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Probability Theory on Trees and Networks

Exercise Sheet 1

Submission is due on 11/09/2020 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file. Please hand in as a single person or in a group of 2. You can use the learnweb forum "Diskussionsforum" to build a group.

In the following we will always consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$.

Definition: A stochastic process $(X_n)_{n\in\mathbb{N}_0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale if the following properties are satisfied for each $n \in \mathbb{N}_0$:

- X_n is $\mathcal{F}_n$-measurable.
- $X_n \in L^1(\mathbb{P})$ (i.e. $\int |X_n| \mathrm{d}\mathbb{P} < \infty$).
- $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ almost surely.

Exercise 1 (6 points)

Prove each of the following statements:

(a) Let $\{\xi_j\}_{j\in\mathbb{N}}$ be independent and identically distributed random variables with $\mathbb{E}[\xi_j] = 0$ for all j. Then the partial sum $(S_n)_n$ with $S_0 = 0$ and $S_n = \xi_1 + ... + \xi_n$ is a martingale w.r.t. the natural filtration $(\mathcal{F}_n)_n$

(b) Let $\{\xi_j\}_{j\in\mathbb{N}}$ be independent and identically distributed random variables with $\mathbb{E}[\xi_j] = 1$ for all j. Then $(M_n)_n$ with $M_n = \prod_{j=1}^n \xi_j$ is a martingale w.r.t. the natural filtration.

(c) Let $\{Z_j\}_j$ be a sequence of i.i.d. Gaussian random variables such that $Z_j \sim$ Normal(0,1) (that is, for any j, $\mathbb{P}(Z_j \in A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{|z|^2}{2}} \mathrm{d}z$). Then for any sequence of real numbers $(\beta_j)_j \subset \mathbb{R}$, we set

$$M_n := \exp\left\{ \sum_{j=1}^n \beta_j Z_j - \frac{1}{2} \sum_{j=1}^n \beta_j^2 \right\}.$$

It follows that $(M_n)_n$ is a martingale w.r.t. the canonical filtration.

Exercise 2 (5 points)

The following model describes the evolution of a population:

Let $(Y_{n,k})_{n\in\mathbb{N}_0, k\in\mathbb{N}}$ be iid random variables in $\mathbb{N}_0$, where $Y_{n,k}$ is the number of children of the k-th individual in the n-th generation. We assume $\mathbb{E}[Y_{n,k}] < \infty$ for all $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. After one step every individual of the last generation dies such that we can define the number of living individuals by

$$S_0 = 1 \quad S_n = \sum_{k=1}^{S_{n-1}} Y_{n-1,k}, \quad n \geq 1.$$

Prove that

$$Z_n := \frac{S_n}{\mu^n}, \quad n \geq 1$$

is a martingale with respect to $\mathcal{F}_n = \sigma(S_0, ..., S_n)$.

Exercise 3 (3 points)

Construct a non-negative martingale $(X_n)_n$, such that $\mathbb{E}[X_n] = 1$ for all $n \in \mathbb{N}$ but $X^\star = \sup_{n\in\mathbb{N}} X_n \notin L^1$.

Hint: Use the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, $\mathbb{P} =$Unif$(0,1)$ with filtration $\mathcal{F}_n = \sigma$-alg. generated by intervals ending with $j/2^n$ for some positive integer j and the random variables

$$X_n = \begin{cases} 2^n, & \text{if } 0 \leq x \leq 2^{-n} \\ 0, & \text{if } 2^{-n} \leq x \leq 1 \end{cases}$$

Definition: Let T be a random variable, which is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{N}_0 \bigcup \{\infty\}$. Then T is called a stopping time (with respect to the filtration $(\mathcal{F}_n)_n$), if the following condition holds:

$$\{T \leq n\} \in \mathcal{F}_n \quad \text{for all } n \in \mathbb{N}_0.$$

Exercise 4 (6 points)

(a) Let $(X_n)_{n\in\mathbb{N}_0}$ be a martingale and T be a stopping time. Show that the stopped process $(X_{T\wedge n})_{n\in\mathbb{N}_0}$ again is a martingale.

(b) Show that $\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_0]$.

Hint: If $(X_n)_n$ is a martingale and $(C_n)_n$ is a sequence of random variables that is previsible (i.e. $C_n \in \mathcal{F}_{n-1}$ for all n) and bounded (i.e. for any n there is some K_n, s.t. $\sup_\omega |C_n(\omega)| \leq K_n$), prove that the process $(Y_n)_n$ with $Y_0 = 0$ and $Y_n = \sum_{k=1}^n C_k(X_k - X_{k-1})$ for $n \in \mathbb{N}$ defines a martingale.