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Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 5

Due: Monday, 31.05.2020

Exercise 1 (3+1 Punkte)

Let Ω be a simply connected domain and let f be a nowhere-vanishing holomorphic function on Ω.

(i) Show that there exists a holomorphic function g on Ω such that

$$\exp(g(z)) = f(z) \quad \text{for every } z \in \Omega. \tag{1}$$

Hint: You may use the fact that simple connectedness of Ω implies that every holomorphic function on Ω has a (holomorphic) primitive.

(ii) Prove that the function g from part (i) is unique up to an additive constant $c \in 2\pi i\mathbb{Z}$. In other words, show that if g_1 and g_2 are two holomorphic functions on Ω satisfying (1), then there exists $m \in \mathbb{Z}$ such that $g_1 = g_2 + 2\pi i m$.

Exercise 2 (4+4 Punkte)

A map of the form

$$f\colon \mathbb{C} \longrightarrow \mathbb{C},\ z \mapsto \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$, is called *Möbius transformation* or *linear fractional transformation.*

(i) Determine all linear fractional transformations which map $\mathbb{H}$ onto $\mathbb{D}$ and also map $\partial\mathbb{H}$ to $\partial\mathbb{D}$.

(ii) Determine all linear fractional transformations which map $\mathbb{H}$ onto $\mathbb{H}$.

Exercise 3 (4+4 Punkte)

Let Ω be any domain. An *analytic automorphism* of Ω is a bijective and holomorphic map

$$\phi\colon \Omega \longrightarrow \Omega.$$

The set of analytic automorphisms of Ω forms a group (w.r.t. composition) and is denoted $\mathrm{Aut}(\Omega)$.

For $a \in \Omega$, denote the *stabilizer of a in* $\mathrm{Aut}(\Omega)$ by

$$\mathrm{Aut}_a(\Omega) := \{\phi \in \mathrm{Aut}(\Omega) : \phi(a) = a\}.$$

Now consider a subgroup $G \subseteq \mathrm{Aut}(\mathbb{D})$ such that $\mathrm{Aut}_0(\mathbb{D}) \subseteq G$.

(i) For $c \in \mathbb{D}$ and $\theta \in \mathbb{R}$ define the function

$$\varphi_{c,\theta} \colon \mathbb{C} \longrightarrow \mathbb{C}, \ z \mapsto e^{i\theta} \frac{z - c}{1 - \overline{c}z}.$$

Suppose that $\varphi_{c_0,\theta_0} \in G$ for some $c_0 \in \mathbb{C}$ and $\theta_0 \in \mathbb{R}$. Show that $\varphi_{c,\theta} \in G$ for every $\theta \in \mathbb{R}$ and every $c \in \mathbb{C}$ with $|c| = |c_0|$.

(ii) Show that if $G \neq \mathrm{Aut}_0(\mathbb{D})$, then $G = \mathrm{Aut}(\mathbb{D})$.