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Probability Theory on Trees and Networks

Exercise Sheet 5

Submission is due on 12/07/2020 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Exercise 1 (5 points)

Let G be a countable and directed network. Fix $a \in V(G)$. Show that

$$\inf\left\{\sum_{e\in\Pi} c(e) : \Pi \text{ a cutset separating } a \text{ from } \infty\right\}$$
$$\leq \max\left\{\text{strength}(\theta) : \theta \text{ flows from } a \text{ to } \infty \text{ and } 0 \leq \theta(e) \leq c(e) \text{ for all } e\right\}.$$

Exercise 2 (5 points)

Let (G, c, D) be a network. Show that

$$\Phi_E(G, c, D)$$
$$= \max\left\{\alpha \geq 0 : \text{there exists } \theta \text{ such that } |\theta(e)| \leq c(e) \text{ and } d^\star\theta(x) \geq \alpha D(x) \text{ for all } x \in V(G)\right\}$$

with θ ranging over all antisymmetric functions on E.

Definition: Let G be a graph and $D : V(G) \to \mathbb{R}_+$ be any function. Let $\partial_V K := \{x \notin K : \text{there exists } y \in K \text{ with } y \sim x\}$ and

$$\Phi_V(G) = \Phi_V(G, D) := \inf\left\{\frac{|\partial_V K|_D}{|K|_D} : \emptyset \neq K \subset V \text{ finite}\right\}.$$

We say that (G, D) is *vertex amenable* if $\Phi_V(G) = 0$. Moreover, if f_1, f_2 are two functions on the same domain such that $\inf \frac{f_1}{f_2} > 0$ and $\sup \frac{f_1}{f_2} < \infty$, we write $f_1 \asymp f_2$. Note that if a network (G, c, D) satisfies $c \asymp \mathbf{1} \asymp D \asymp$ degree, then (G, c, D) is edge amenable if and only if (G, D) is vertex amenable. We will call (G, c, D) simply *amenable* if it is both, edge amenable and vertex amenable.

Exercise 3 (5 points)

Suppose G is a graph such that for some $o \in V$,

$$\liminf_{n\to\infty} \#\{x \in V : \text{ dist}(o, x) \leq n\}^{1/n} = 1$$

(i.e. we have sub-exponential growth of balls). Show that $(G, \mathbf{1})$ is vertex amenable.

Exercise 4 (5 points)

Show that every Cayley graph of a finitely generated abelian group is amenable.