Extremal and near-extremal black holes

Mihalis Dafermos
(Cambridge/Princeton)

Münster Midterm Conference
27 March 2024

Plan of the lecture

1. ***What is a black hole?***

2. ***What makes a black hole "extremal"?***

3. ***Can non-extremal black holes become extremal?***

4. ***How exceptional are extremal black holes and what do dynamics look like nearby?***

5. ***Epilogue: A new picture of the moduli space of gravitational collapse***

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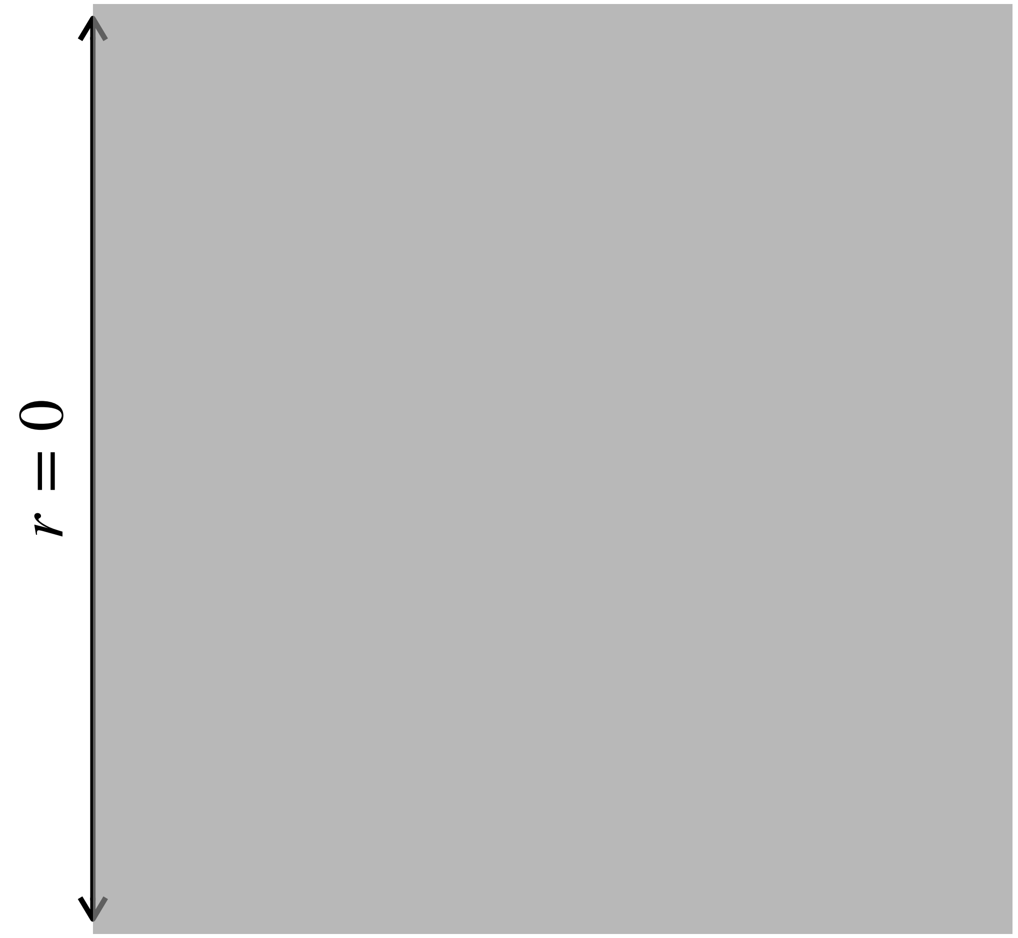
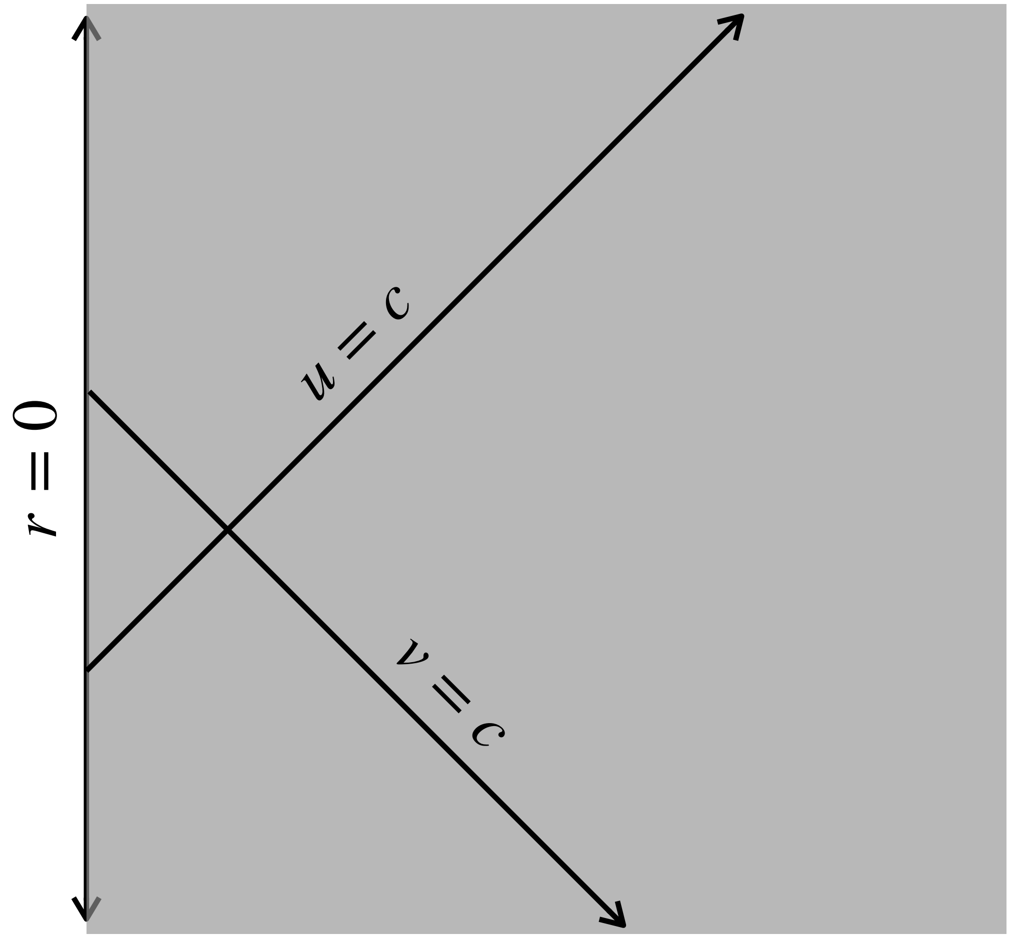
Minkowski space



$$-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Minkowski space



$$u = t - r$$
$$v = t + r$$

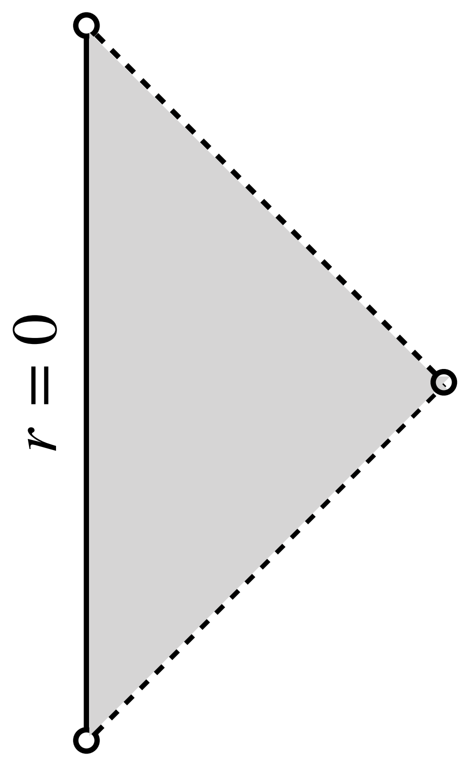
$$-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$-du dv + r^2(u, v)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Minkowski space

$U = U(u)$

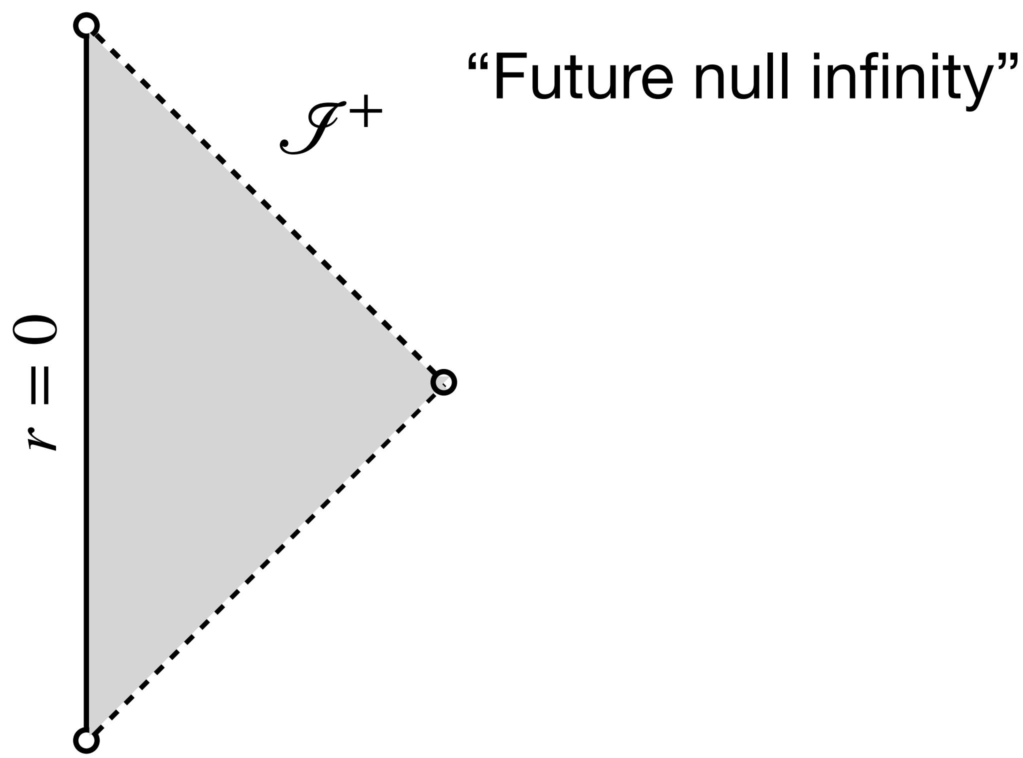
$V = V(v)$



$$-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$-\Omega^2(U,V)dUdV + r^2(U,V)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Minkowski space



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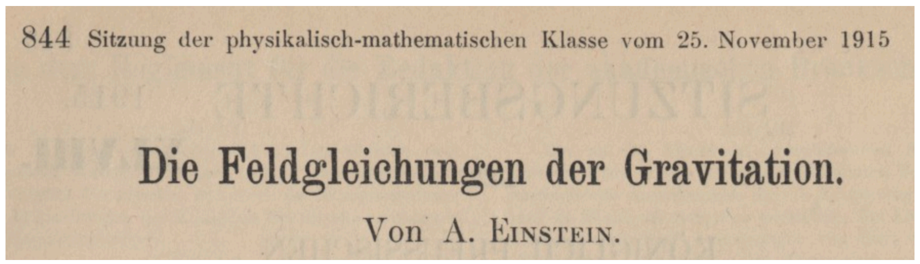
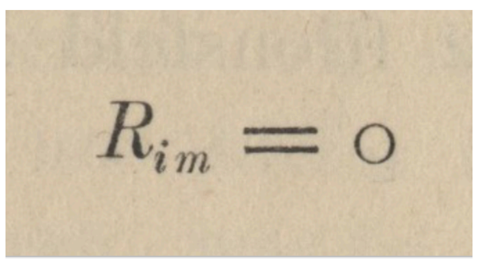
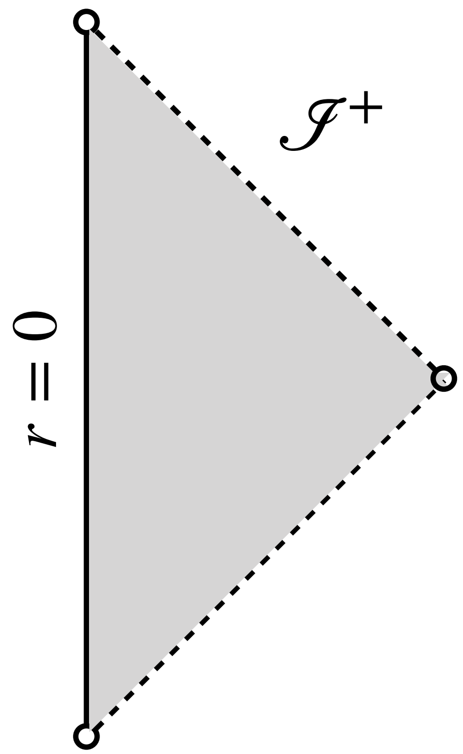
Minkowski space


844 Sitzung der physikalisch-mathematischen Klasse vom 25. November 1915

Die Feldgleichungen der Gravitation.

