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Winter Semester 2019/20
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Probability theory II

Exercise Sheet 9

Submission is due on 12/11/2019 2 p.m.
Box 133

Please note that only the 4 best exercises will count for your marks. But we recommend to do all exercises, since they are all relevant for the exam.

Exercise 1 (5 points)

Let T be measure-preserving on $(\Omega, \mathcal{F}, \mathbb{P})$. If $A \in \mathcal{F}$ such that $\mathbb{P}(A \Delta T^{-1}A) = 0$, show that there exists $B \in \mathcal{F}$ such that $\mathbb{P}(A \Delta B) = 0$ and $B = T^{-1}B$.

Exercise 2 (5 points)

Let $\Omega = \mathbb{Z}$ and $\mathcal{F} = 2^{\Omega}$. Let $T : \mathbb{Z} \to \mathbb{Z}$ such that $Tn = n + 1$ for all $n \in \mathbb{Z}$. If $\mathcal{M}_s(\Omega)$ is the set of all T-invariant probability measures, prove that $\mathcal{M}_s(\Omega) = \emptyset$.

Exercise 3 (5 points)

Let $A : H \to H$ be a bounded linear operator on a Hilbert space such that $A^\star A = AA^\star = I$ where $A^\star$ is the adjoint of A. Such an operator is called unitary. Show that

$$\begin{aligned} A \text{ is unitary} \quad &\Leftrightarrow \quad A \text{ is surjective and } \langle Ax, Ay\rangle_H = \langle x, y\rangle_H \text{ for all } x, y \in H \\ &\Leftrightarrow \quad \text{range of } A \text{ is dense in } H \text{ and } \langle Ax, Ay\rangle_H = \langle x, y\rangle_H \text{ for all } x, y \in H. \end{aligned}$$

Exercise 4 (5 points)

Suppose $\mu_1, \mu_2 \in \mathcal{M}_e(\Omega)$, i.e. μ_1 and μ_2 are ergodic invariant probability measures on $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a measure-preserving transformation T. Show that there exists $A \in \mathcal{F}$ such that $A = T^{-1}A$, $\mu_1(A) = 1$ and $\mu_2(A) = 0$ (i.e. μ_1 and μ_2 are orthogonal on the invariant σ-algebra $\mathcal{I}$).

Exercise 5 (5 points)

Let $(a_n)_n$ be a sequence of real numbers. Let $\Omega = \{\pm 1\}^{\mathbb{N}}$. Let $\omega^{(n)} \in \Omega$ such that for all $n \in \mathbb{N}$ $\mathbb{P}(\{\omega_i^{(n)} = 1\}) = \mathbb{P}(\{\omega_i^{(n)} = -1\}) = \frac{1}{2}$ where $\omega_i^{(n)}$ is the i-th component of the infinite sequence $\omega^{(n)}$. Prove that

$$\mathbb{P}\Big(\sum_{n\in\mathbb{N}} a_n \omega^{(n)} \text{ converges}\Big) \in \{0, 1\}.$$

Hint: You may use Kolmogorov's 0-1 law.