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Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 7

Due: Monday, 14.06.2021

Exercise 1 (5+5 Punkte)

Let $D \subset \mathbb{C}$ be a Greenian domain. Moreover, let (X_n) be a sequence of iid standard Gaussians and (f_n) be an orthonormal basis of $H_0^1(D)$. For $f \in H_0^1(D)$ and $N \geq 1$ recall that we defined $h_N := \sum_{n=1}^N X_n f_n$. Now set

$$\langle h_N, f\rangle_\nabla := \sum_{n=1}^{N} X_n \langle f_n, f\rangle_\nabla .$$

(i) Show that as $N \to \infty$ the sequence $(\langle h_N, f\rangle_\nabla)$ converges in $L^2(\mathbb{P})$ and almost surely.

(ii) Determine the law of the limit.

Exercise 2 (4 Punkte)

Let $D, D' \subset \mathbb{C}$ be proper simply connected domains and $\varphi : D \longrightarrow D'$ a conformal map. Show that the Dirichlet inner product is conformally invariant in the sense that for every $f, g \in H_0^1(D)$

$$\langle f, g \rangle_\nabla = \langle f \circ \varphi^{-1}, g \circ \varphi^{-1}\rangle_\nabla$$

Exercise 3 (6 Punkte)

In the lectures, we proved that, if $D \subset \mathbb{R}^d$ is a bounded domain for arbitrary dimension $d \geq 1$, (X_n) is a sequence of iid standard Gaussians and (f_n) is an orthonormal basis of $H_0^1(D)$, then the series

$$\sum_{n=1}^{\infty} X_n f_n$$

converges almost surely in $H_0^s(D)$ for every $s = 1 - d/2 - \epsilon$ with $\epsilon > 0$. In the proof, the boundedness assumption for D was crucial in order to apply Weyl's Lemma for the eigenvalues of $-\Delta$. Note that the above result implies in particular, that the above series converges $\mathbb{P}$-almost surely in the space of distributions $\mathcal{D}_0'(D)$.

Now assume that $D \subset \mathbb{C}$ is a simply connected proper domain (and not necessarily bounded). Prove that the above series converges $\mathbb{P}$-almost surely in $\mathcal{D}_0'(D)$.

Hint: Use the Riemann mapping theorem and Exercise 2.