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Probability Theory on Trees and Networks

Exercise Sheet 12

Submission is due on 02/08/2021 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Exercise 1 (5 points)

Let T be an infinite tree (locally finite, rooted) such that the walk (T, P) is transient. Then

$$\{\text{ends}\} \simeq \{\text{rays coming out a fixed root}\} =: \partial T.$$

Define the boundary map b as in the lecture. Show that $(\partial T, \mathcal{B}_{\partial T}, b)$ is a compactification boundary of (T, P).

Exercise 2 (6 points)

Prove the following statements:

(a) For any $a \in \mathbb{C}$ such that $|a| < 1$, the map

$$\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$$

is a Möbius transformation. Also, for every such a, φ_a is a self-map of the disc $\mathbb{D}$. Likewise, $(\varphi_a)^{-1} = \varphi_{a^{-1}}$ is also a self-map of $\mathbb{D}$.

(b) Conversely, any Möbius transformation of the unit disc onto itself is a combination of a rotation and one such φ_a (i.e. any Möbius transformation which is a self-map of $\mathbb{D}$ can be written as $f(z) = \varphi_b(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$ and some b).

Recall parts (iii)-(iv) of today's lecture:

Let $(\Theta, \mathcal{F}_\Theta, b_\Theta)$ be a boundary of the irreducible Markov chain (V, P) with harmonic measures $(\vartheta_x)_{x \in V}$. Fix $o \in V$. Then, (i)-(iv) are equivalent, where

(iii) For all $A \in \mathcal{I}$ there exists $E \in \mathcal{F}_\Theta$ such that $\mathbb{P}_o\big(A \,\Delta\, b^{-1}(E)\big) = 0$ (equivalently, one can write $b^{-1}(\mathcal{F}_\Theta) = \mathcal{I} \bmod \mathbb{P}_o$).

(iv) For every other boundary $(\Theta', \mathcal{F}_{\Theta'}, b'_{\Theta'})$ there is $\pi : \Theta \to \Theta'$ such that $b' = \pi \circ b$.

Exercise 3 (9 points)

(a) Show that the map π in part (iv) of the theorem is unique up to a set of ϑ_o measure zero.

Hint: Use the invariant sigma algebra, namely part (iii).

(b) Suppose $(\Theta, \mathcal{F}_\Theta, b_\Theta)$ and $(\Theta', \mathcal{F}_{\Theta'}, b'_{\Theta'})$ are two Poisson boundaries of (V, P). Then show that the map π in part (iv) is 1 - 1 on a set $\Theta_1 \in \mathcal{F}_\Theta$, such that $\vartheta_o(\Theta_1) = 1$ and moreover π^{-1} is measurable.

(c) Suppose both Poisson boundaries above are Γ-boundaries. Then the map $\pi : \Theta \to \Theta'$ is Γ-equivariant (i.e. $(\pi \circ \gamma)(\theta) = (\gamma \circ \pi)(\theta)$ for all $\theta \in \Theta$ and for all $\gamma \in \Gamma$).