Von A. Einstein.


$$R_{im} = 0$$



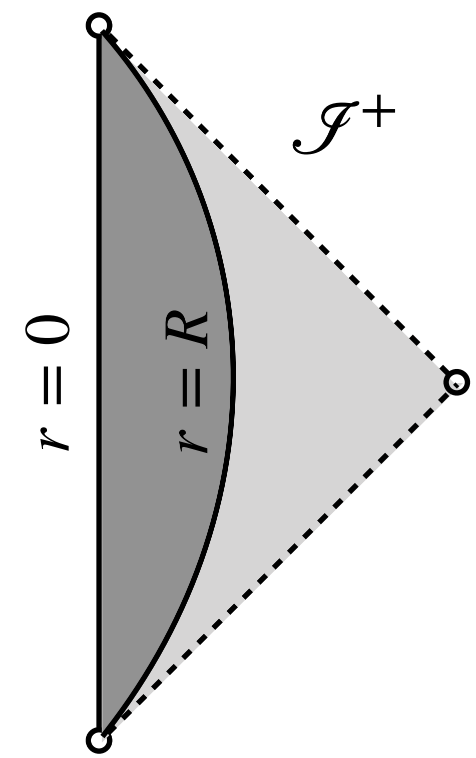
"Future null infinity"

$$-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$-\Omega^2(U,V)dUdV + r^2(U,V)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Stars in General Relativity

$$R_{im} = -\varkappa\left(T_{im} - \frac{1}{2} g_{im} T\right)$$



"Future null infinity"

$$-\left(1 - \frac{2M(r)}{r}\right) dt^2 + \left(1 - \frac{2M(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$-\Omega^2(U,V) dU dV + r^2(U,V)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Star collapsing to a black hole

SEPTEMBER 1, 1939 PHYSICAL REVIEW VOLUME 56

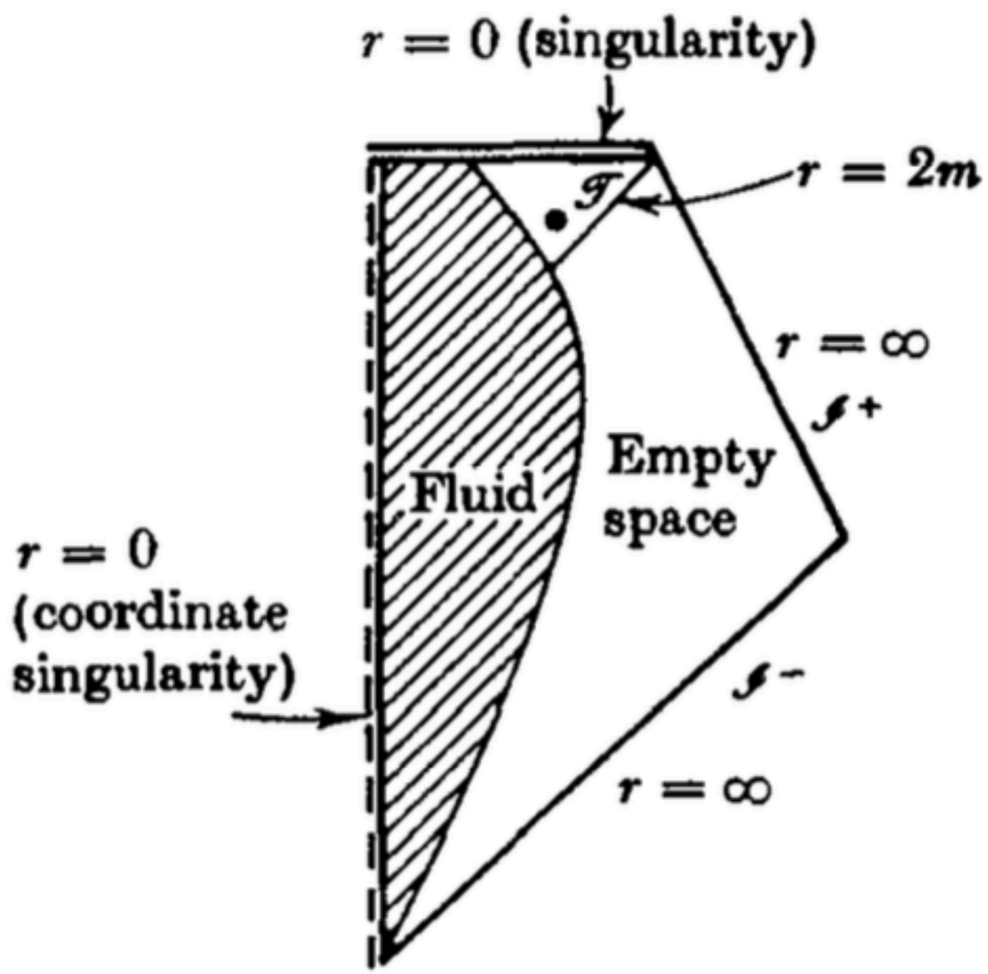
On Continued Gravitational Contraction

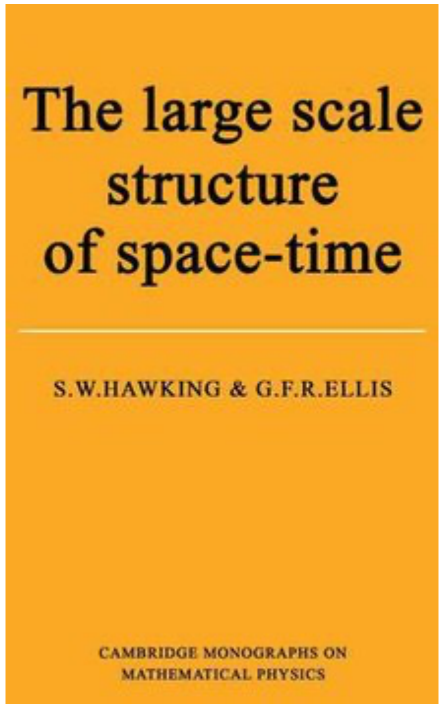
J. R. Oppenheimer and H. Snyder
University of California, Berkeley, California

(Received July 10, 1939)

When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. Unless fission due to rotation, the radiation of mass, or the blowing off of mass by radiation, reduce the star's mass to the order of that of the sun, this contraction will continue indefinitely. In the present paper we study the solutions of the gravitational field equations which describe this process. In I, general and qualitative arguments are given on the behavior of the metrical tensor as the contraction progresses: the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles. In II, an analytic solution of the field equations confirming these general arguments is obtained for the case that the pressure within the star can be neglected. The total time of collapse for an observer comoving with the stellar matter is finite, and for this idealized case and typical stellar masses, of the order of a day; an external observer sees the star asymptotically shrinking to its gravitational radius.

Star collapsing to a black hole

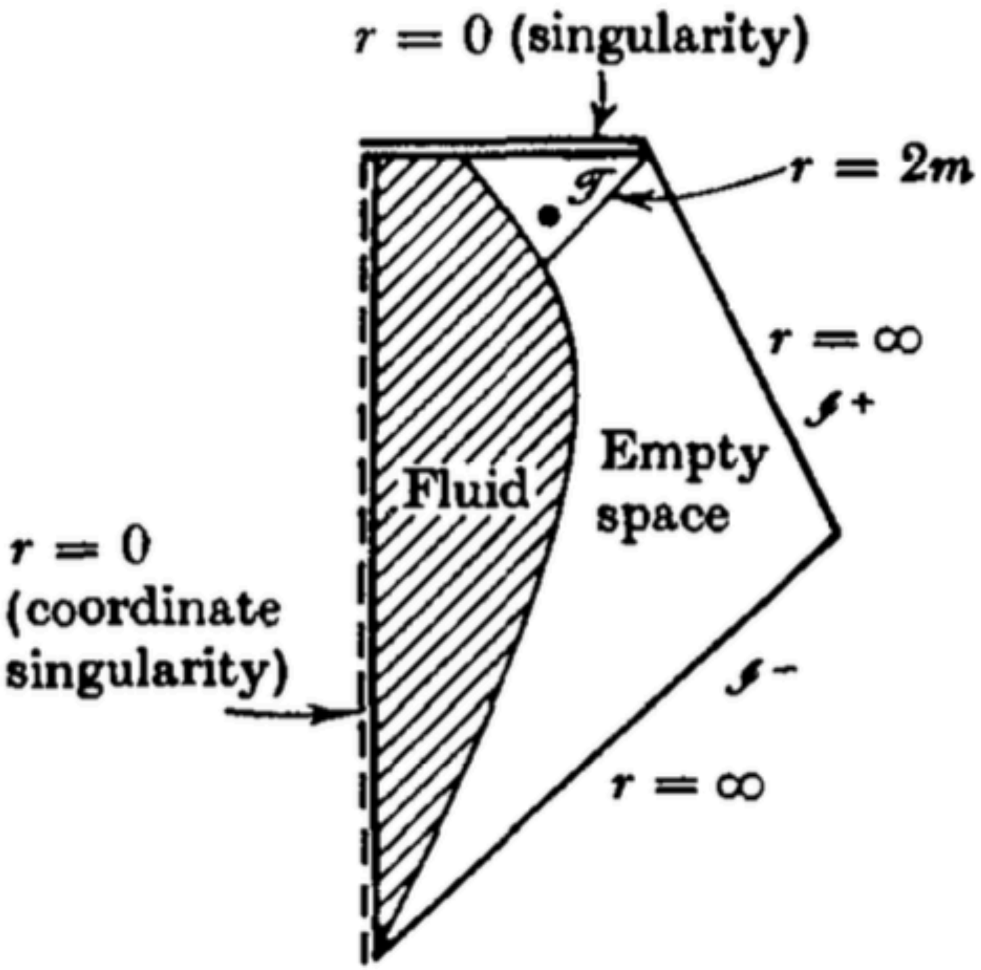


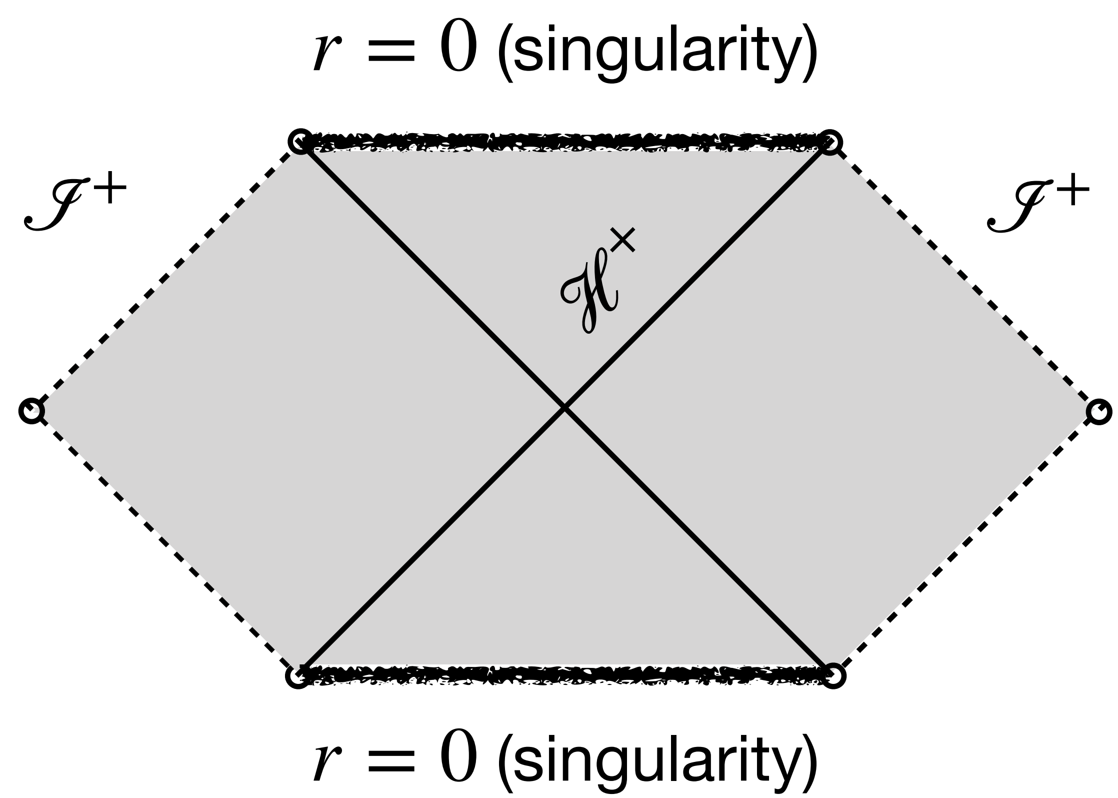


$$-\Omega^2(U,V)dUdV + r^2(U,V)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Schwarzchild spacetime

$$R_{im} = 0$$





$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$-\Omega^2(U,V)dUdV + r^2(U,V)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

