Westfälische
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Advanced Financial Mathematics

Lecture Notes Summer Term 2021

Contents

1 Informal Introduction

This chapter gives a preliminary introduction to financial markets and its ingredients.

1.1 Financial securities

On financial markets so called financial securities are traded. These are for example

- stocks (VW, Telekom, Apple, Google, Tesla, etc.)
- bonds (government bonds, corporate bonds, etc.)
- foreign currencies (Dollar, Euro, British Pound, etc.)
- commodities (oil, electricity, noble metals like gold, silver etc. , agricultural commodities etc.)

Based on these assets further financial contracts can be derived. Examples of these derivatives are

- Options
- Swaps, Floating Rate Note (FRN), Swaptions, Caps, Floors
- forwards, futures

Market places where these assets are traded are so called spot markets and futures markets. Examples are

- Exchanges (Stock Exchange, Currency Exchange, Commodities Exchange)
- futures exchange (German Futures Exchange, Chicago Board of Trade, etc.)
- derivatives exchange (German Futures and Derivatives Exchange)

The financial securities traded on these market places are

- normalised contracts. These are standardised securities that allow an efficient and very cheap trading.
- OTC contracts. These are tailor-made and highly specific.

1.1.1 Options

The so called plain vanilla options are puts and calls. These are simple derivatives on an underlying and can be explained in the following. Ingredients are

- the running time T, also called maturity,
- the strike K,
- an underlying denoted by S,

1. The **call** gives its holder the right to buy the underlying at the initially predetermined strike price K at maturity T.
2. The **put** gives its holder the right to sell the underlying at the initially predetermined strike K at maturity T.

If the price $S(T)$ at maturity of the underlying exceeds K, then the call holder can use his option to buy the underling at K and sell it immediately at $S(T)$. He would achieve a payoff

$$C(T) = (S(T) - K)^+.$$

If $S(T) < K$ then the holder of a put can buy the underlying at a price $S(T)$ and uses his put-option to sell it immediately at a price $S(T)$. Hence he receives at T a payoff

$$P(T) = (K - S(T))^+.$$

Mathematically speaking, put and call can be seen as derivatives that achieve a payoff

$$C(T) = (S(T) - K)^+ \quad \text{, resp.} \quad , P(T) = (K - S(T))^+.$$

1.1.2 long, short

A trader initially buys and sells financial assets and builds a portfolio. During trading time he changes his positions and balances his portfolio. He takes a

- long position in an asset, if he owns the asset.
- short position in an asset, if he has sold the asset.

For example

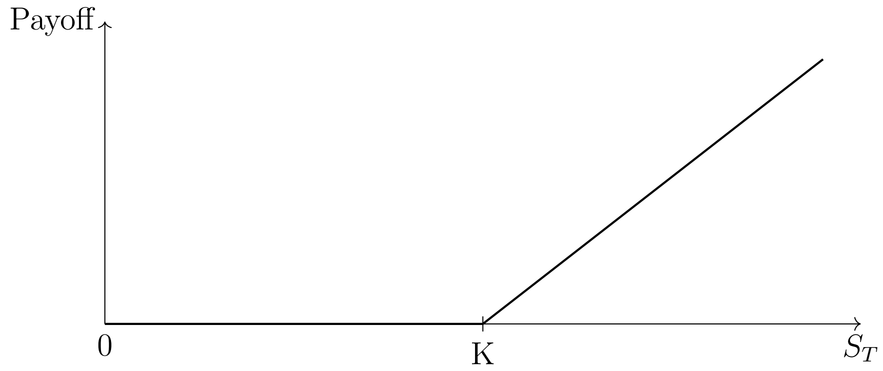
- A long call position pays the call value at buying time and receives the payoff at maturity.
- A short call position receives the value of the call contract when selling and has to deliver the payoff at maturity.

- A long stock position pays the stock value at buying time, gets all benefits like dividends during the holding time and receives the changed stock value at selling time.

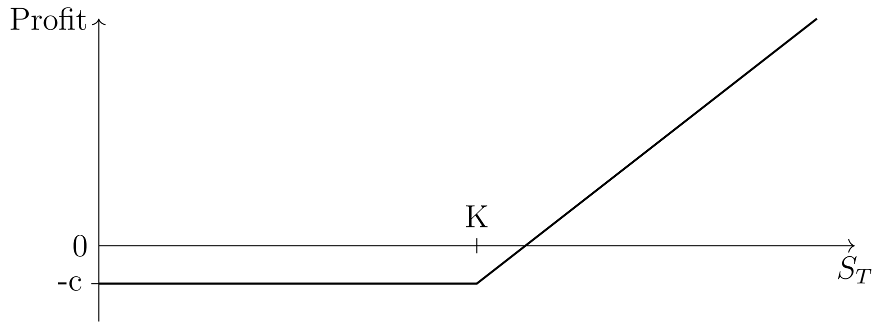
- A short stock position receives the stock value at selling time and has to pay the changed stock value in order to neutralise his position at a future time point.

