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Winter Semester 2020/21
12/14/2020

Probability Theory on Trees and Networks

Exercise Sheet 7

Submission is due on 12/21/2020 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Exercise 1 (4 points)

Recall Lectures 11-12 for the earlier notation. That is, (G, c, D) is a network with $G = (V, E)$ and with π being a probability measure on the vertices. We write

$$d : \ell^2(V) \to \ell^2_-(E) \quad \text{with} \quad (df)(e) = f(e^-) - f(e^+)$$

and

$$d^* : \ell^2_-(E) \to \ell^2(V) \quad \text{with} \quad (d^*\theta)(x) = \sum_{e:e^-=x} \theta(e).$$

Further we write $D_{00} = \{\text{all functions on } V \text{ with finite support}\}$.

(i) Prove that for any $f \in D_{00}$ it is true that

$$d^*(c\ df) = \pi(f - Pf)$$

where $(Pf)(x) = \sum_{y\sim x} f(y)p_{xy}$ with $p_{xy} = \frac{c_{xy}}{\sum c_{xy'}}$.

(ii) Prove that

$$\sup\left\{\frac{\|Pf\|_\pi}{\|f\|_\pi} :\ f \neq 0\right\} = \sup\left\{\frac{\langle Pf, f\rangle_\pi}{\langle f, f\rangle_\pi} :\ f \in D_{00} \setminus \{0\}\right\}.$$

We will now define relative entropy on arbitrary probability spaces. Let S be a complete separable metric space. By $\mathcal{M}_1(S)$ we denote the set of all probability measures on S. Then, for $\mu, \nu \in \mathcal{M}_1(S)$,

$$H(\mu|\nu) = \begin{cases} \int \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)\mathrm{d}\mu, & \text{if } \mu \ll \nu \\ \infty, & \text{else.} \end{cases}$$

Exercise 2 (6 points)

(i) Give an example of $\mu, \nu \in \mathcal{M}_1(\mathbb{R})$ such that

$$H(\mu|\nu) \neq H(\nu|\mu).$$

This implies that $H(\cdot, \cdot)$ is, in general, not a metric.

(ii) Define

$$d(\mu, \nu) = \frac{1}{2}\big(H(\mu|\nu) + H(\nu|\mu)\big) \quad \mu, \nu \in \mathcal{M}_1(\mathbb{R}).$$

Show that $d(\cdot, \cdot)$ is not a metric either.

Exercise 3 (4 points)

Let S be a complete separable metric space and $\nu \in \mathcal{M}_1(S)$. Show that for any bounded Borel measurable real valued function f on S, we have

$$-\log \int_S e^{-f} \mathrm{d}\nu = \inf_{\mu \in \mathcal{M}_1(S)} \left\{ H(\mu|\nu) + \int_S f \mathrm{d}\mu \right\}.$$

Exercise 4 (6 points)

Let Normal$(0,1)$ denote the standard Gaussian on $\mathbb{R}$. Show that

$$\inf \left\{ H\Big(\nu \,\Big|\mathrm{Normal}(0,1)\Big) : \nu \in \mathcal{M}_1(\mathbb{R}), \int x \; \nu(\mathrm{d}x) = z \right\} = \frac{1}{2} z^2.$$