Jun.-Prof. Dr. C. Mukherjee
Yannic Bröker

Winter Semester 2019/20
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Probability theory II

Exercise Sheet 5

Submission is due on 11/13/2019 2 p.m.
Box 133

Please note that only the 4 best exercises will count for your marks. But we recommend to do all exercises, since they are all relevant for the exam.

Exercise 1 (5 points)

Let $(X_n)_n$ be a Markov chain on a state space E. If E is finite, show that there exists at least one closed communication class.

Exercise 2 (5 points)

(a) Show that every recurrent communication class is closed and every finite closed communication class is recurrent.

(b) Consider an irreducible and recurrent Markov chain with an arbitrary initial distribution α. Then show that for every $j \in E$, the number of visits to the j is infinite with probability 1.

Exercise 3 (5 points)

Without using Stirling's formula, show that a simple random walk is transient in $d \geq 3$.

Hint: Choose $v(x) = \frac{1}{|x|^a}$ for $a \in (0, d-2)$. Show that for $|x|$ sufficiently large, $(\mathbf{P}v)(x) - v(x) \leq 0$. Now if $|x| > L$ and τ_L is the first entrance time into the ball of radius L, show that $\mathbb{P}_x(\tau_L < \infty) \leq \frac{v(x)}{\inf\limits_{|y| \leq L} v(y)}$. Now let $|x| \to \infty$, keeping L fixed.

Exercise 4 (5 points)

Without using Stirling's formula, show that a simple random walk is recurrent in $d = 2$.

Hint: A similar argument as in exercise 3 will work for the choice $v(x) = \log|x| - \frac{c}{|x|}$ for an appropriate constant $c > 0$.

Exercise 5 (5 points)

Let $(X_n)_n$ be a Markov chain on a state space E. Let $u : E \to \mathbb{R}$ such that $\inf u > 0$ and $\sup u < \infty$. Show that

$$\sup_{x \in E, n \in \mathbb{N}} \mathbb{E}_x\left[\exp\left\{\sum_{j=0}^{n-1} \log\left(\frac{u(X_j)}{(\mathbf{P}u)(X_j)}\right)\right\}\right] \leq \frac{\sup u}{\inf u}.$$