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Probability Theory on Trees and Networks

Exercise Sheet 2

Submission is due on 11/16/2020 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Exercise 1 (4 points)

Let $n \in \mathbb{N}$ and let Γ be the reduced residue class of n (i.e. Γ consists of all positive integers $< n$ which are coprime to n) equipped with the operation $*$ defined in the lecture (i.e. for all $x, y \in \Gamma$, $x * y =$ remainder of $x \times y$ w.r.t. n). Prove that $(\Gamma, *)$ is a group.

Hint: To prove that every element has an inverse w.r.t. the $*$-operation you need the following fact that you can assume:

If a, b are coprime to each other, then there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$.

Exercise 2 (6 points)

Let f be a bounded μ-harmonic function on a group Γ. We define

$$u_n(x) := \sum_{y_1,\dots,y_n} \left(f\left(x + \sum_{i=1}^n y_i\right) - f\left(x + \sum_{i=2}^n y_i\right)\right)^2 \prod_{i=1}^n \mu(y_i).$$

(i) Show that $u_n(x) \leq u_{n+1}(x)$.

(ii) Show that

$$u_n(x) = \sum_{y_1,\dots,y_n} f\left(x + \sum_{i=1}^n y_i\right)^2 \prod_{i=1}^n \mu(y_i) - \sum_{y_2,\dots,y_n} f\left(x + \sum_{i=2}^n y_i\right)^2 \prod_{i=2}^n \mu(y_i).$$

(iii) Show that $\sum_n u_n(x) < \infty$.

(iv) Show that $u_1(x) = 0$.

Exercise 3 (4 points)

Show that if G is a Cayley graph of a finitely generated abelian group Γ and μ is a symmetric probability measure on Γ (i.e. $\mu(g) = \mu(g^{-1})$ for all $g \in \Gamma$) with finite support that generates Γ, then there are no nonconstant μ-harmonic functions h whose growth is sublinear in distance, that is, such that $h(x)/\mathrm{dist}_G(0, x) \to 0$ as $\mathrm{dist}_G(0, x) \to \infty$.

Exercise 4 (2 points)

Let $(X_n)_n$ be a Markov chain taking values in E, which is finite or countable, with transition probabilities $P = (p_{i,j})$. Define

$$f_{i,j} = \mathbb{P}(\tau_j < \infty)$$

where $\tau_j = \inf\{n \geq 1 : Z_n = j\}$. Show that

$$f_{i,j} = 1 \, \forall i, j \in E \Leftrightarrow \text{every non-negative } P-\text{superharmonic function on } E \text{ is constant.}$$

Exercise 5 (4 points)

Let $(S_n)_{n\geq 0}$ be a random walk defined as $S_n = \xi_1 + ... + \xi_n$ with $S_0 = 0$ and $\mathbb{P}(\xi_1 = \pm 1) = 1/2$.

(a) Show that there exists $\sigma > 0$ such that $\mathbb{E}_0[e^{\sigma \tau_R}] < \infty$, where $\tau_R = \inf\{n \geq 1 : S_n \notin (-R, R)\}$ and $R > 0$.

(b) Is the estimate true for all $\sigma > 0$?