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Probability Theory on Trees and Networks

Exercise Sheet 8

Submission is due on 01/11/2021 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Exercise 1 (3 points)

Prove that for every $k \in \mathbb{Z}$ there exists a unique polynomial $Q_k \in \mathbb{R}[X]$ of degree k such that

$$Q_k(\cos \xi) = \cos(k\xi) \qquad \text{for all } \xi \in \mathbb{R}.$$

Exercise 2 (6 points)

Let S_n be the simple random walk on $\mathbb{Z}$. Show that for all $a > 0$,

$$\mathbb{P}_0(|S_n| > a) \leq 2 \exp\bigg(- \frac{a^2}{2n} \bigg).$$

Exercise 3 (6 points)

Show the following statements.

(i) Let $X_1, ..., X_n$ be independent random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\mathbb{R}$, and with $\mathbb{E}[X_k] = 0$ and $\mathbb{V}(X_k) < \infty$ for all $k = 1, ..., n$. Then, for all $\lambda > 0$,

$$\mathbb{P}\bigg(\max_{1 \leq k \leq n} |S_k| \geq \lambda \bigg) \leq \frac{1}{\lambda^2} \mathbb{V}(S_n),$$

where $S_k = X_1 + ... + X_k$.

(ii) The entropy on the nearest neighbor random walk on the lamplighter group is equal to 0.

Hint: Use part (i) to prove part (ii).

Exercise 4 (5 points)

Let Γ be a finitely generated group with respect to a symmetric generating set S and let μ be a probability measure with support on S. Let h be the Avez entropy of the simple random walk and ρ is the spectral radius. Show that

$$h \geq -2 \log \rho.$$

For the proof you may not use the statement from the lecture that says, positive entropy is equivalent to positive speed.