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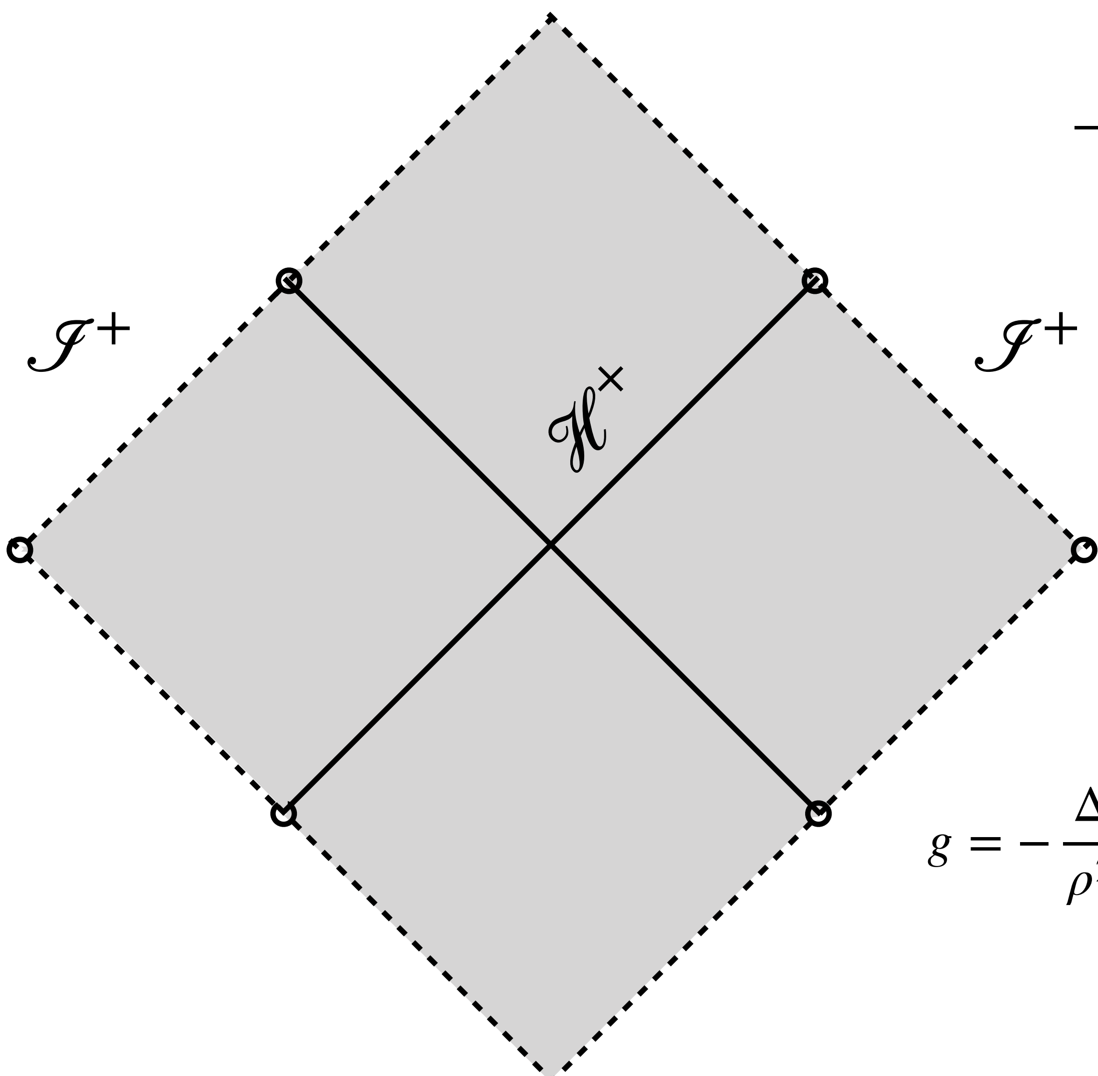
Reissner-Nordström and Kerr



add charge Q (Reissner–Nordström)

$$-\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Black hole: $|Q| \leq M$

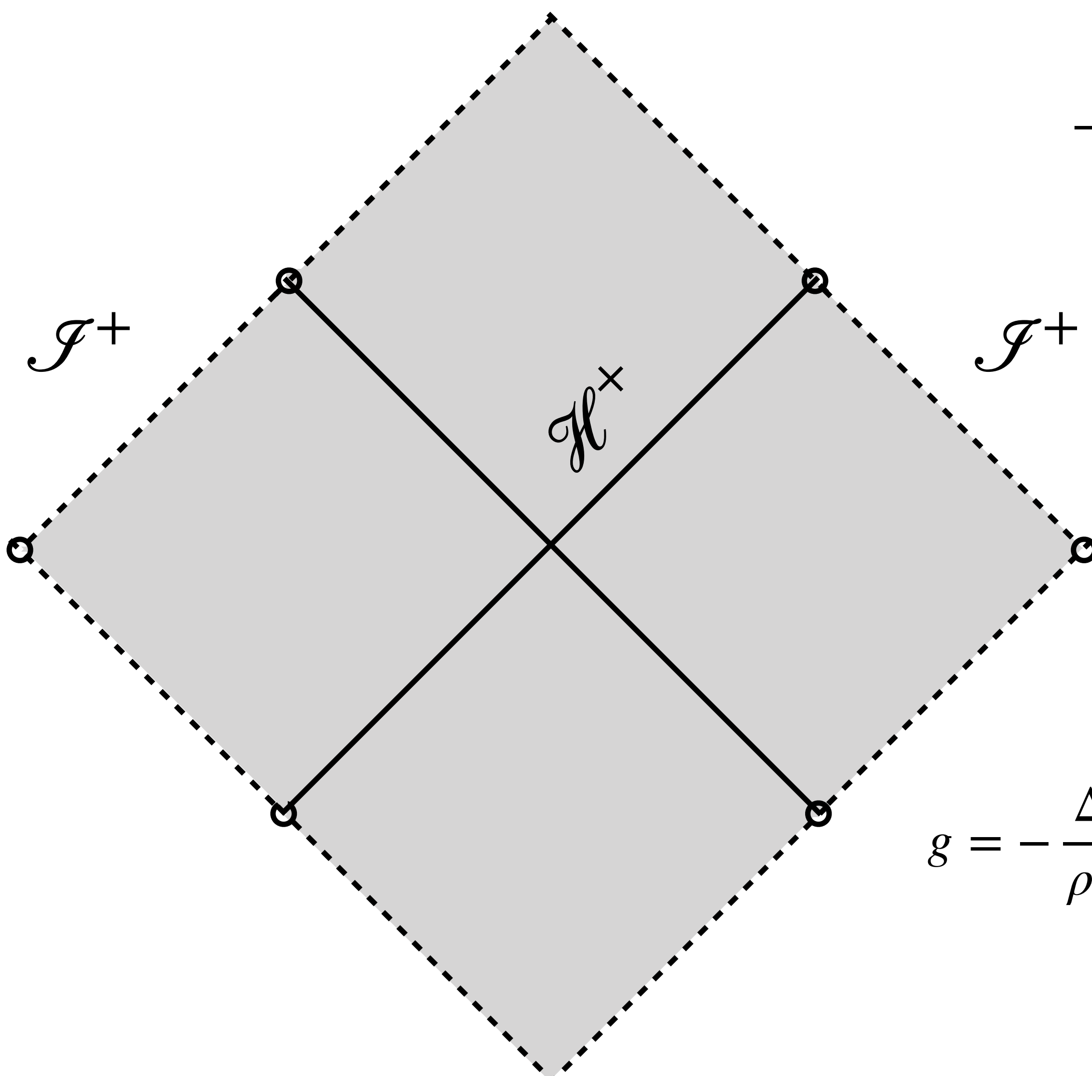
add rotation a (Kerr)

$$g = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}(a\,dt - (r^2+a^2)\,d\phi)^2$$

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \quad \Delta = (r - r_+)(r - r_-), \quad \rho^2 = r^2 + a^2\cos^2\theta$$

Black hole: $|a| \leq M$

Reissner-Nordström and Kerr



add charge Q (Reissner–Nordström)

$$-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

Black hole: $|Q| \leq M$

add rotation a (Kerr)

$$g=-\frac{\Delta}{\rho^2}(dt-a\sin^2\theta d\phi)^2+\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2+\frac{\sin^2\theta}{\rho^2}(a\,dt-(r^2+a^2)\,d\phi)^2$$

$$r_\pm = M \pm \sqrt{M^2-a^2}, \quad \Delta=(r-r_+)(r-r_-), \quad \rho^2=r^2+a^2\cos^2\theta$$

Black hole: $|a| \leq M$

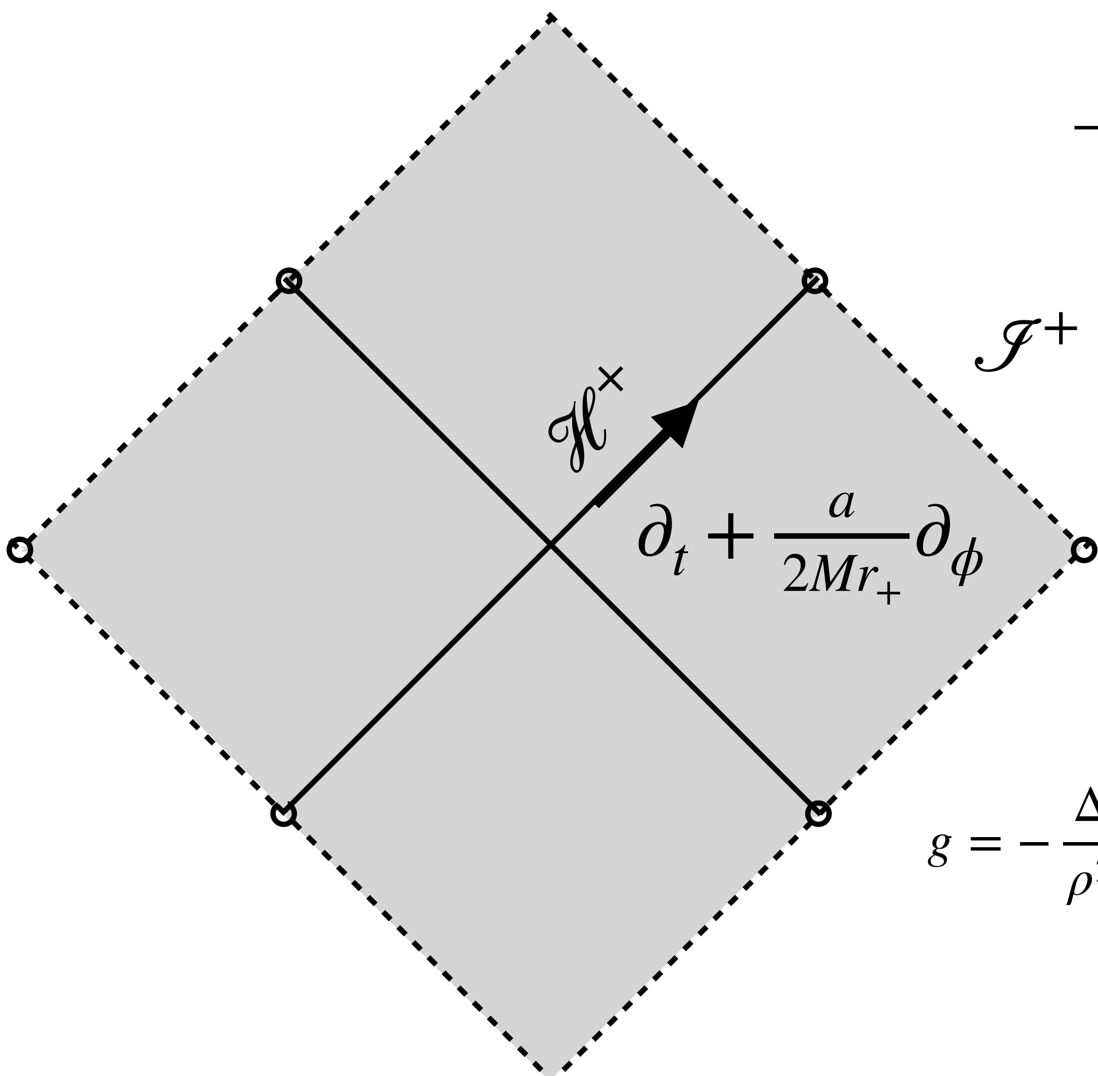
$|Q|>M?$, $|a|>M?$ **"naked singularity" !**

