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Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 9

Due: Monday, 28.06.2021

Exercise 1 (10 Punkte)

Consider a Gaussian Free Field h on the unit disc $\mathbb{D}$. For $r > 0$ let

$$C_r := \{z \in \mathbb{C} : |z| = r\}$$

be the circle of radius r. A function f on $\mathbb{D}$ is called *radially symmetric* if it is constant on every circle C_r for $r \in (0,1)$. We denote by H_{rad} be the closure in $H_0^1(\mathbb{D})$ of the subspace of radially symmetric test functions. Moreover, we let H_{circ} denote the closure in $H_0^1(\mathbb{D})$ of the subspace of test functions which have mean zero on every circle C_r for $r \in (0,1)$.

Prove that we may decompose

$$h = h_{\text{rad}} + h_{\text{circ}},$$

where

(i) h_{rad} is a radially symmetric function.

(ii) h_{circ} is a distribution and, as a distribution, the limit of elements of H_{circ}.

(iii) h_{rad} and h_{circ} are independent.

Hint: Compare with the proof of the Domain Markov property of the Gaussian Free Field.

In the following, let D be a bounded domain, h a Gaussian Free Field on D and $(h_\varepsilon(z))$ a jointly continuous version of the circle average defined in the lectures.

Exercise 2 (5 Punkte)

Prove that for each fixed $\gamma \in [0, \infty)$ and $z \in D$, the random variables

$$e^{\gamma h_\varepsilon(z)} \varepsilon^{\gamma^2/2}$$

form a martingale as a function of ε.

Exercise 3 (5 Punkte)

Let $\varepsilon > 0$ and $\delta = \varepsilon/2$. Prove that $h_\varepsilon(x) - h_\delta(x)$ and $h_\varepsilon(y) - h_\delta(y)$ are independent if $|x - y| \geq 2\varepsilon$.