq \mathrm{id}_{R_p^\wedge} \textit{ and } \widetilde{H}^i(R,\mathbb{Z}) = 0 \textit{ for } i > -1. \end{array} \right\}$$

Conjecture

$$C_*(-,\mathbb{S}) : \left\{ \begin{array}{l} \textit{simply} \\ \textit{conn. spaces} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{l} \textit{simply conn. } \mathbb{E}_\infty\textit{-coalgebras } C \textit{ with} \\ \textit{coherent trivializations } \varphi_p \simeq \mathrm{id}_{C_p^\wedge} \end{array} \right\}$$

The Frobenius homomorphism...
...in ordinary & higher algebra

Commutative Frobenius

R commutative ring

$$\varphi_p : R \to R/p \qquad r \mapsto [r^p]$$

Map of commutative rings

Tate valued Frobenius

R commutative ring spectrum

$$\varphi_p : R \to R^{tC_p}$$

Map of commutative ring spectra

Associative Frobenius

R associative ring

$$\varphi_p : R/[R,R] \to (R/[R,R])/pR \qquad [r] \mapsto [r^p]$$

Map of abelian groups

Frobenius on THH

R associative ring spectrum

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

Map of spectra

The Frobenius homomorphism...
...in ordinary & higher algebra

Commutative Frobenius

R commutative ring

$$\varphi_p : R \to R/p \qquad r \mapsto [r^p]$$

Map of commutative rings

Tate valued Frobenius

R commutative ring spectrum

$$\varphi_p : R \to R^{tC_p}$$

✓

Map of commutative ring spectra

Associative Frobenius

R associative ring

$$\varphi_p : R/[R,R] \to (R/[R,R])/pR \qquad [r] \mapsto [r^p]$$

Map of abelian groups

Frobenius on THH

R associative ring spectrum

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

Map of spectra

Example

R associative ring, then

$$R/[R, R] = R \otimes_{R \otimes_{\mathbb{Z}} R^{op}} R$$

as abelian groups.

Topological Hochschild homology

Example

R associative ring, then

$$R/[R,R] = R \otimes_{R \otimes_{\mathbb{Z}} R^{op}} R$$

as abelian groups.

Definition

*R associative ring spectrum. **Topological Hochschild homology** is the spectrum*

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R\,.$$

Topological Hochschild homology

Example

R associative ring, then

$$R/[R,R] = R \otimes_{R \otimes_{\mathbb{Z}} R^{op}} R$$

as abelian groups.

Definition

R associative ring spectrum. Topological Hochschild homology is the spectrum

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R\,.$$

Facts:

- R commutative ring spectrum $\Rightarrow$ $\mathrm{THH}(R)$ commutative ring spectrum

Topological Hochschild homology

Example

R associative ring, then

$$R/[R, R] = R \otimes_{R \otimes_{\mathbb{Z}} R^{op}} R$$

as abelian groups.

Definition

R associative ring spectrum. Topological Hochschild homology is the spectrum

$$THH(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R\,.$$

Facts:

- R commutative ring spectrum $\Rightarrow$ THH(R) commutative ring spectrum
  $\Rightarrow THH_*(R) := \pi_*(THH(R))$ graded commutative ring

Topological Hochschild homology

Example

R associative ring, then

$$R/[R,R] = R \otimes_{R\otimes_{\mathbb{Z}} R^{op}} R$$

as abelian groups.

Definition

R associative ring spectrum. Topological Hochschild homology is the spectrum

$$\mathrm{THH}(R) = R \otimes_{R\otimes_{\mathbb{S}} R^{op}} R\,.$$

Facts:

- R commutative ring spectrum $\Rightarrow$ THH(R) commutative ring spectrum
  $\Rightarrow \mathrm{THH}_*(R) := \pi_*(\mathrm{THH}(R))$ graded commutative ring
- There is an S^1-action on THH(R) (Connes '83)

Topological Hochschild homology

Example

R associative ring, then

$$R/[R,R] = R \otimes_{R \otimes_{\mathbb{Z}} R^{op}} R$$

as abelian groups.