The different effects these positions cause can be visualised by payoff resp. profit diagrams. These are plots at a specific time point, usually maturity, in dependence of the underlying value. Examples are

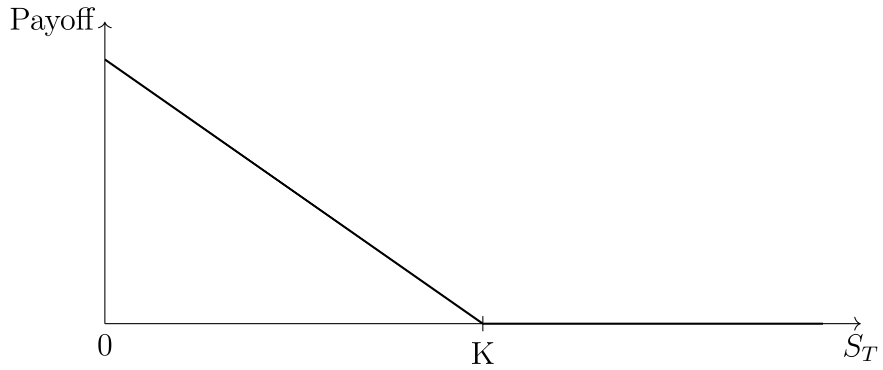
a) long call with strike K and maturity T. Payoff $(S(T) - K)^+$



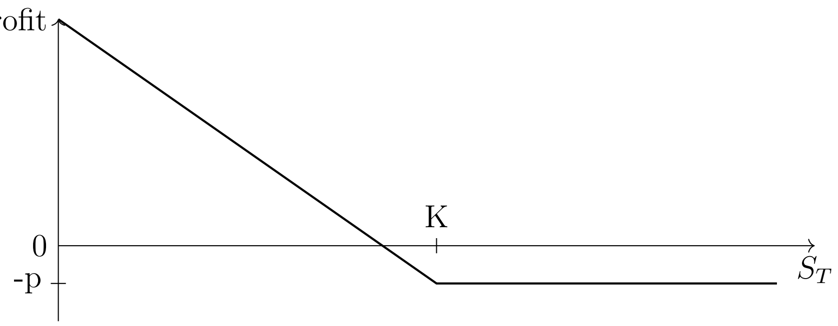
Costs: Initial call price: $c > 0$
Profit: $(S_T - K)^+ - c$



b) long put with strike K and maturity T, payoff $(K - S(T))^+$



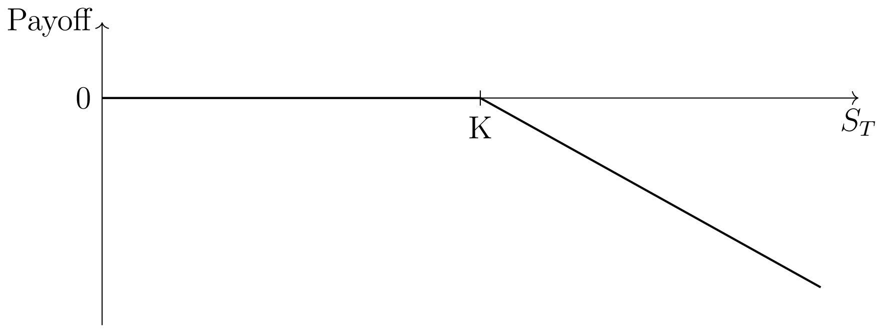
Costs: Initial put price: $p > 0$
Profit: $(S_T - K)^+ - p$