Surface gravity

add charge Q (Reissner–Nordstöm)

$$-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

Black hole: $|Q| \leq M$



add rotation a (Kerr)

$$g=-\frac{\Delta}{\rho^2}(dt-a\sin^2\theta d\phi)^2+\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2+\frac{\sin^2\theta}{\rho^2}(a\,dt-(r^2+a^2)\,d\phi)^2$$

$$r_\pm = M \pm \sqrt{M^2-a^2}, \quad \Delta=(r-r_+)(r-r_-), \quad \rho^2=r^2+a^2\cos^2\theta$$

$$\nabla_{\partial_t+\frac{a}{2Mr_+}\partial_\phi}\left(\partial_t+\frac{a}{2Mr_+}\partial_\phi\right)=\kappa\left(\partial_t+\frac{a}{2Mr_+}\partial_\phi\right)$$

$\kappa \geq 0$ (“surface gravity”)

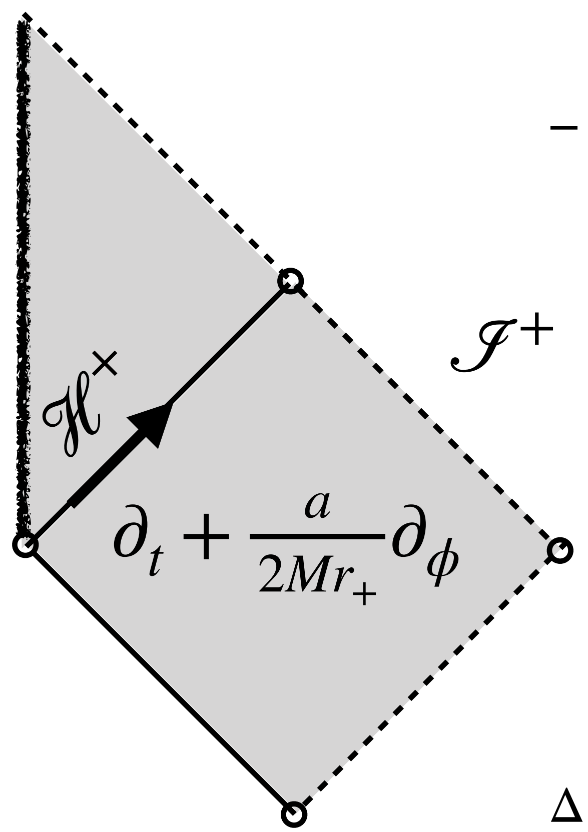
Black hole: $|a| \leq M$

Extremal black holes

add charge Q (Reissner–Nordstöm)

$$-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

Extremal case: $|Q|=M$



add rotation a (Kerr)

$$g=-\frac{\Delta}{\rho^2}(dt-a\sin^2\theta d\phi)^2+\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2+\frac{\sin^2\theta}{\rho^2}(a\,dt-(r^2+a^2)\,d\phi)^2$$

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$$\nabla_{\partial_t+\frac{a}{2Mr_+}\partial_\phi}(\partial_t+\frac{a}{2Mr_+}\partial_\phi)=\kappa(\partial_t+\frac{a}{2Mr_+}\partial_\phi)$$

$\kappa=0$ (“surface gravity”)

Extremal case: $|a|=M$

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Black hole thermodynamics

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Abstract. Expressions are derived for the mass of a stationary axisymmetric solution of the Einstein equations containing a black hole surrounded by matter and for the difference in mass between two neighboring such solutions. Two of the quantities which appear in these expressions, namely the area A of the event horizon and the "surface gravity" κ of the black hole, have a close analogy with entropy and temperature respectively. This analogy suggests the formulation of four laws of black hole mechanics which correspond to and in some ways transcend the four laws of thermodynamics.

Commun. math. Phys. 31, 161–170 (1973)

The “third law” paradigm

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

“Extremal black holes are a physically inaccessible ideal limit”

The “third law” paradigm

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It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

“Extremal black holes are a physically inaccessible ideal limit”

The “third law” paradigm

Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity. If this were to happen there would be a breakdown of the assumption of asymptotic predictability which is the basis of many results in black hole theory, including the law that A cannot decrease.

The third law is false!

arXiv:2211.15742v1 [gr-qc] 28 Nov 2022

Gravitational collapse to extremal black holes
and the third law of black hole thermodynamics

Christoph Kehle*[1] and Ryan Unger†[2]

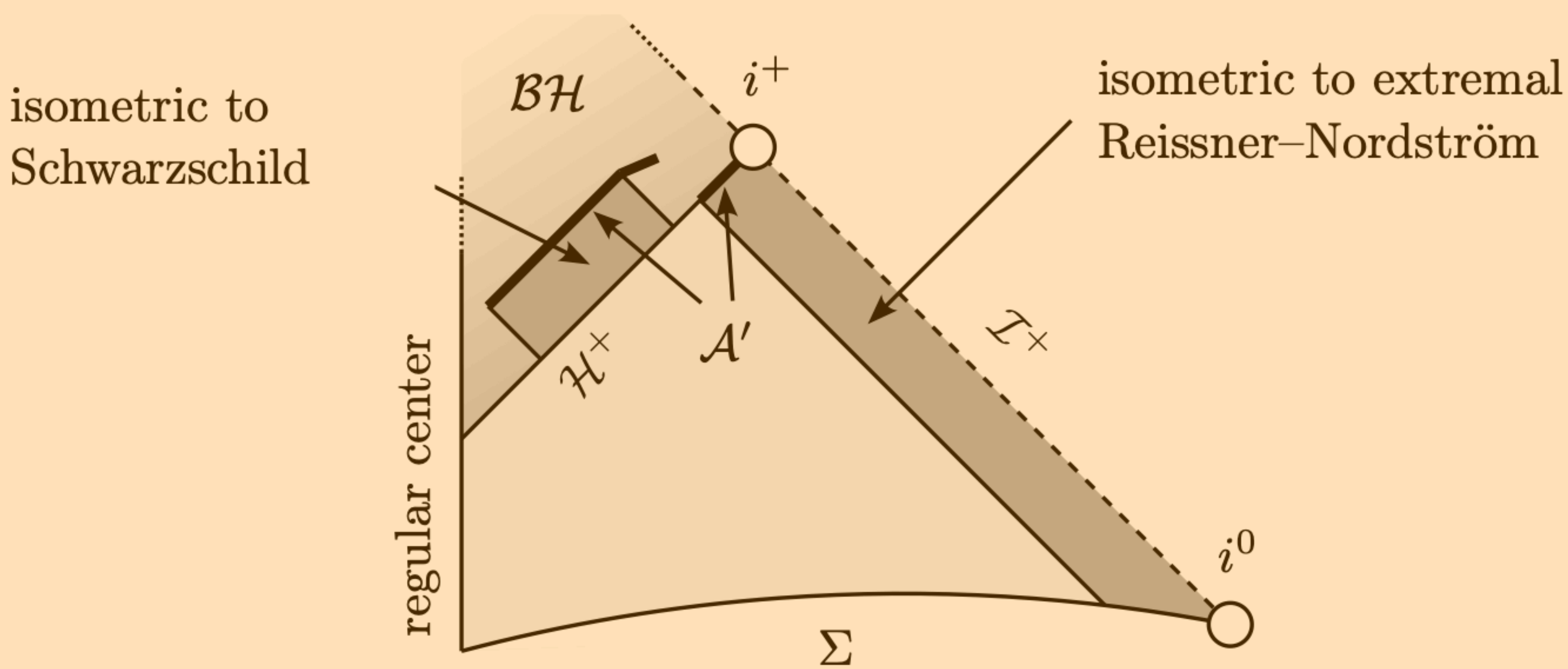
[1] *Institute for Theoretical Studies & Department of Mathematics, ETH Zürich, 8092 Zürich, Switzerland*

[2] *Department of Mathematics, Princeton University,*
Washington Road, Princeton NJ 08544, United States of America

The third law is false!

Theorem 1. *Subextremal black holes can become extremal in finite time, evolving from* <u>*regular*</u> *initial data. In fact, there exist regular one-ended Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Reissner–Nordström event horizon at a later advanced time.*

In particular, <u>*the "third law of black hole thermodynamics" is false.*</u>



~~Extending the analogy even further one would postulate:~~

~~*The Third Law*~~

~~It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.~~

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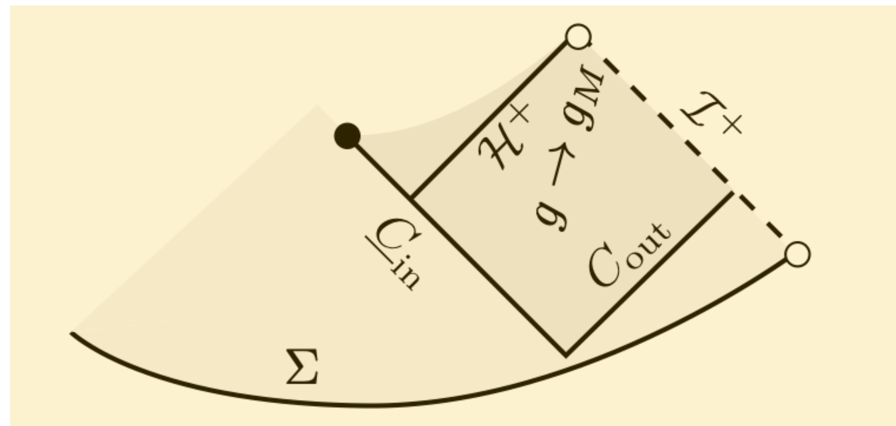
Back to Schwarzschild



Theorem I.3.1 (The nonlinear asymptotic stability of Schwarzschild in full co-dimension). *For all characteristic initial data prescribed on* (I.3.2)*, assumed sufficiently close to Schwarzschild data with mass* M_{init} *and lying on a codimension-3 "submanifold"* $\mathfrak{M}_{\text{stable}}$ *of the moduli space* $\mathfrak{M}$ *of initial data, the maximal Cauchy development* $\mathcal{M}$ *contains a region* $\mathcal{R}$ *which can be covered by appropriate (teleologically normalised) global double null gauges* (I.1.1) *and which*

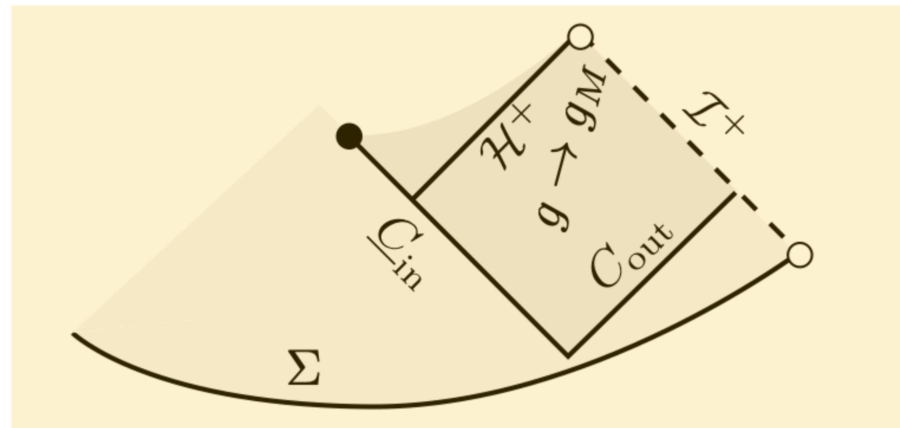
(i) possesses a complete future null infinity $\mathcal{I}^+$ *such that* $\mathcal{R} \subset J^-(\mathcal{I}^+)$*, and in fact the future boundary of* $\mathcal{R}$ *in* $\mathcal{M}$ *is a regular, future affine complete "event horizon"* $\mathcal{H}^+$*. Moreover,*

(ii) the metric remains close to the Schwarzschild metric with mass M_{init} *in* $\mathcal{R}$ *(moreover, measured with respect to an energy at the same order as a suitable "initial" energy), and*

(iii) asymptotes, inverse polynomially, to a Schwarzschild metric with mass $M_{\text{final}} \approx M_{\text{init}}$ *as* $u \to \infty$ *and* $v \to \infty$*, in particular along* $\mathcal{I}^+$ *and* $\mathcal{H}^+$*.*

[M.D.–Holzegel–Rodnianski–Taylor, 2021]

Back to Schwarzschild



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Could a similar statement hold for extremal black holes?

Problem:
the Aretakis instability

ADV. THEOR. MATH. PHYS.
Volume 19, Number 3, 507–530, 2015

Horizon instability of extremal black holes

Stefanos Aretakis

We show that axisymmetric extremal horizons are unstable under scalar perturbations. Specifically, we show that translation invariant derivatives of generic solutions to the wave equation do not decay along such horizons as advanced time tends to infinity, and in fact, higher order derivatives blow up. This instability holds in particular for extremal Kerr–Newman and Majumdar–Papapetrou spacetimes and is in stark contrast with the subextremal case for which decay is known for all derivatives along the event horizon.

