An introduction to representations of p-adic groups

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Universität Bonn

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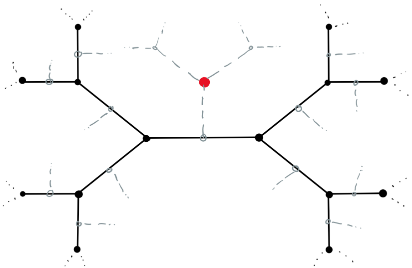

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$p =$ prime number, e.g. 2, 3, 5, 7, 11, ...

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The **p-adic integers** $\mathbb{Z}_p$ are the completion of the integers $\mathbb{Z}$ by the absolute value $|\cdot|_p$, i.e. a p-adic integer is of the form

$$a_0 + a_1 \cdot p + a_2 \cdot p^2 + a_3 \cdot p^3 + \dots \text{ for some integers } a_i \ (0 \leqslant a_i < p).$$

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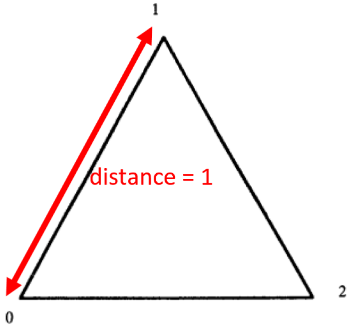
The **p-adic numbers** $\mathbb{Q}_p$ are

- the fraction field of the p-adic integers $\mathbb{Z}_p$
- the completion of the rational numbers $\mathbb{Q}$ with respect to $|\cdot|_p$

What do 3-adic integers look like?

$$a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3 + \dots \quad (p = 3)$$



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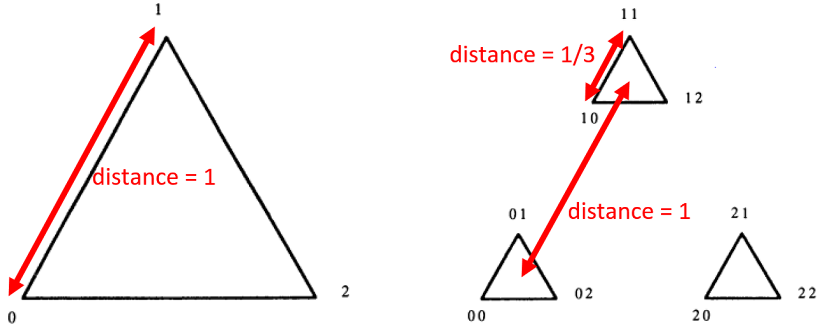

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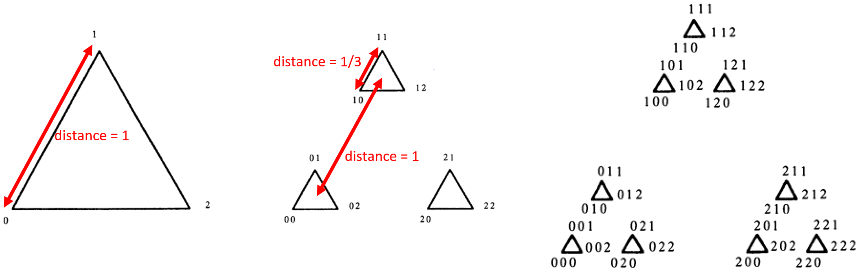

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real vs. p-adic numbers: so similar and yet so different

real numbers $\mathbb{R}$

- completion of $\mathbb{Q}$ with respect to usual absolute value $|\cdot|$

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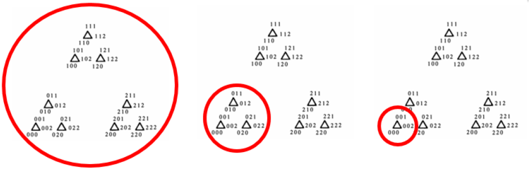
- infinitely many compact subgroups under addition: $\mathbb{Z}_p, p \cdot \mathbb{Z}_p, p^{15} \cdot \mathbb{Z}_p, \frac{1}{p}\mathbb{Z}_p, \ldots$



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$\mathbb{Q}_p$ (p-adic numbers)

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Representations of p-adic groups

Notation: $F = \mathbb{Q}_p$ or $\mathbb{F}_p((t))$
$G = p$-adic group, e.g. $\mathrm{GL}_n(F)$, $\mathrm{GL}_n(F)$, $\mathrm{SO}_n(F)$, or $\mathrm{Sp}_{2n}(F)$

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Construct all supercuspidal representations.

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Vague answer to Problem 1

We can do this under minor assumptions.

Show me a representation (not yet a supercuspidal one)

$$G = \mathrm{SL}_2(\mathbb{Q}_p),\ B = \begin{pmatrix} \mathbb{Q}_p & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p \end{pmatrix}_{\det=1}$$

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$$V = \left\{ f : G \to \mathbb{C} \ \middle|\ \begin{array}{l} f(bg) = f(g)\, \forall\, b \in B, g \in G \\ f \text{ locally constant} \end{array} \right\}$$

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for some choice of number and sizes of blocks

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for some choice of number and sizes of blocks and some $g \in G$.

