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Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 1

Due: Monday, 26.04.2020

Exercise 1 (3+4 Punkte)

Let $n \geq 1$ and $Z_1, \ldots, Z_n$ be independent and identically distributed standard Gaussians on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ and a constant $Z_{n,\beta} > 0$ define a new measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{Q}(\mathrm{d}\omega) := \frac{1}{Z_{n,\beta}} \exp\Big(\sum_{i=1}^{n} \beta_i Z_i(\omega) \Big) \mathbb{P}(\mathrm{d}\omega),$$

that is to say for every $A \in \mathcal{F}$

$$\mathbb{Q}(A) := \frac{1}{Z_{n,\beta}} \int_A \exp\Big(\sum_{i=1}^{n} \beta_i Z_i(\omega) \Big) \mathbb{P}(\mathrm{d}\omega).$$

(i) Determine the value of $Z_{n,\beta}$ which makes $\mathbb{Q}$ a probability measure.

(ii) For the above choice of $Z_{n,\beta}$, determine the distribution of the vector $(Z_1, \ldots, Z_n)$ under $\mathbb{Q}$.

For the following exercises let $X = (V, E)$ be a countable, locally finite graph, $\partial \subseteq V$ and $\hat{V} := V \setminus \partial$. For vertices $x, y \in V$, write $d(x) := \deg(x)$ and $x \sim y$ if x and y are connected by an edge. Moreover let P be the transition matrix of the *graph random walk* on X, i.e. for $x, y \in V$

$$P(x, y) = \mathbb{1}_{\{x \sim y\}} \frac{1}{d(x)},$$

and denote by G the corresponding Green function defined in the lectures.

Exercise 2 (2+3 Punkte)

(i) Show that $d(\cdot)$ defines a reversible invariant measure for P.

(ii) Assuming that X is finite, show that the matrix G is nonnegative semidefinite, i.e. for every real vector $\lambda = (\lambda_x)_{x \in V}$ we have

$$\lambda^T G \lambda = \sum_{x,y \in V} \lambda_x \lambda_y G(x, y) \geq 0.$$

Exercise 3 (4+4 Punkte)

(i) Let $\hat{P}$ be the restriction of P to $\hat{V}$, let $\hat{G}$ be the restriction of G to $\hat{V}$ and let I be the identity matrix of dimension $|\hat{V}|$. Assuming that $\hat{V}$ is finite, show that the matrix $I - \hat{P}$ is invertible with inverse given by

$$(I - \hat{P})^{-1} = \hat{G}D,$$

where D is the diagonal $\hat{V} \times \hat{V}$-matrix with diagonal entries $d(x)$ for $x \in \hat{V}$.

(ii) In the countable set-up, the infinite matrix P induces a bounded self-adjoint operator $\mathbf{P}$ on the Hilbert space

$$\ell^2(V, d) = \Big\{ f \colon V \longrightarrow \mathbb{R} \ : \ \sum_{x \in V} f(x)^2 d(x) < \infty \Big\},$$

with the inner product

$$\langle f, g \rangle = \sum_{x \in V} f(x) g(x) d(x),$$

by defining for $f \in \ell^2(V, d)$ and $x \in V$

$$(\mathbf{P}f)(x) = \sum_{y \in V} P(x, y) f(y).$$

Let $\mathbf{I}$ be the identity operator on $\ell^2(V, d)$. Prove that if $\|\mathbf{P}\| < 1$ (as an $\ell^2(V, d)$-operator), then the operator $\mathbf{I} - \mathbf{P}$ is invertible and its inverse satisfies for all $x, y \in \hat{V}$

$$G(x, y) d(y) = d(x)^{-1} \langle \mathbb{1}_x, (\mathbf{I} - \mathbf{P})^{-1} \mathbb{1}_y \rangle.$$

Hint: Neumann series.