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Winter Semester 2019/20
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Probability theory II

Exercise Sheet 8

Submission is due on 12/04/2019 2 p.m.
Box 133

Please note that only the 4 best exercises will count for your marks. But we recommend to do all exercises, since they are all relevant for the exam.

Exercise 1 (4 points)

(a) Prove that if $T : V \to V$ is a linear transformation of a finite dimensional vector space, then either one of the following statements holds:

- $\forall v \in V, \exists u \in V$ s.t. $Tu = v$ (i.e. T is surjective)
- $\dim(\ker T) > 0$ where $\ker T = \{v \in V : Tv = 0\}$.

(b) Let A be an $n \times n$ matrix over the field of complex numbers $\mathbb{C}$. Prove that if b is a column vector $b \in \mathbb{C}^n$, then

$$b \in \text{ range } (A) \qquad \iff \qquad b \in (\ker\, (A^{\mathrm{T}}))^{\perp}$$

Exercise 2 (4 points)

Let H be a Hilbert space (normed, linear space with an inner product $\langle \cdot, \cdot \rangle_H$) and $T : H \to H$ is a bounded operator (i.e. $\|Tx\| \leq \|x\|$ for all $x \in H$). Prove that,

$$(\ker\, T)^{\perp} = \overline{\text{range } (T^{\star})}$$
$$\text{and} \qquad \text{range } (T)^{\perp} = \ker\, (T^{\star})$$

where $T^{\star}$ is the adjoint operator given by $\langle Tx, y \rangle = \langle x, T^{\star} y \rangle$.

Exercise 3 (4 points)

It has been shown in the lectures that if $(X_n)_n$ is a stationary and ergodic Markov chain taking values in a finite state space E with unique invariant probability measure μ, then for all $f : E \to \mathbb{R}$ with $\int f \mathrm{d}\mu = 0$ and $\mathbb{E}^{\mu}[f(X_1)^2] < \infty$, $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(X_j) \xrightarrow{\text{(d)}} \mathcal{N}(0, \sigma^2)$ for some $\sigma^2 > 0$.

Show that σ^2 admits a representation in terms of the Dirichlet form $\sigma^2 = \langle u, -Lu \rangle_{L^2(\mu)}$ where as usual $L = P - I$ and u is the solution of $-Lu = f$.

Exercise 4 (5 points)

Let $(X_n)_n$ be a Markov chain taking values in $\{1, 2\}$ and with transition probabilities

$$p_{11} = p_{22} = p > 0, \quad p_{12} = p_{21} = q > 0 \quad \text{and } p + q = 1.$$

(a) Find out the invariant probability measure.

(b) Let $A_n = \#$ visits to the state $\{1\}$ by time n and $B_n = \#$ visits to the state $\{2\}$ by time n and $S_n = A_n - B_n$. Prove that $\frac{S_n}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0, \sigma^2(p))$ and compute $\sigma^2(p)$ as a function of p.

(c) What happens to $\sigma^2(p)$ as $p \to 0$ or $p \to 1$? Without doing the computation before, can you guess $\sigma^2(1/2)$? Justify the answer.

Exercise 5 (8 points)

Consider a random walk taking values on $\mathbb{Z}_+$ with transition probability

$$p_{xy} = \begin{cases} 1/2, & \text{if } x = y \geq 0 \\ \frac{1-\delta}{4}, & \text{if } y = x+1, x \geq 1 \\ \frac{1+\delta}{4}, & \text{if } y = x-1, x \geq 1 \\ 1/2, & \text{if } x = 0, y = 1. \end{cases}$$

(a) Write down the transition matrix and prove that the chain is positive recurrent, aperiodic and irreducible.

(b) Compute the invariant probability measure μ explicitly.

(c) If $f : \mathbb{Z}_+ \to \mathbb{R}$ is a function with compact support, solve the equation $-Lu = f$ for u explicitly (where $L = P - I$).

(d) Show that either u grows exponentially at infinity or is a constant for large x. Moreover, u is a constant outside a finite region and only if

$$\sum_{j \in \mathbb{Z}_+} \mu(j) f(j) = 0.$$

(e) What is the limiting distribution of $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(X_j)$ if $\sum_{j \in \mathbb{Z}_+} \mu(j) f(j) = 0$?