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Probability theory II

Exercise Sheet 14

Submission is due on 01/22/2020 2 p.m.
Box 133

In what follows, $\{p(x,\cdot)\}_{x\in S}$ is a transition probability function, μ is an invariant measure and if $\mathbb{P}$ is a stationary Markov chain with 1-dimensional marginal distribution μ and transition probability $p(x,\mathrm{d}y)$, it is customary to denote such a stationary Markov chain by $\mathbb{P}_\mu$.

Exercise 1 (5 points)

If $\mathcal{I}$ is the T-invariant σ-algebra and $A \in \mathcal{I}$, then show that there exists $B \in \mathcal{F}$, such that $p(x,B) = 1_B(x)$ for μ-a.e. x and

$$A = \bigcap_{n\geq 1}\bigcup_{m\geq n}\{X_m \in B\} = \{X_0 \in B\}.$$

Exercise 2 (5 points)

Let $\mathcal{M}_{\text{inv}}^{(p)} = \{\mu : \mu(A) = \int p(x,A)\mu(\mathrm{d}x)\}$. Show that

(a) $\mu \in \operatorname{extr}(\mathcal{M}_{\text{inv}}^{(p)})$ if and only if, for all $B \in \mathcal{F}$ such that $p(x,B) = 1_B(x)$ for μ-a.e. x, $\mu(B) \in \{0,1\}$.

(b) $\mu \in \mathcal{M}_{\text{inv}}^{(p)}$ can be written as a convex combination of extreme points of $\mathcal{M}_{\text{inv}}^{(p)}$, i.e. for all $\mu \in \mathcal{M}_{\text{inv}}^{(p)}$ there exists a probability measure Γ_μ on $\operatorname{extr}(\mathcal{M}_{\text{inv}}^{(p)})$ such that $\mu = \int_{\operatorname{extr}(\mathcal{M}_{\text{inv}}^{(p)})} \nu\, \Gamma_\mu(\mathrm{d}\nu)$.

Exercise 3 (5 points)

(a) Let $\rho_n = \sup_{x,y\in S}\sup_{A\in\mathcal{F}} |p^{(n)}(x,A) - p^{(n)}(y,A)|$. Suppose the transition probability $p(x,\mathrm{d}y) = \pi(x,y)\alpha(\mathrm{d}y)$ for some reference measure $\alpha \in \mathcal{M}_1(S)$, and $\inf_{x\in S}\pi(x,y) \geq q(y) \geq 0$ for all $y \in S$ and $\int q(y)\alpha(\mathrm{d}y) \geq \delta > 0$. Then show that $\rho_n \leq (1-\delta)^n$.

(b) Show that if $\rho_n \to 0$ as $n \to \infty$, then the Markov chain has a unique invariant probability measure.

Exercise 4 (5 points)

Suppose $\mathbb{P} \in \mathcal{M}_s(\Omega)$. $\mathbb{P}$ is called *strongly mixing* if

$$|\mathbb{P}(A \cap T^{-n}B) - \mathbb{P}(A)\mathbb{P}(B)| \to 0$$

for all $A, B \in \mathcal{F}$. Show if $\mathbb{P}$ is strongly mixing, any $\mathbb{P} \in \mathcal{M}_e(\Omega)$.