This result provides a entirely new aspect of the evolution of solutions to the wave equation along degenerate horizons and has a wealth of new applications.

Weak stability

Nonetheless, in the case of extremal Reissner–Nordstrom we have weak stability results.

In particular, we have boundedness and decay results away from the horizon:

arXiv:2211.09182v1 [gr-qc] 16 Nov 2022

Instability of gravitational and electromagnetic perturbations of extremal Reissner–Nordström spacetime

Marios Antonios Apetroaie *

Theorem. *(Rough version)* Let $\alpha, \mathfrak{f}, \tilde{\beta}$ and $\underline{\alpha}, \underline{\mathfrak{f}}, \underline{\tilde{\beta}}$ be solutions to the generalized Teukolsky system of $\pm$ spin on the extreme Reissner–Nordström exterior, and let Y denote a transversal invariant derivative, then for generic initial data

i) Away from the event horizon $H^+ \equiv \{r = M\}$, i.e. $\{r \geq r_0\}$ for any $r_0 > M$, Teukolsky solutions **decay** with respect to the time function of a suitable foliation of the exterior,

ii) The following pointwise decay, non-decay and blow-up estimates hold asymptotically along the event horizon $\mathcal{H}^+$ [1]

(a) For the positive spin solutions, we have

- $\|\nabla_Y^m \mathfrak{f}\|_\infty(\tau)$, $\left\|\nabla_Y^m \tilde{\beta}\right\|_\infty(\tau)$, and $\|\nabla_Y^n \alpha\|_\infty(\tau)$ **decay** for any $m \leq 2, n \leq 4$.
- $\left\|\nabla_Y^3 \mathfrak{f}\right\|_{S^2_{\tau,M}}$, $\left\|\nabla_Y^3 \tilde{\beta}\right\|_{S^2_{\tau,M}}$, and $\left\|\nabla_Y^5 \alpha\right\|_{S^2_{\tau,M}}$ do **not** decay along $\mathcal{H}^+$.
- $\left\|\nabla_Y^{k+3} \xi\right\|_{S^2_{\tau,M}} \sim_k \tau^k$, and $\left\|\nabla_Y^{k+5} \alpha\right\|_{S^2_{\tau,M}} \sim_k \tau^k$, as $\tau \to \infty$, for any $k \in \mathbb{N}$, $\xi \in \left\{\mathfrak{f}, \tilde{\beta}\right\}$.

(b) For the negative spin solutions, we have

- Decay, $\|\underline{\mathfrak{f}}\|_\infty(\tau) + \left\|\underline{\tilde{\beta}}\right\|_\infty(\tau) \xrightarrow{\tau \to \infty} 0$
- $\|\nabla_Y \underline{\mathfrak{f}}\|_{S^2_{\tau,M}}$, $\left\|\nabla_Y \underline{\tilde{\beta}}\right\|_{S^2_{\tau,M}}$, and $\|\underline{\alpha}\|_{S^2_{\tau,M}}$ do **not** decay along $\mathcal{H}^+$,
- $\left\|\nabla_Y^{k+1} \xi\right\|_{S^2_{\tau,M}} \sim_k \tau^k$, and $\left\|\nabla_Y^{k} \underline{\alpha}\right\|_{S^2_{\tau,M}} \sim_k \tau^k$, as $\tau \to \infty$, for any $k \in \mathbb{N}$, $\xi \in \left\{\underline{\mathfrak{f}}, \underline{\tilde{\beta}}\right\}$.

It gets worse for Kerr: Azimuthal instabilities

arXiv:2302.06636v2 [gr-qc] 17 Apr 2023

AZIMUTHAL INSTABILITIES ON EXTREMAL KERR

DEJAN GAJIC

ABSTRACT. We prove the existence of instabilities for the geometric linear wave equation on extremal Kerr spacetime backgrounds, which describe stationary black holes rotating at their maximally allowed angular velocity. These instabilities can be associated to non-axisymmetric azimuthal modes and are stronger than the axisymmetric instabilities discovered by Aretakis in [Are15]. The existence of non-axisymmetric *instabilities* follows from a derivation of very precise *stability* properties of solutions: we determine therefore the precise, global, leading-order, late-time behaviour of solutions supported on a bounded set of azimuthal modes via energy estimates in both physical and frequency space. In particular, we obtain sharp, uniform decay-in-time estimates and we determine the coefficients and rates of inverse-polynomial late-time tails everywhere in the exterior of extremal Kerr black holes. We also demonstrate how non-axisymmetric instabilities leave an imprint in the radiation on future null infinity via the coefficients appearing in front of slowly decaying and oscillating late-time tails.

Weak stability for extremal Kerr?

For extremal Kerr, outside of axisymmetry,
the only positive stability result is mode stability

Commun. Math. Phys. 378, 705–781 (2020)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-020-03796-z

Communications in **Mathematical Physics**

Mode Stability for the Teukolsky Equation on Extremal and Subextremal Kerr Spacetimes

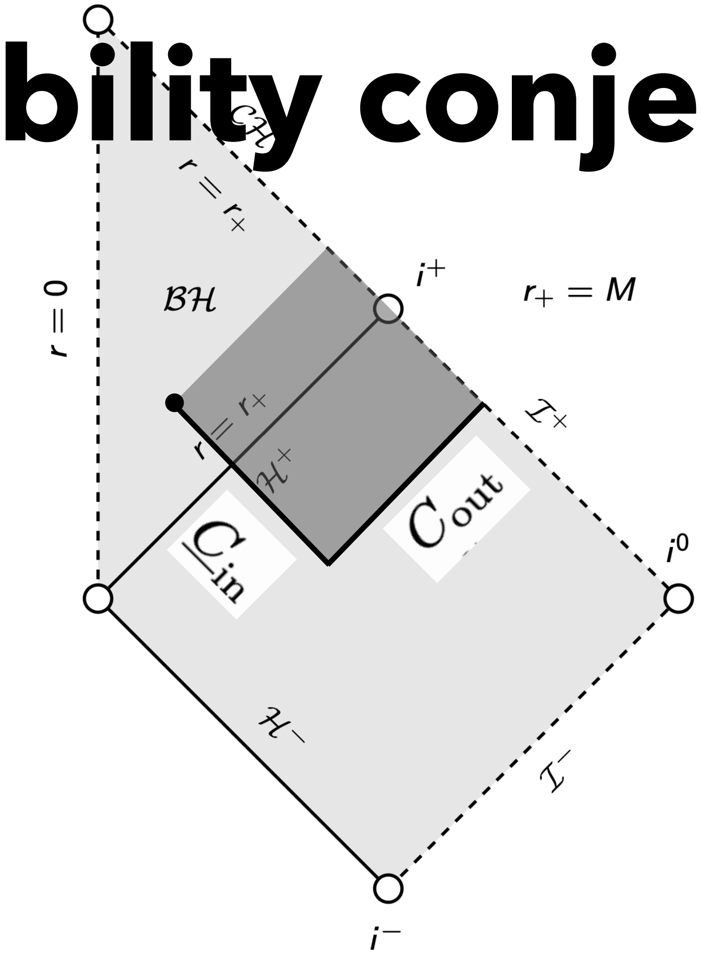
Rita Teixeira da Costa

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK. E-mail: rita.t.costa@dpmms.cam.ac.uk

In particular, even for the scalar wave equation,
the following most basic stability question remains open:

Do solutions of $\Box_g \psi = 0$ remain bounded in the exterior of subextremal Kerr?

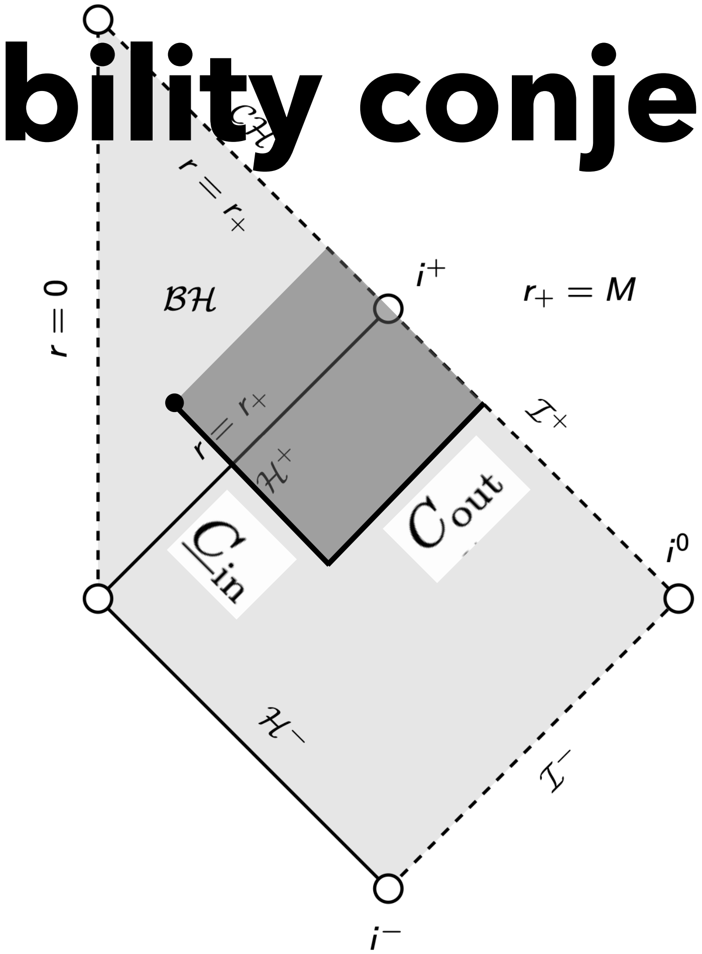
A stability conjecture?



Conjecture IV.2 (Asymptotic stability of extremal Reissner–Nordström but with growing horizon "hair")**.** *For all characteristic initial data for the Einstein–Maxwell system prescribed on* (I.3.2), *assumed sufficiently close to extremal Reissner–Nordstrom data with mass* M_{init} *and* $Q_{\text{init}} = M_{\text{init}}$ *and lying on a codimension-*4 *"submanifold"* $\mathfrak{M}_{\text{stable}}$ *of the moduli space* $\mathfrak{M}$ *of initial data, the maximal Cauchy development* $\mathcal{M}$ *contains a region* $\mathcal{R}$ *which can be covered by appropriate (teleologically normalised) global double null gauges* (I.1.1) *and where the analogues of (i), (ii) and (iii) of Theorem I.3.1 are satisfied with an extremal Reissner–Nordström metric with parameters* $M_{\text{final}} = Q_{\text{final}}$ *in the place of Schwarzschild. Along* $\mathcal{H}^+$*, however, one has decay to extremal Reissner–Nordström only in a weaker sense, in particular, for generic data lying on* $\mathfrak{M}_{\text{stable}}$*, suitable higher order quantities in the arising solution blow up polynomially along* $\mathcal{H}^+$ *(growing horizon "hair").*

[M.D.–Holzegel–Rodnianski–Taylor, 2021]

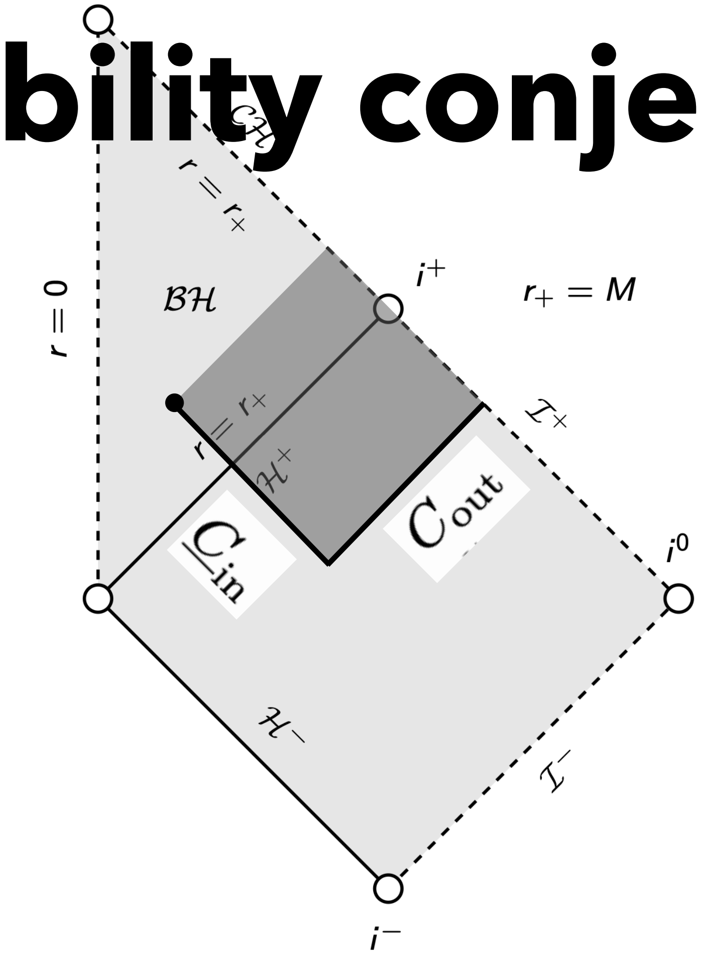
A stability conjecture?