Definition

R associative ring spectrum. Topological Hochschild homology is the spectrum

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R\,.$$

Facts:

- R commutative ring spectrum $\Rightarrow$ THH(R) commutative ring spectrum
  $\Rightarrow \mathrm{THH}_*(R) := \pi_*(\mathrm{THH}(R))$ graded commutative ring
- There is an S^1-action on THH(R) (Connes '83)

Theorem (Bökstedt '85)

There is an isomorphism

$$\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[x] \qquad |x| = 2$$

Proposition (Bökstedt–Hsiang–Madsen ’93,...,N.–Scholze ’17)

For every prime p there is a S^1-equivariant map of spectra

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

The Frobenius on THH

Proposition (Bökstedt–Hsiang–Madsen ’93,...,N.–Scholze ’17)

For every prime p there is a S^1-equivariant map of spectra

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

- Constructed from the Tate-diagonal

The Frobenius on THH

Proposition (Bökstedt–Hsiang–Madsen ’93,...,N.–Scholze ’17)

For every prime p there is a S^1-equivariant map of spectra

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

- Constructed from the Tate-diagonal
- R commutative ring spectrum, then have commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{THH}(R) \\ \downarrow{\scriptstyle \varphi_p} & & \downarrow{\scriptstyle \varphi_p} \\ R^{tC_p} & \longleftarrow & \mathrm{THH}(R)^{tC_p} \end{array}.$$

The Frobenius on THH

Proposition (Bökstedt–Hsiang–Madsen '93,...,N.–Scholze '17)

For every prime p there is a S^1-equivariant map of spectra

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

- Constructed from the Tate-diagonal
- R commutative ring spectrum, then have commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{THH}(R) \\ \downarrow{\scriptstyle \varphi_p} & & \downarrow{\scriptstyle \varphi_p} \\ R^{tC_p} & \longleftarrow & \mathrm{THH}(R)^{tC_p} \end{array} .$$

- φ_p is equivalence after p-completion (in positive degrees) for $R = \mathbb{S}, H\mathbb{F}_p, H\mathbb{Z}, MU, BP, \ldots$

Recall

For $K_*(\mathbb{Z}/p^k)$ *we use* ***trace methods***:

$$K_*(R)$$

$$\uparrow$$

$$TC_*(R)$$

$$\uparrow$$

$$THH_*(R)$$

Recall

For $K_*(\mathbb{Z}/p^k)$ *we use* ***trace methods****:*

$$\begin{array}{c} K_*(R) \\ \uparrow \\ TC_*(R) \\ \uparrow \\ THH_*(R) \end{array}$$

- TC(R) defined by Bökstedt-Hsiang-Madsen ’94

Topological cyclic homology

Recall

For $K_*(\mathbb{Z}/p^k)$ *we use trace methods:*

$$\begin{array}{c} K_*(R) \\ \uparrow \\ TC_*(R) \\ \uparrow \\ THH_*(R) \end{array}$$

- $TC(R)$ defined by Bökstedt-Hsiang-Madsen '94
- There is a map $K(R) \to TC(R)$ called cyclotomic trace

Topological cyclic homology

Recall

For $K_*(\mathbb{Z}/p^k)$ *we use* *trace methods:*

$$\begin{array}{c} K_*(R) \\ \uparrow \wr \\ TC_*(R) \\ \uparrow \wr \\ THH_*(R) \end{array}$$

- $TC(R)$ defined by Bökstedt-Hsiang-Madsen ’94
- There is a map $K(R) \to TC(R)$ called cyclotomic trace
- Often an iso, e.g. $K_*(\mathbb{Z}/p^k)^\wedge_p \cong TC_*(\mathbb{Z}/p^k)$ for $* > 0$

Topological cyclic homology

Recall

For $K_*(\mathbb{Z}/p^k)$ *we use* *trace methods*:

$$\begin{array}{c} K_*(R) \\ \uparrow \\ TC_*(R) \\ \uparrow \\ THH_*(R) \end{array}$$

- $TC(R)$ defined by Bökstedt-Hsiang-Madsen '94
- There is a map $K(R) \to TC(R)$ called cyclotomic trace
- Often an iso, e.g. $K_*(\mathbb{Z}/p^k)^\wedge_p \cong TC_*(\mathbb{Z}/p^k)$ for $* > 0$

Theorem (N.-Scholze '17)

If R *is connective, then* $TC(R)$ *can be computed from* $THH(R)$ *with its* S^1*-action and maps* φ_p.

