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Probability theory II

Exercise Sheet 13

Submission is due on 01/15/2020 2 p.m.
Box 133

Exercise 1 (6 points)

Let T be measure preserving on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{I} \subset \mathcal{F}$ being the set of all T-invariant sets in $\mathcal{F}$, i.e. $\mathcal{I}$ consists of those $A \in \mathcal{F}$ with $\mathbb{P}(A \Delta T^{-1}(A)) = 0$. Prove that

(a) $\mathcal{I}$ is a σ-algebra.

(b) if X is an $\mathcal{F}$-measurable, real-valued random variable, then X is $\mathcal{I}$-measurable if and only if X is invariant in the sense $X \circ T = X$ P-almost surely.

Exercise 2 (4 points)

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Let $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$. We consider the transformation $T : \mathbb{T} \to \mathbb{T}$ given by $T(z) = e^{2\pi i \vartheta} z$. Prove that $\{T^k(1) : k \in \mathbb{Z}\}$ is dense in $\mathbb{T}$.

Exercise 3 (4 points)

Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$T(x) := \begin{cases} \frac{1}{2}\left(x - \frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Prove that T is measure preserving for the Cauchy-distribution with parameter 1, i.e. the distribution that has density $f(x) = \frac{1}{\pi(1+x^2)}$.

Exercise 4 (6 points)

Let $\Omega = [0, 1)$ and $\mathcal{F} = \mathcal{B}$. We consider the following transformations $T : \Omega \to \Omega$

(a) $T(x) = \lambda x$ with $0 < \lambda < 1$.

(b) $T(x) = x^2$.

Prove that there exists no probability measure $\mathbb{P}$, such that T is measure preserving and $\mathbb{P}(\{\omega\}) = 0$ for all $\omega \in [0, 1)$.