Conjecture IV.2 (Asymptotic stability of extremal Reissner–Nordström but with growing horizon "hair"). *For all characteristic initial data for the Einstein–Maxwell system prescribed on* (I.3.2)*, assumed sufficiently close to extremal Reissner–Nordstrom data with mass* M_{init} *and* $Q_{\text{init}} = M_{\text{init}}$ *and lying on a codimension-4 "submanifold"* $\mathfrak{M}_{\text{stable}}$ *of the moduli space* $\mathfrak{M}$ *of initial data, the maximal Cauchy development* $\mathcal{M}$ *contains a region* $\mathcal{R}$ *which can be covered by appropriate (teleologically normalised) global double null gauges* (I.1.1) *and where the analogues of (i), (ii) and (iii) of Theorem I.3.1 are satisfied with an extremal Reissner–Nordström metric with parameters* $M_{\text{final}} = Q_{\text{final}}$ *in the place of Schwarzschild. Along* $\mathcal{H}^+$*, however, one has decay to extremal Reissner–Nordström only in a weaker sense, in particular, for generic data lying on* $\mathfrak{M}_{\text{stable}}$*, suitable higher order quantities in the arising solution blow up polynomially along* $\mathcal{H}^+$ *(growing horizon "hair").*

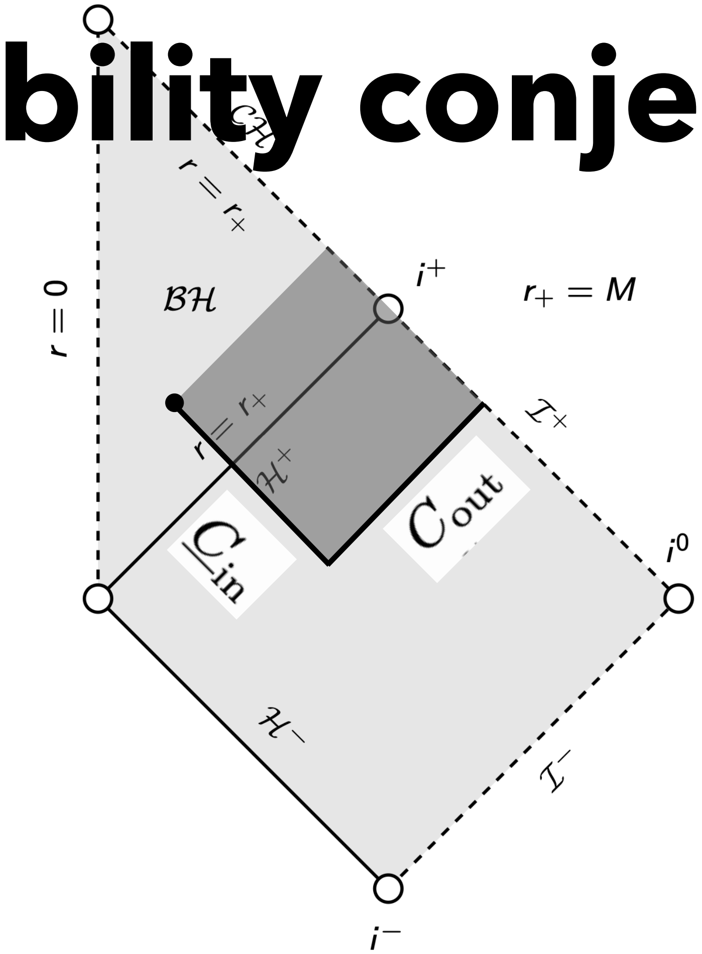
(Of course, one can already conjecture the analogue of Conjecture IV.2 for extremal Kerr as a family of the Einstein vacuum equations; we emphasise, however, that the dynamics near this phase transition in that case may be considerably more complicated!)

A stability conjecture?



Conjecture IV.2 (Asymptotic stability of extremal Kerr but with growing horizon "hair"). *For all characteristic initial data for the Einstein vacuum equations prescribed on* (I.3.2)*, assumed sufficiently close to extremal* Kerr *data with mass $M_{\rm init}$ and $a_{\rm init} = M_{\rm init}$ and lying on a codimension-1 "submanifold" $\mathfrak{M}_{\rm stable}$ of the moduli space $\mathfrak{M}$ of initial data, the maximal Cauchy development $\mathcal{M}$ contains a region $\mathcal{R}$ which can be covered by appropriate (teleologically normalised) global double null gauges* (I.1.1) *and where the analogues of (i), (ii) and (iii) of Theorem I.3.1 are satisfied with an extremal* Kerr *metric with parameters $M_{\rm final} = a_{\rm init}$ in the place of Schwarzschild. Along $\mathcal{H}^+$, however, one has decay to extremal* Kerr *only in a weaker sense, in particular, for generic data lying on $\mathfrak{M}_{\rm stable}$, suitable higher order quantities in the arising solution blow up polynomially along $\mathcal{H}^+$ (growing horizon "hair").*

A stability conjecture?

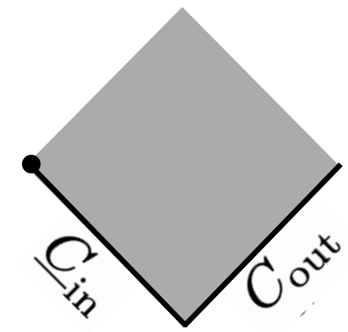


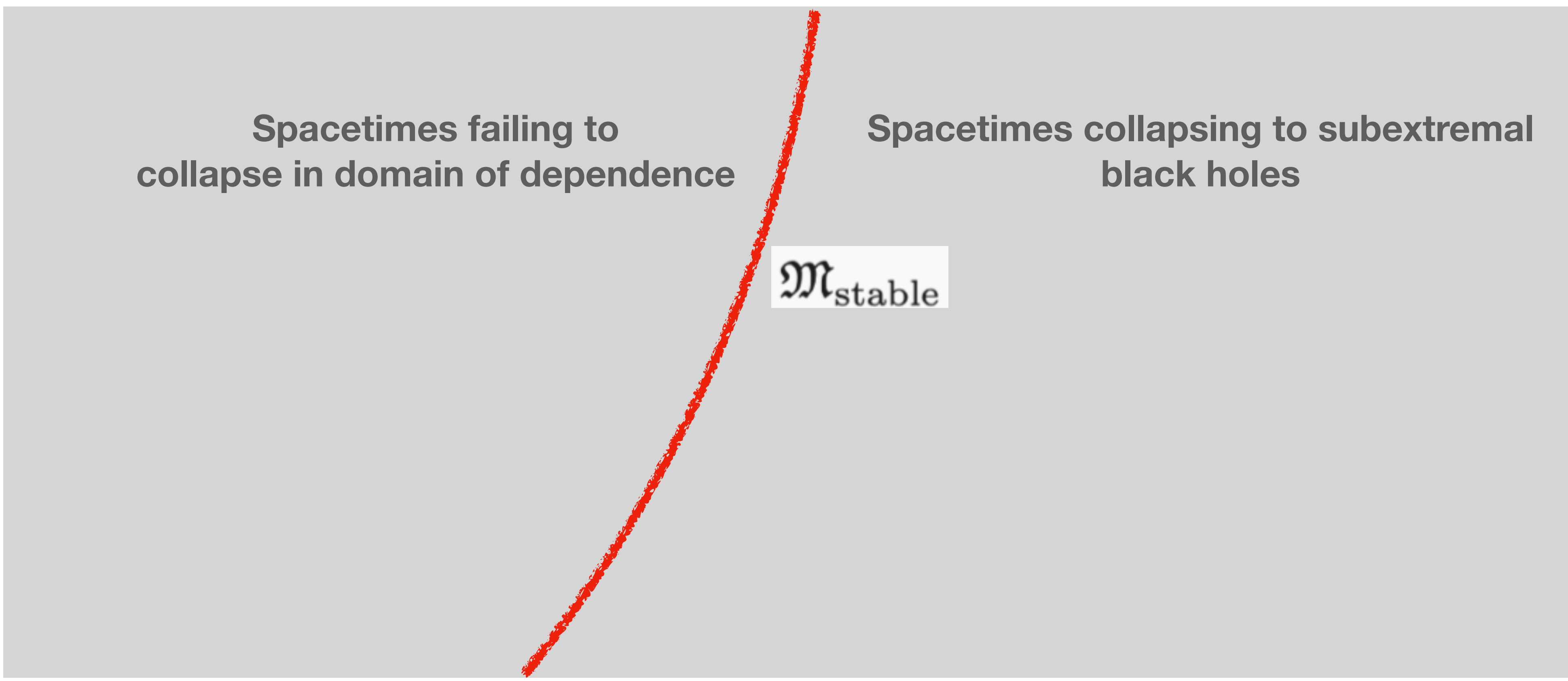
Let's say that we believe the codimension-1 stability conjecture:

For extremal Kerr and there exists a codimension-1 submanifold $\mathfrak{M}_{\text{stable}}$ of moduli space consisting of solutions asymptotic back to extremal Kerr.

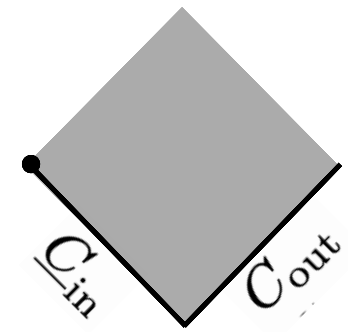
What does this submanifold $\mathfrak{M}_{\text{stable}}$ separate?

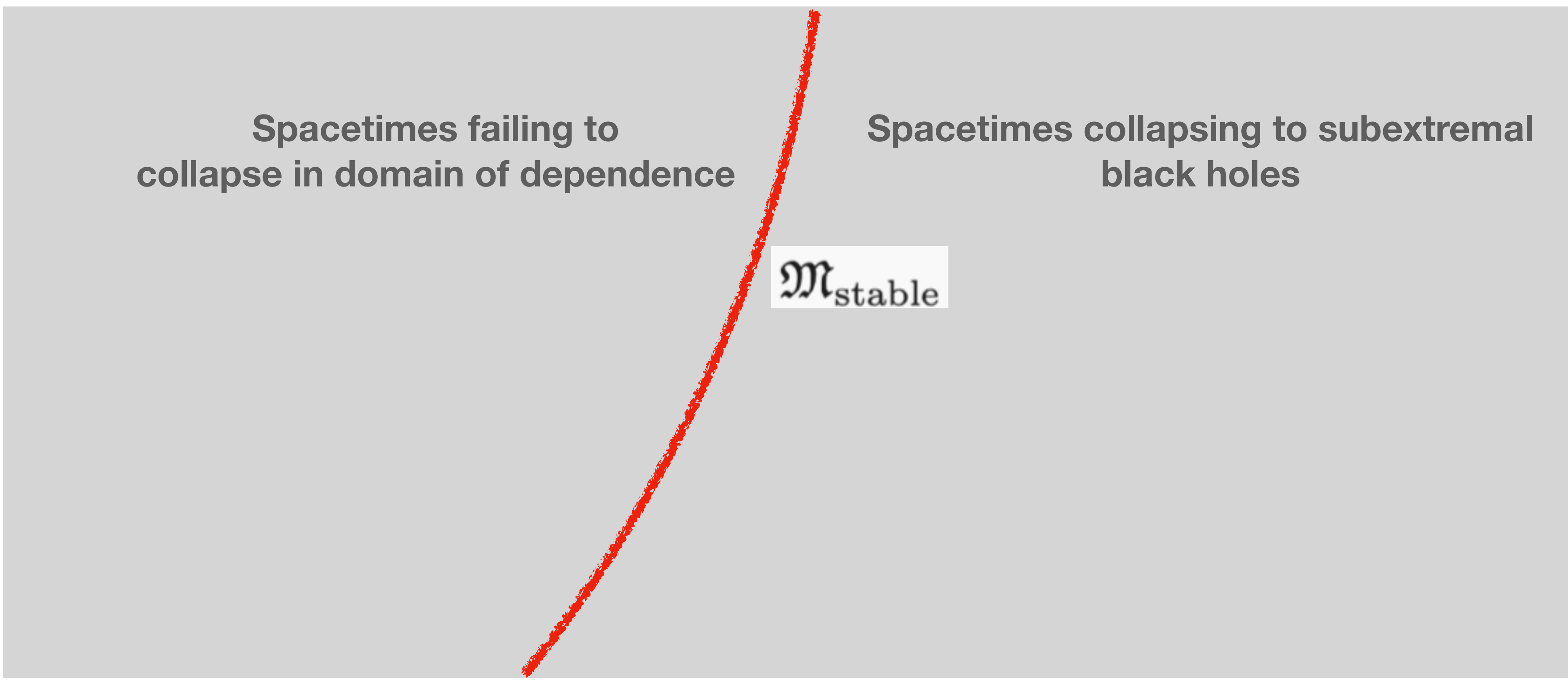
Moreover, one could hope to prove that this submanifold $\mathfrak{M}_{\text{stable}}$ delimits the boundary signifying a phase transition between two very different open regions of moduli space $\mathfrak{M}$: (1) the set of data leading to spacetimes failing to collapse (i.e. those for which $C_{\text{in}} \subset J^-(\mathcal{I}^+)$) and (2) the set of data leading to a black hole exterior settling down to a subextremal *Kerr*.