Topological cyclic homology

Recall

For $K_*(\mathbb{Z}/p^k)$ we use trace methods:

$$\begin{array}{c} K_*(R) \\ \uparrow \\ TC_*(R) \\ \uparrow \\ THH_*(R) \end{array}$$

- $TC(R)$ defined by Bökstedt-Hsiang-Madsen '94
- There is a map $K(R) \to TC(R)$ called cyclotomic trace
- Often an iso, e.g. $K_*(\mathbb{Z}/p^k)^\wedge_p \cong TC_*(\mathbb{Z}/p^k)$ for $* > 0$

Theorem (N.-Scholze '17)

If R is connective, then $TC(R)$ can be computed from $THH(R)$ with its S^1-action and maps φ_p.

More precisely: CycSp the ∞-category of spectra with S^1-action and S^1-equiv. maps $\varphi_p : X \to X^{tC_p}$. Then

$$TC(R) \simeq \mathrm{map}_{\mathrm{CycSp}}(\mathbb{1}, THH(R))$$

Prisms and Bökstedt periodicity

For $\mathrm{TC}(\mathbb{Z}/p^k)$, we compute TC relative to $\mathbb{S}[\![z]\!]_p$ and use descent along $\mathbb{S} \to \mathbb{S}[\![z]\!]_p$

Prisms and Bökstedt periodicity

For $\mathrm{TC}(\mathbb{Z}/p^k)$, we compute TC relative to $\mathbb{S}[\![z]\!]_p$ and use descent along $\mathbb{S} \to \mathbb{S}[\![z]\!]_p$

Proposition

Let $\mathbb{S}_A$ be a p-complete $\mathbb{E}_\infty$-ring, flat over $\mathbb{S}$ and $A := \pi_0(\mathbb{S}_A)$.

Prisms and Bökstedt periodicity

For $TC(\mathbb{Z}/p^k)$, we compute TC relative to $\mathbb{S}[\![z]\!]_p$ and use descent along $\mathbb{S} \to \mathbb{S}[\![z]\!]_p$

Proposition

Let $\mathbb{S}_A$ be a p-complete $\mathbb{E}_\infty$-ring, flat over $\mathbb{S}$ and $A := \pi_0(\mathbb{S}_A)$. Then

$$\varphi_p : \mathbb{S}_A \to \mathbb{S}_A^{tC_p} \simeq \mathbb{S}_A$$

induces on π_0 a lift of Fronbenius, i.e. A is a δ-ring.

Prisms and Bökstedt periodicity

For $\mathrm{TC}(\mathbb{Z}/p^k)$, we compute TC relative to $\mathbb{S}[\![z]\!]_p$ and use descent along $\mathbb{S} \to \mathbb{S}[\![z]\!]_p$

Proposition

Let $\mathbb{S}_A$ be a p-complete $\mathbb{E}_\infty$-ring, flat over $\mathbb{S}$ and $A := \pi_0(\mathbb{S}_A)$. Then

$$\varphi_p : \mathbb{S}_A \to \mathbb{S}_A^{tC_p} \simeq \mathbb{S}_A$$

induces on π_0 a lift of Fronbenius, i.e. A is a δ-ring.

Theorem (Antieau–Krause–N.)