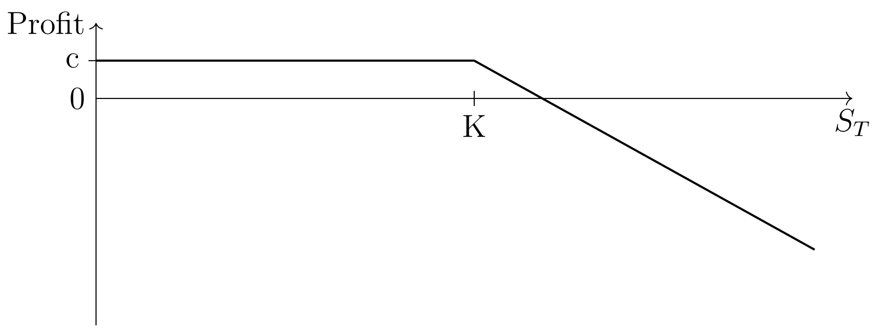


c) short call with strike K and maturity T

Payoff: $-(S_T - K)^+$

Profit: $c - (S_T - K)^+$

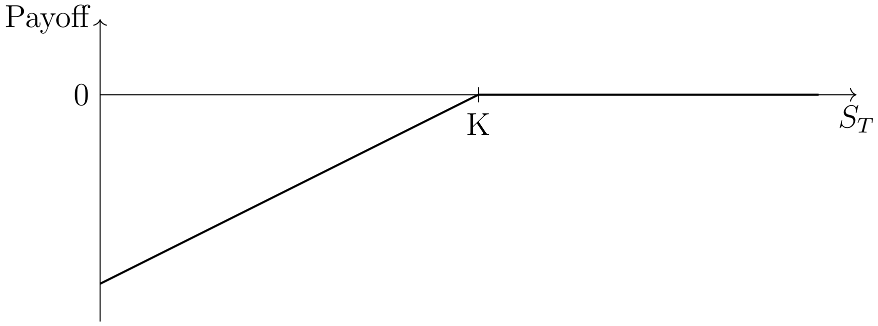


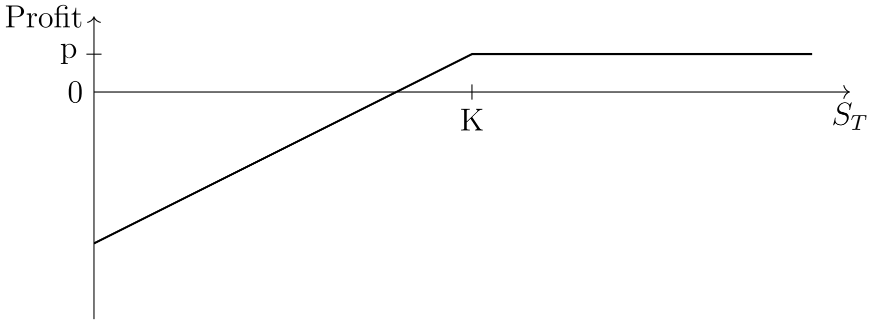


d) short put with strike K and maturity T

Payoff: $-(K - S_T)^+$

Profit: $p - (K - S_T)^+$





1.1.3 Zero Coupon Bond

A zero coupon bond denotes a financial security that delivers a payoff of 1 money unit (Euro) at maturity T. The holder of a zero coupon bond receives no coupons during the running time. This contract can be seen as loan. The holder pays initially the bond price $B(0, T) < 1$ to the seller and receives at maturity the loan sum 1 Euro. The difference $1 - B(0, T)$ can be seen as coupon resp. interest which is paid at maturity. Zero coupon bonds do not have a great volume as traded bonds in bond markets. Its importance relies in the fact that their prices can be computed from prices of traded bonds. They are easier to understand and give easier information on the state of the bond market resp. its evolution. A zero-coupon bond with maturity T is also called T-bond shortly. Its price-process will be denoted by $(B(t, T))_{t \leq T}$. The initial-state of the bond market can be expressed by the so called term-structure of bond prices. This is the bond price as function of maturity $(B(0, T))_{T>0}$. The evolution of the bond-market with time can be modelled by the change of the term-structure of bond prices with time.

1.2 Arbitrage

An arbitrage denotes an opportunity for a trader to achieve a risk-less profit. For example, this means that he may receive a positive payoff without any initial capital.

Thus he is able to get a free lunch which stands as an alternative expression for an arbitrage opportunity. The main assumption is:

Financial markets are free of arbitrage.

The whole pricing of financial securities rely on this basic assumption and this is justified in efficient and transparent markets where no barriers on trading exist. If an arbitrage occurs, for example due to miss-pricing of a financial security, the efficient price-building in markets would cancel this miss-pricing in very short time.
Based on this no-arbitrage principle several conclusions can be drawn which are of great importance for pricing derivatives.

Theorem 1.2.1 (Replication Principle). *We consider a financial market with dividend-free assets. If two self-financing trading strategies with value processes V and W coincide at a time point T they coincide at each time point $t \in [0, T]$ in between. Hence*

$$V(T) = W(T) \Longrightarrow V(t) = W(t) \quad \text{for all} \;\; 0 \le t \le T.$$

Note that the replication principle is no mathematical theorem, since we have not established a mathematical model so far and cannot state mathematical claims. But we can give arguments why the replication-principle can be deduced from the no-arbitrage principle.

