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Winter Semester 2020/21
12/07/2020

Probability Theory on Trees and Networks

Exercise Sheet 6

Submission is due on 12/14/2020 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Recall Lecture 5 for definition of the Taxicab lattice (or Manhattan lattice). That is, with $a = (1, 0)$ and $b = (0, 1)$ look at the Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$ with respect to $S = \{a, b, a^{-1}, b^{-1}\}$.

Exercise 1 (5 points)

(i) Show that the canonical map $\mathbb{R}^2 \to \{\text{Manhattan lattice}\}$ is a quasi-isometry.

(ii) Show that quasi-isometry is an equivalence relation.

Exercise 2 (5 points)

(i) Show that different Cayley graphs of the same finitely generated group are quasi-isometric.

(ii) If G_1, G_2 are two graphs which are quasi-isometric and G_1, G_2 both have bounded degrees with $D(G_i) \asymp c(G_i) \asymp 1$ for $i = 1, 2$ (recall definition of $\asymp$ from Exercise Sheet 5), then

$$G_1 \text{ is amenable} \quad \Leftrightarrow \quad G_2 \text{ is amenable.}$$

Exercise 3 (5 points)

(i) Let Γ be a finitely generated group and G be one of its Cayley graphs. Assume that G is amenable. Show that Γ is an amenable group.

Remark: A sketch of the proof already has been given in the lecture. We now want to have a proof with all details.

(ii) Give an example (with justification) of an amenable group which has exponential growth.

Exercise 4 (5 points)

Assume that G is a connected graph with conductances $c(e) > 0$ for all $e \in E$. Further, take

$$\pi(x) = \sum_{e \text{ incident to the vertex } x} c(e)$$

and for all x, y choose

$$p(x, y) = \frac{c_{xy}}{\pi(x)} = \frac{c_{xy}}{\sum\limits_{z \sim x} c_{xz}}.$$

$G = (G, c, \pi)$ defines a network for which we define $\rho(G) := \limsup_{n\to\infty} p_n(x, y)^{1/n}$, with $p_n(x, y) = \mathbb{P}_x(X_n = y)$ as in the lecture.

Show that the definition of $\rho(G)$ does not depend on the choices of x and y.