R a nice A-algebra. Then $\mathrm{TC}(R/\mathbb{S}_A)$ admits a complete filtration with i-th graded given by an extension

$$\mathbb{Z}_p(i)\big(R/A\big)[2i]\ .$$

of Bhatt-Scholze's syntomic cohomology relative to δ-rings

Prisms and Bökstedt periodicity

For $TC(\mathbb{Z}/p^k)$, we compute TC relative to $\mathbb{S}[\![z]\!]_p$ and use descent along $\mathbb{S} \to \mathbb{S}[\![z]\!]_p$

Proposition

Let $\mathbb{S}_A$ be a p-complete $\mathbb{E}_\infty$-ring, flat over $\mathbb{S}$ and $A := \pi_0(\mathbb{S}_A)$. Then

$$\varphi_p : \mathbb{S}_A \to \mathbb{S}_A^{tC_p} \simeq \mathbb{S}_A$$

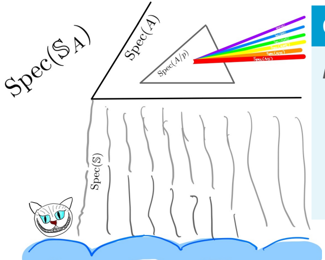
induces on π_0 a lift of Fronbenius, i.e. A is a δ-ring.

Theorem (Antieau–Krause–N.)

R a nice A-algebra. Then $TC(R/\mathbb{S}_A)$ admits a complete filtration with i-th graded given by an extension

$$\mathbb{Z}_p(i)(R/A)[2i] \,.$$

of Bhatt-Scholze's syntomic cohomology relative to δ-rings



Corollary (Ultimate Bökstedt periodicity)

For $R = A/I$ with (A, I) a prism we have

$$THH_*(R/\mathbb{S}_A) \cong \begin{cases} I^n/I^{n+1} & \text{for } * = 2n \\ 0 & \text{else} \end{cases}$$

THH-Perfectness

Definition

We say that $\mathrm{THH}(R)$ is *eventually* p*-perfect* if the map $\varphi_p : \mathrm{THH}(R)^{\wedge}_p \to \mathrm{THH}(R)^{tC_p}$ is an isomorphism on π_* for $* \gg 0$.

Definition

We say that $\mathrm{THH}(R)$ is *eventually* p*-perfect* if the map $\varphi_p : \mathrm{THH}(R)^\wedge_p \to \mathrm{THH}(R)^{tC_p}$ is an isomorphism on π_* for $* \gg 0$.

This is true for $R = \mathbb{S}, \mathbb{F}_p, \mathbb{Z}$, MU, BP, ...

Theorem (Antieau–N. ’18)

1. *There is a* t*-structure on* $\mathrm{CycSp}^\wedge_p$ *whose connective objects are those* (X, φ_p) *such that* X *is connective.*
2. *Every t-truncated object* X *is eventually* p*-perfect and has truncated* TC.

Definition

We say that $\mathrm{THH}(R)$ is *eventually* p*-perfect* if the map $\varphi_p : \mathrm{THH}(R)^{\wedge}_p \to \mathrm{THH}(R)^{tC_p}$ is an isomorphism on π_* for $* \gg 0$.

This is true for $R = \mathbb{S}, \mathbb{F}_p, \mathbb{Z}, \mathrm{MU}, \mathrm{BP}, \ldots$

Theorem (Antieau–N. ’18)

1. *There is a* t*-structure on* $\mathrm{CycSp}^{\wedge}_p$ *whose connective objects are those* (X, φ_p) *such that* X *is connective.*
2. *Every t-truncated object* X *is eventually* p*-perfect and has truncated* TC.

Hahn–Wilson ’20 prove a converse to (2) and deduce that

$$\mathrm{TC}(\mathrm{BP}\langle n\rangle) \to L^f_{n+1}(\mathrm{TC}(\mathrm{BP}\langle n\rangle))$$

$$\mathrm{K}(\mathrm{BP}\langle n\rangle) \to L^f_{n+1}(\mathrm{K}(\mathrm{BP}\langle n\rangle))$$

are after p-completion equivalences in degrees $* \gg 0$ (redshift).