Moduli space of spacetimes evolving from 



Moreover, one could hope to prove that this submanifold $\mathfrak{M}_{\text{stable}}$ delimits the boundary signifying a phase transition between two very different open regions of moduli space $\mathfrak{M}$: (1) the set of data leading to spacetimes failing to collapse (i.e. those for which $C_{\text{in}} \subset J^{-}(\mathcal{I}^{+})$) and (2) the set of data leading to a black hole exterior settling down to a subextremal *Kerr* .

Moduli space of spacetimes evolving from 



Moreover, a sufficient condition to lie on the right hand side is the presence of a single trapped or marginally trapped surface.

Plan of the lecture

1. *What is a black hole?*

2. *What makes a black hole "extremal"?*

3. *Can non-extremal black holes become extremal?*

4. *How exceptional are extremal black holes and what do dynamics look like nearby?*

5. ***Epilogue: A new picture of the moduli space of gravitational collapse***

Living Rev. Relativity, **10**, (2007), 5
http://www.livingreviews.org/lrr-2007-5
(Update of lrr-1999-4)

LIVING REVIEWS
in relativity

Critical Phenomena in Gravitational Collapse

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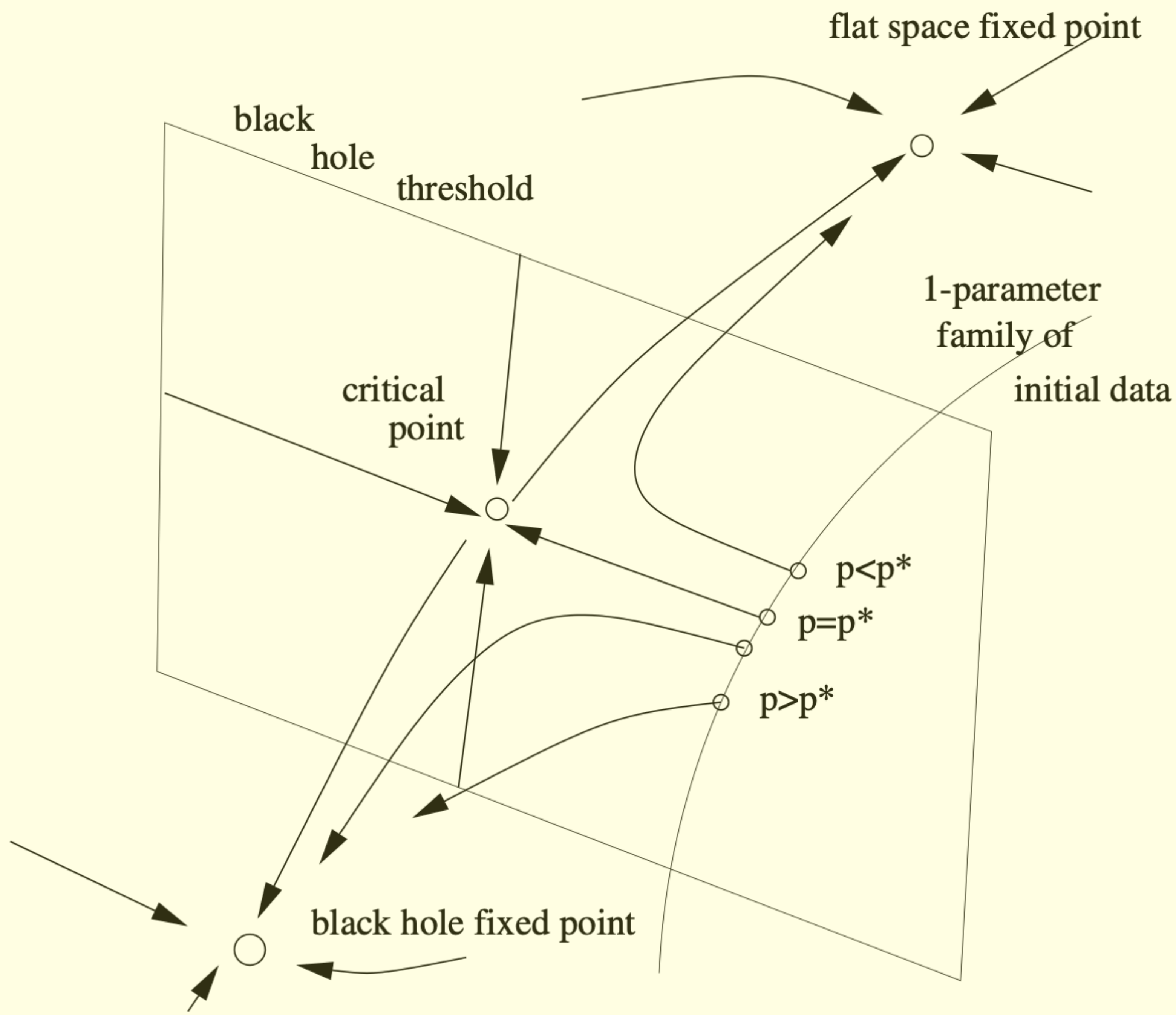
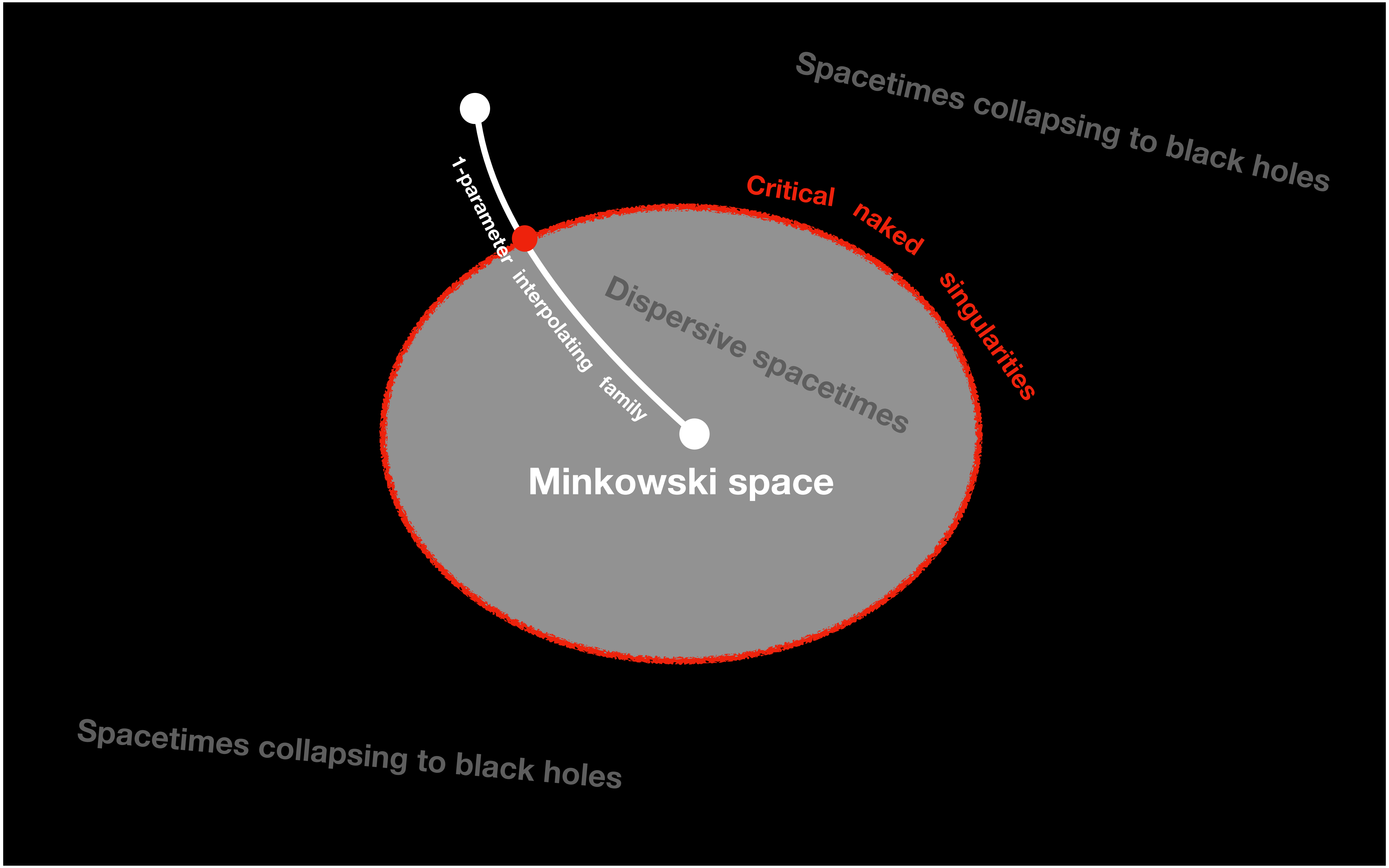


Figure 1: The phase space picture for the black hole threshold in the presence of a critical point. Every point correspond to an initial data set, that is, a 3-metric, extrinsic curvature, and matter fields. (In type II critical collapse these are only up to scale). The arrow lines are solution curves, corresponding to spacetimes, but the critical solution, which is stationary (type I) or self-similar (type II) is represented by a point. The line without an arrow is not a time evolution, but a 1-parameter family of initial data that crosses the black hole threshold at $p = p_*$. The 2-dimensional plane represents an $(\infty - 1)$-dimensional hypersurface, but the third dimension represents really only one dimension.