Proof. We would like to show that the initial prices $V(0)$ and $W(0)$ of both strategies coincide and assume first that $V(0) > W(0)$.
But then we can

- initially sell V buy W,
- follow the trade of W and trade opposite to V in $(0, T)$,
- take the payoff $W(T)$ of W at the end to neutralise the obligation of $V(T)$ at the end.

This strategy would provide an arbitrage opportunity, the risk-less profit $V(0) - W(0)$ from the beginning.
In the case $W(0) > V(0)$ the same arguments work the other way round.

□

1.2.1 Put-Call Parity

As application of the replication principle we will show that there is a correspondence between put and call price of an underlying with the same maturity and strike. This is the so called put-call parity.

Theorem 1.2.2 (Put-Call Parity). *We consider a put and a call with same strike K and maturity T on a dividend-free underlying. Let S_0, c, p denote the inital price of the underling, call and put. Then*

$$p + S_0 = c + KB(0, T). \tag{1.1}$$

Proof. To show the assertion we consider two trading strategies.

(i) long in put and long in the underlying,

(ii) long in call and $K\times$ long in a T-Bond.

Both strategies get a payoff $\max\{S(T), K\}$ at T due to

$$(K - S(T))^+ + S(T) = \max\{S(T), K\} = (S(T) - K)^+ + K.$$

The replication principle implies that their initial prices coincide. But this is the claimed equation (1.1) above. $\square$

The first strategy above underlines the importance of the put-option in risk management. If you buy a stock you face the risk of a downside stock movement. To cover this risk you can buy in addition a put with strike K. Then your payoff will exceed at least the strike K. A put can be seen as an insurance contract protecting against downside stock movements.

1.2.2 Chooser-Option

A further application of the replication principle can be given in order to express the price of a so called chooser-option by a suitable call and put price.
We consider a financial market with

- deterministic, constant interest rate $r > 0$. This means that the price of a T bond is given by $B(t,T) = e^{-r(T-t)}$,
- an underlying S,
- puts and calls of all maturities and strikes.

A chooser-option gives its holder the right to choose at $T_1 < T$ a put or a call with strike K and maturity T. Let $ch(T_1, T, K)$, $c(S_0, T, K)$, $p(S_0, T_1, Ke^{-r(T-T_1)})$ denote the initial price of the chooser-option, the call with maturity T, strike K and the put with maturity T_1 and strike $Ke^{-r(T-T_1)}$. Then

Proposition 1.2.3.

$$ch(T_1, T, K) = c(S_0, T, K) + p(S_0, T_1, Ke^{-r(T-T_1)})$$

Proof. The holder of a chooser-option will take a call in T_1 if its more valuable than the corresponding put. If we denote by $c(S(T_1), T, K)$ and $p(S(T_1), T, K)$ their prices in T_1, then the chooser-option can be seen as a derivative with payoff

$$C = (S(T) - K)^+ \mathbb{1}_{\{c(S(T_1),T,K) \geq p(S(T_1),T,K)\}} + (K - S(T)^+ \mathbb{1}_{\{p(S(T_1),T,K) > c(S(T_1),T,K)\}}.$$

The question is how to replicate this payoff. The key observation is that we can reformulate the choose condition by applying the put-call parity. It holds

$$c(S(T_1), T, K) + Ke^{-r(T-T_1)} = p(S(T_1), T, K) + S(T_1)$$

and therefore

$$c(S(T_1), T, K) \geq p(S(T_1), T, K) \Longleftrightarrow S(T_1) \geq Ke^{-r(T-T_1)}.$$

This leads to the replicating strategy:

1. At the beginning:
   - Take a long position in a call with strike K and maturity T
   - Take a long position in a put with strike $Ke^{-r(T-T_1)}$ and maturity T_1
2. at T_1, if $S(T_1) < Ke^{-r(T-T_1)}$:
   - exercise the put and receive $Ke^{-r(T-T_1)} - S(T_1)$
   - sell the call and receive $c(S(T_1), K, T)$
   - use the received money to buy a put option with strike K and maturity T.
3. at T_1, if $S(T_1) \geq Ke^{-r(T-T_1)}$: hold the call until T.

