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Probability theory II

Exercise Sheet 12

Submission is due on 01/08/2020 2 p.m.
Box 133

In what follows, S is a complete separable metric space, and $\Omega = S^{\mathbb{Z}}$ equipped with the translation map $T : \Omega \to \Omega$ such that if $\omega = (x_n)_{n\in\mathbb{Z}} \in \Omega$, $T\omega = (x_{n+1})_{n\in\mathbb{Z}}$. Also, we will write, for all $\omega \in \Omega$, $X_n(\omega) := \omega(n)$ for the co-ordinate mapping process.

Exercise 1 (4 points)

Let $\mathbb{P} \in \mathcal{M}_s(\Omega)$. Show that $\mathbb{P} \in \mathcal{M}_e(\Omega)$ if and only if

$$\frac{1}{n}\sum_{j=1}^{n} Y_j \to \mathbb{E}[Y]$$

where $Y : \Omega \to \mathbb{R}$ is measurable such that $\mathbb{E}[Y]$ exists and $Y_j(\omega) = Y(T^{j-1}\omega)$.

Exercise 2 (4 points)

(a) Let $I = [0, 1)$ and $\mathbb{P}$ =Lebesgue and $T : I \to I$ such that $T(x) = 2x \pmod 1$. Show that T is $\mathbb{P}$ - preserving and ergodic

(b) Let $X : I \to \mathbb{R}$ such that $X(\omega) = \omega$. Show that the proportion of 1's, in the expansion of X to base 2, equals 1/2 almost surely.

Exercise 3 (4 points)

Let $g : \mathbb{R} \to \mathbb{R}$ be periodic with period 1, and uniformly continuous and integrable over $(0, 1)$. Let $X \sim$ Unif([0,1]) and α is irrational. If $\xi_n = g(X + (n-1)\alpha)$ for $n \geq 1$, show that $\frac{1}{n}\sum_{j=1}^{n} \xi_j \to \int_0^1 g(x)\mathrm{d}x$ almost surely.

Exercise 4 (8 points)

Let $\{\xi_j\}_j$ be iid random variables taking values on $\mathbb{Z}$. Let $S_0 = 0$ and $S_n = \xi_1 + ... + \xi_n$ be the random walk and R_n = range of $S_n \overset{\text{def}}{=} \#\{\text{distinct sites visited by } S_1, S_2, ..., S_n\}$.

(a) Show that $\frac{1}{n}\mathbb{E}[R_n] \to \mathbb{P}(\text{ no return }) = \mathbb{P}(S_k \neq 0 \text{ for all } k \geq 1)$.

(b) Prove that almost surely, $\limsup_n \left[\frac{R_n}{n}\right] \leq \mathbb{P}(\text{ no return })$.

(c) Let

$$V_k = \begin{cases} 1, & \text{if } S_j \neq S_k \text{ for all } j > k \\ 0, & \text{else.} \end{cases}$$

Show that $\frac{1}{n}\sum_{k=1}^{n} V_k \to \mathbb{E}[V_1]$ almost surely.

(d) Use part (c) to prove, that almost surely,

$$\liminf_{n} \left[\frac{R_n}{n}\right] \geq \mathbb{E}[V_1].$$

(e) Show that $\mathbb{E}[V_1] = \mathbb{P}(\text{ no return })$ and hence

$$\lim_{n\to\infty} \frac{R_n}{n} = \mathbb{P}(\text{ no return })$$

almost surely.