Show me more representations (still not supercuspidal)

Levi decomposition

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M is called a **Levi subgroup**

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Parabolic induction

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Notation: $(\pi, V) =: (\mathrm{Ind}_P^G\,\sigma, \mathrm{Ind}_P^G\,V_\sigma)$

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Construct all supercuspidal representations.

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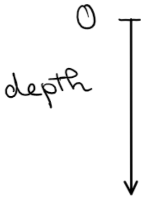
1994/96 A. Moy and G. Prasad



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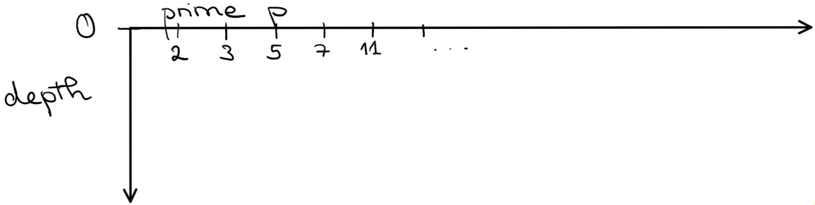

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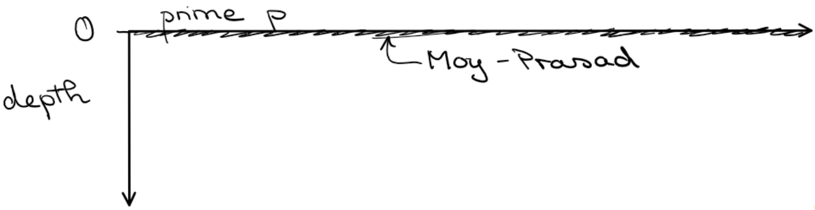

Construction of supercuspidal representations for general G

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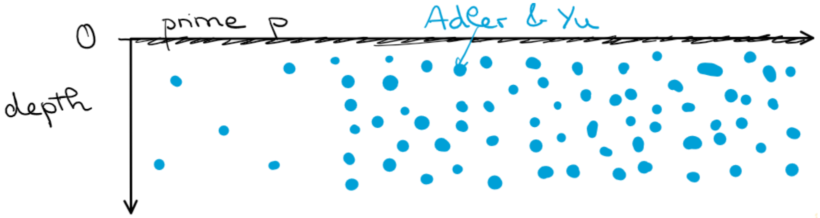

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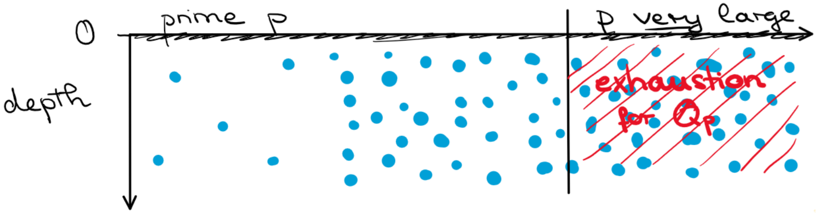

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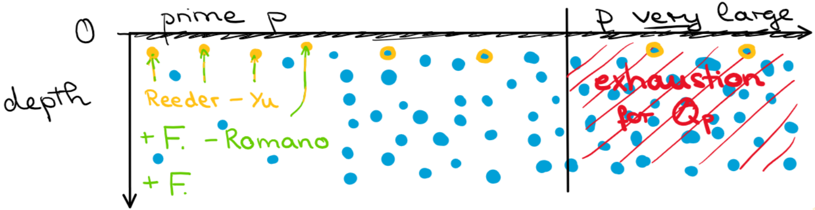

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Construct all supercuspidal representations.

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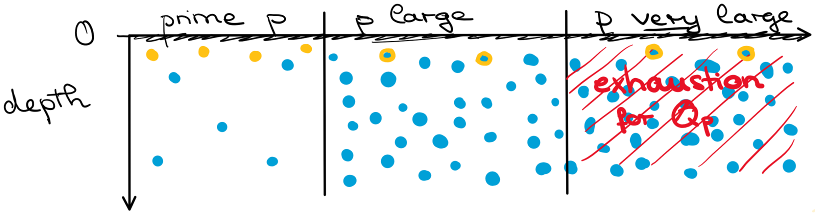

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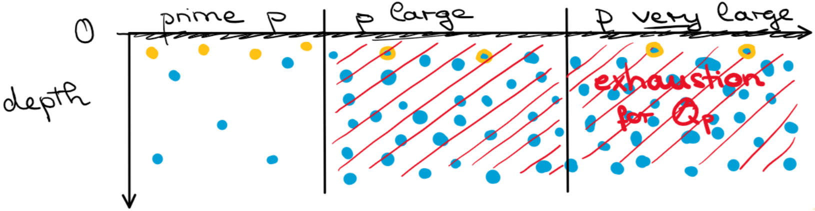