Moduli space of gravitational collapse

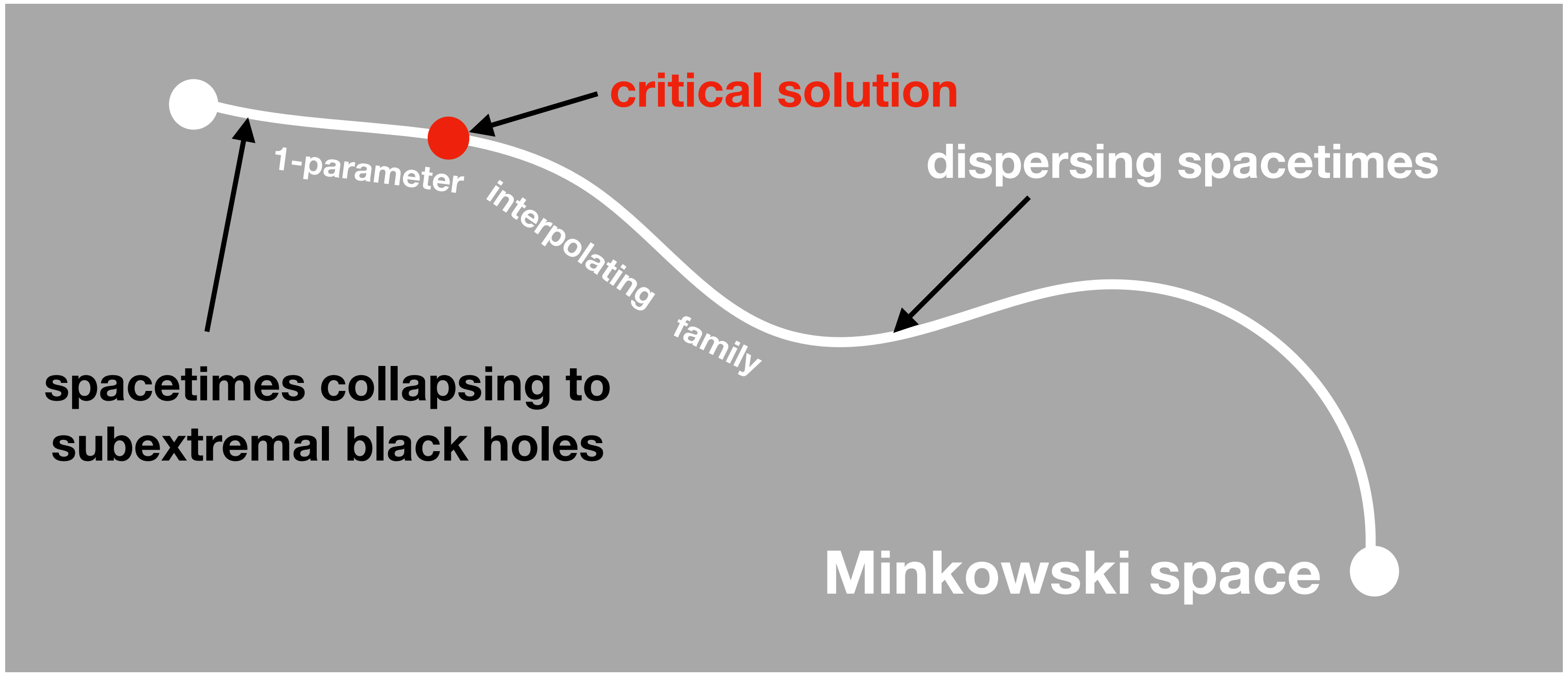


arXiv:2402.10190v1 [gr-qc] 15 Feb 2024

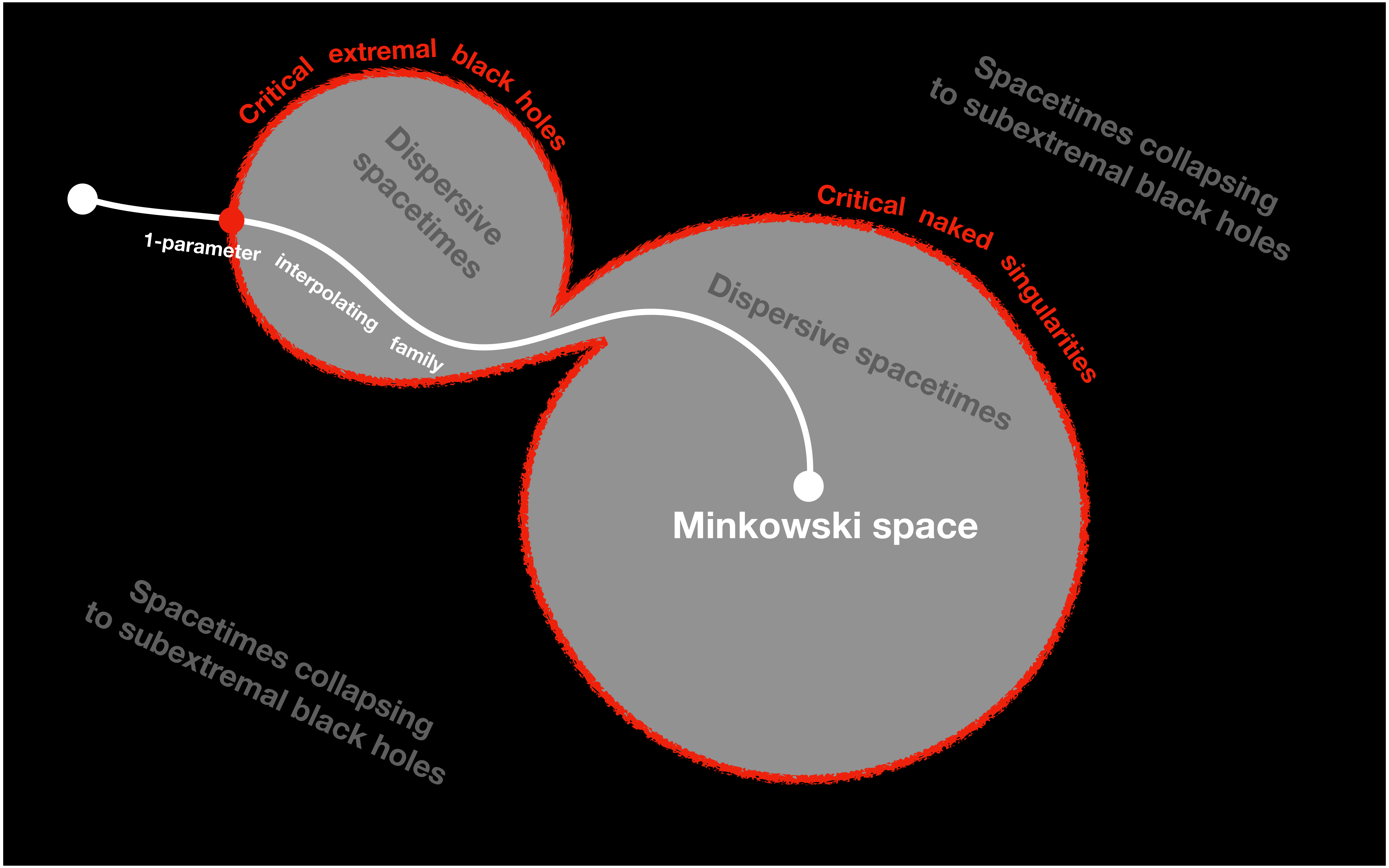
Extremal black hole formation as a critical phenomenon

Christoph Kehle*[1] and Ryan Unger†[2]

Theorem. *Extremal black holes occur at the threshold of black hole formation: There exist one-parameter families of initial data for the Einstein–Maxwell–charged Vlasov system interpolating between collapse and dispersion where the critical solution is a spacetime collapsing to an extremal Reissner–Nordström solution.*



Moduli space of gravitational collapse revised!



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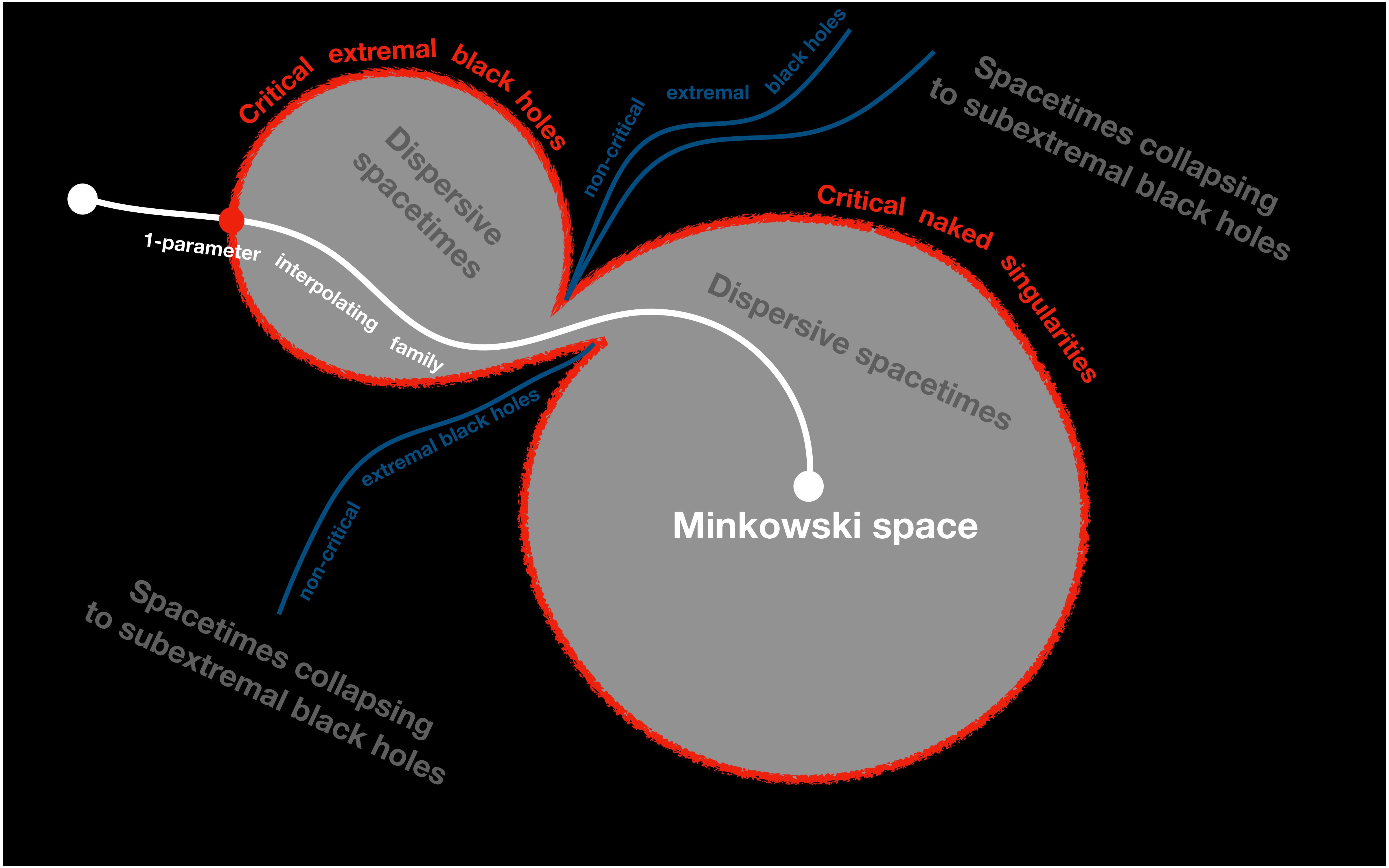

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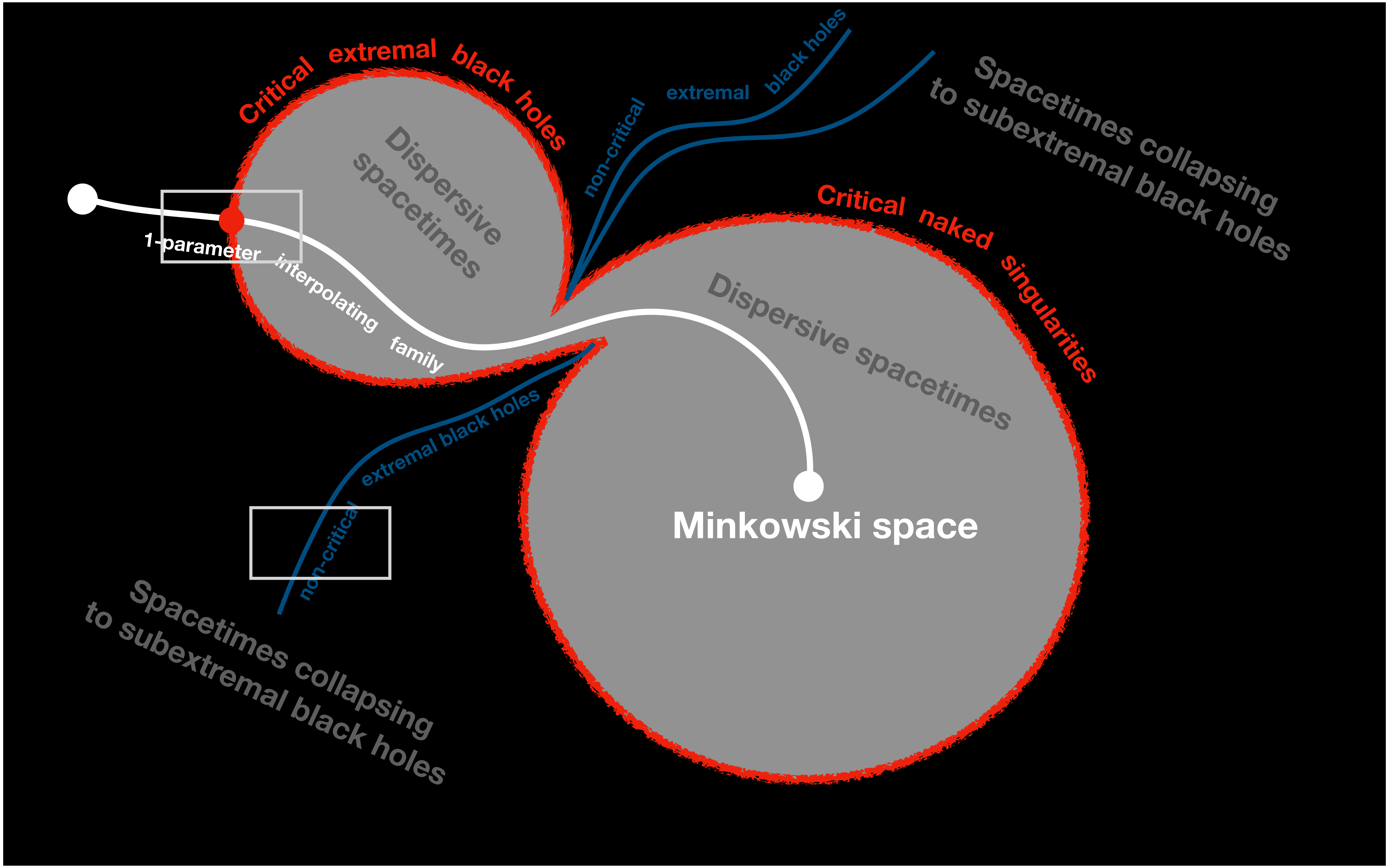

Moduli space of gravitational collapse revised!