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Sommersemester 2021
26. April 2021

Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 2

Due: Monday, 03.05.2020

Exercise 1 (Moments of Gaussian Distribution) (3 Punkte)

Let X be a Gaussian random variable with mean zero and variance $\sigma^2 > 0$. Show that for every $n \geq 1$

$$\mathbb{E}[X^{2n}] = c_n \sigma^{2n},$$

where $c_n \coloneqq 1 \cdot 3 \cdot \ldots \cdot (2n-1)$. What happens to the odd moments?

Exercise 2 (Hölder continuity of Brownian motion) (5 Punkte)

Let $(B_t)_{t\in[0,1]}$ be a one-dimensional Brownian motion. Prove that, almost surely, its sample paths do not satisfy a Hölder condition for Hölder exponent $\gamma \geq 1/2$ by showing that for any constant $C > 0$

$$\mathbb{P}\left[\sup_{0\leq s<t\leq 1} \frac{|B_t - B_s|}{\sqrt{t-s}} \leq C\right] = 0.$$

Exercise 3 (5 Punkte)

Consider a finite graph $X = (V, E)$ with boundary ∂ and write $V = \{x_1, \ldots, x_n\}$. Again let P be the transition matrix of the graph random walk, $\hat{P}$ its restriction to $\hat{V} = V \setminus \partial$ and $\hat{D} = \operatorname{diag}(d(x) : x \in \hat{V})$. It was proved in the lecture that the law of the discrete GFF $(h(x))_{x\in V}$ is absolutely continuous with respect to $|\hat{V}|$-dimensional Lebesgue measure.

Moreover, for every Borel set $A \subseteq \mathbb{R}^n$ such that $(y_1, \ldots, y_n) \in A$ implies that $y_i = 0$ for all indices i corresponding to $x_i \in \partial$, we have

$$\mathbb{P}\big[(h(x_1), \ldots, h(x_n)) \in A\big] = \frac{1}{Z} \int_A \exp\Big(-\frac{1}{4} \sum_{i,j\in\{1,\ldots,n\}: x_i\sim x_j} (y_i - y_j)^2\Big) \prod \mathrm{d}y_i$$

for an appropriate constant $Z > 0$. Show that $Z = (2\pi)^{|\hat{V}|/2} (\det \hat{D})^{-1/2} \det(I - \hat{P})^{-1/2}$.

Exercise 4 (3+2+2 Punkte)

Let $(B_t)_{t\geq 0}$ be a 2-dimensional Brownian motion, let

$$\mathbb{H} = \{x + iy : x \in \mathbb{R}, y > 0\} \subset \mathbb{C}$$

be the upper half-plane and let $p_t^{\mathbb{H}}(\cdot,\cdot)$ denote the transition probability densities of Brownian motion $(B_t^{\mathbb{H}})_{t\geq 0}$ killed off the domain $\mathbb{H}$ as defined in the lectures.

(i) Show that for every $z, w \in \mathbb{C}$

$$p_t^{\mathbb{H}}(z,w) = p_t(z,w) - p_t(z,\overline{w}).$$

Hint: Write (B_t) as a sum of two independent one-dimensional Brownian motions and use the reflection principle for the imaginary part.

(ii) Using part (i), deduce that for every $z, w \in \mathbb{C}$ with $z \neq w$

$$G_0^{\mathbb{H}}(z,w) = \log\left|\frac{z-\overline{w}}{z-w}\right|.$$

(iii) Show that $G_0^{\mathbb{H}}(z,\cdot)$ is harmonic in $\mathbb{D} \setminus \{z\}$ by proving that the function $w \mapsto \log|z-w|$ is harmonic in $\mathbb{C} \setminus \{z\}$.