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Probability theory II

Exercise Sheet 10

Submission is due on 12/18/2019 2 p.m.
Box 133

Exercise 1 (4 points)

Let μ be a probability measure on $\mathbb{R}^2$ and α be its marginal distribution for the first co-ordinate (i.e. $\alpha(\mathrm{d}x) = \int_{\mathbb{R}^2} \mu(\mathrm{d}x, \mathrm{d}y)$ or equivalently, for any bounded and measurable function $f : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} f(x)\alpha(\mathrm{d}x) = \int_{\mathbb{R}^2} f(x)\mu(\mathrm{d}x, \mathrm{d}y)$). Show that for any $x \in \mathbb{R}$, there exists a probability measure β_x on $\mathbb{R}^2$ such that

(a) $x \mapsto \beta_x(\cdot)$ is measurable

(b) $\beta_x(\{x\} \times \mathbb{R}) = 1$ (i.e. β_x is supported on the vertical line through $\{(x, y) : y \in \mathbb{R}\}$)

(c) $\mu(A) = \int_{\mathbb{R}} \beta_x(A)\alpha(\mathrm{d}x)$ for all $A \subset \mathbb{R}^2$ measurable.

Remark: Note that the converse is easier, i.e. given any probability measure α and β_x, we can construct a unique probability measure μ on $\mathbb{R}^2$ which "disintegrates" in the above sense.

Hint: You may use the existence and properties of regular conditional probability distributions.

Exercise 2 (4 points)

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $T : \Omega \to \Omega$ is a measure preserving transformation. Show that for any $E \in \mathcal{F}$, almost every point returns infinitely often, i.e. show that for all $E \in \mathcal{F}$,

$$\mu(\omega \in E : \exists N \in \mathbb{N} \text{ and } T^{(n)}(\omega) \notin E \ \forall n > N) = 0.$$

Exercise 3 (6 points)

(a) Let $\Omega = \mathbb{R}^2$ and $\mu =$ Lebesgue measure on $\mathbb{R}^2$. Let $T : \Omega \to \Omega$ be the linear transformation $T(x, y) = (x + y, y)$, i.e. T is given by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Does the conclusion of Exercise 2 hold true in this case?

(b) Let $\Omega = \mathbb{T}^2$, $\mu =$ Lebesgue measure on $\mathbb{T}^2$, where $\mathbb{T}^2$ is the two-dimensional torus (i.e. $\mathbb{T}^2$ is the square $[0, 1]^2$ with the opposite sides identified). Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be defined as $T(x, y) = (x+y \mod 1, y \mod 1)$. Show that for any rectangle $R = [a, b] \times [c, d] \subset \mathbb{T}^2$, all points $(x, y) \in R$ are infinitely recurrent to R (i.e. $\mu((x, y) \in R : T^{(n)}((x, y)) \in R$ for finitely many $n) = 0$).

(c) Let $\Omega = \mathbb{R}$, $\mu =$ Lebesgue and $T(x) = x + 1$ for all $x \in \mathbb{R}$. Then T is measure-preserving. Is there any $x \in \mathbb{R}$ which is recurrent in the above sense?

Exercise 4 (6 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T : \Omega \to \Omega$ is measure-preserving such that $\mathbb{P} \in \mathcal{M}_e(\Omega)$ (i.e. $\mathbb{P}$ is invariant and ergodic). Let $A \in \mathcal{F}$ be such that $\mathbb{P}(A) > 0$. Define $N : A \to \mathbb{N} \cup \{\infty\}$ by

$$N(\omega) = \min\{k \geq 1 : T^{(k)}(\omega) \in A\}.$$

(a) Prove that $N(\cdot) < \infty$ almost surely with respect to $\mathbb{P}$.

(b) If $S : A \to A$ is defined as $S(\omega) = T^{(N(\omega))}(\omega)$ for all $\omega \in A$, then prove that S is invertible and S preserves the conditional measure $\mathbb{P}(\cdot|A)$.

(c) Prove that $\mathbb{P}(\cdot|A)$ is ergodic with respect to S (i.e. if $SB = B$ for some event B, then $\mathbb{P}(B|A) \in \{0, 1\}$).