Construction of supercuspidal representations for general G

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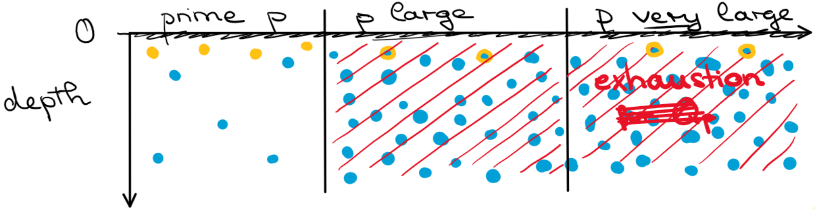
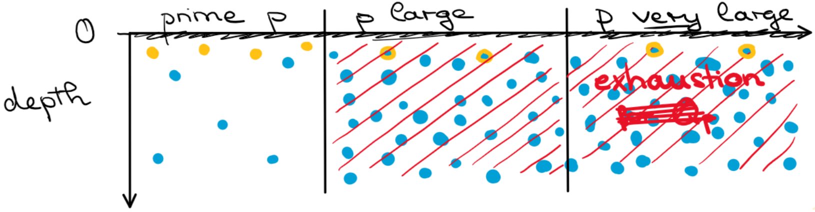


Theorem (F., 2021)

Suppose p is large, then Yu's construction yields all supercuspidal representations.



Recent results

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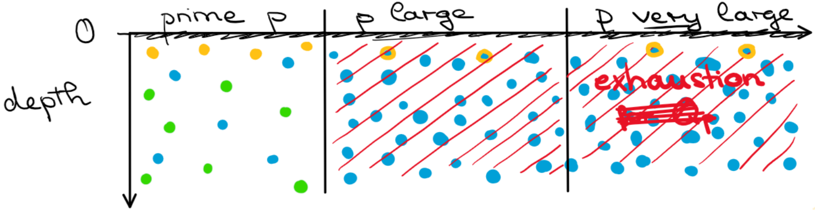
Expected Result (F.–Schwein, work in progress)

We provide a construction that yields new arbitrarily deep supercuspidal representations.



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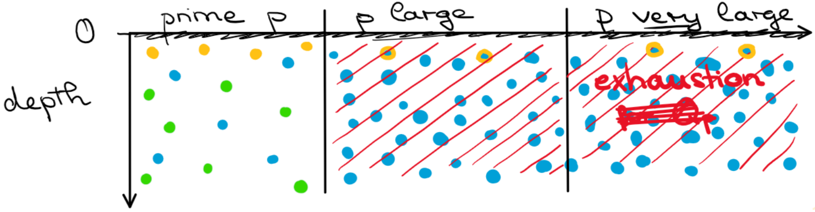

How things look like (proof sketches)

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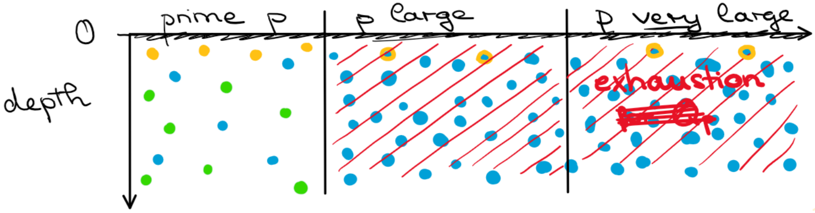
for some compact-mod center open subgroup $K \subset G$, e.g. $\mathrm{SL}_2(\mathbb{Z}_p) \subset \mathrm{SL}_2(\mathbb{Q}_p)$,



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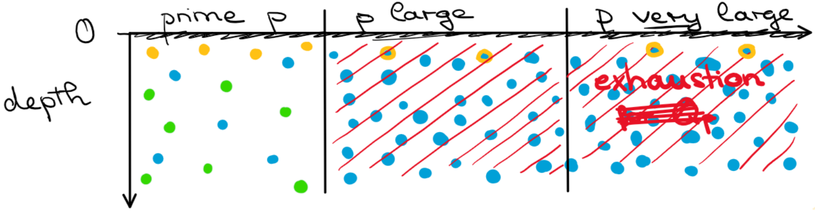

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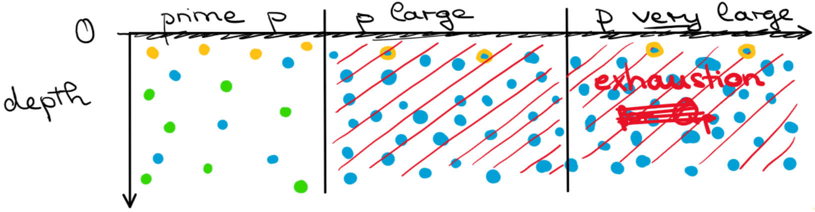

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Everything comes to an end

$$\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathrm{Rep}(G^0)_{[M^0,\sigma_0]} \simeq \mathbb{C}[\Omega,\mu] \ltimes \mathcal{H}_{\mathrm{aff}} - \mathsf{mod}$$