This strategy is self-financing and provides the payoff of the chooser option at T. The replication principle implies the above assertion. □

2 The Black-Scholes Model

Black and Scholes developed in 1971 the approach of pricing derivatives by computing replicating trading strategies. They were able to derive pricing formulas for plain-vanilla options like call and put, in particular the famous Black-Scholes call-price formula. The benefits of the model are the following:

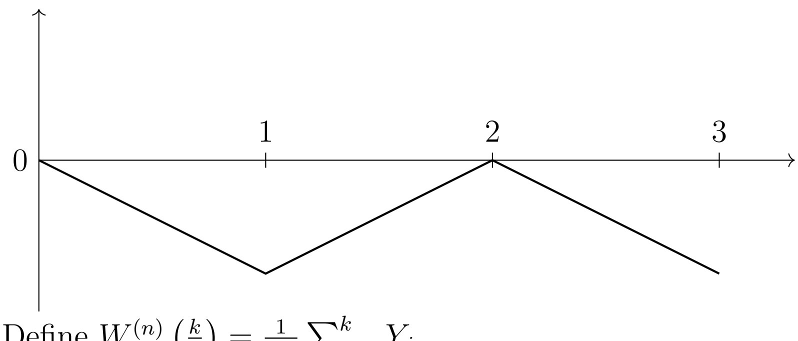
- simple model
- reasonable economic background
- analytically tractable

This will be explained in this chapter.

2.1 Wiener-Process

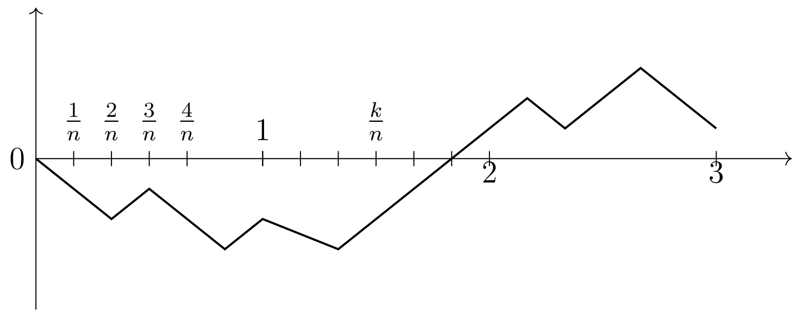
The Wiener-process, also called Brownian-motion, is one of the most important stochastic processes in continuous time with continuous paths. It is the starting point for the development of the stochastic integration theory and many sophisticated models in physics and economics use this process as basic tool. A Wiener-process can be seen as the continuous counterpart of a centered random-Walk and can be constructed as limiting process of suitable normalised centered random-walks.
To be more precise let $(Y_k)_{k\in\mathbb{N}}$ be a sequence of identically distributed independent random variables with

$$\mathbb{P}(Y_k = 1) = \frac{1}{2} = \mathbb{P}(Y_k = -1)$$



Put$W^{(1)}(t) := \sum_{k=1}^{t} Y_k$. By linear interpolation we obtain a continuous time process with continuous paths $(W^{(1)}(t))_{t\geq 0}$. Enlarge the frequency at factor n and compress the height at $\sqrt{n}$

Define $W^{(n)}\left(\frac{k}{n}\right) = \frac{1}{\sqrt{n}}\sum_{j=1}^{k} Y_j$.



Again linear interpolation leads to a continuous time stochastic process $\left(W^{(n)}(t)\right)_{t\geq 0}$. This sequence $W^{(n)}$ converges to a limiting process $(W(t))_{t\geq 0}$, a Wiener-process.
A random-walk is a discrete time stochastic process with independent and stationary increments. This property carries over to the limiting process W and can be used to give a precise definition of a Wiener-process.

Definition 2.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$ a filtration. An adapted stochastic-process $W = (W(t))_{t\geq 0}$ is called standard Wiener-process if it fulfills the following properties*

1. $W(0) = 0$ $\mathbb{P}$-*a.s.*
2. $W(t+s) - W(t)$ *is independent of* $\mathcal{F}_t$ *for all* $s, t \geq 0$
3. $W(t+s) - W(t)$ *has the same law as* $W(s)$ *for all* $s, t \geq 0$
4. $W(t)$ *is* $\mathbb{N}(0,t)$ *distributed for all* $t > 0$
5. *The paths of* W *are* $\mathbb{P}$-*a.s. continuous.*

Usually we omit the adjective standard and speak of a Wiener-process when a standard Wiener-process is meant.
Martingales play an essential role in many fields of probability theory, in particular in finance and stochastic analysis.

Definition 2.1.2. *An adapted process M is called an $(\mathcal{F}_t)_{t\geq 0}$ martingale if the following holds*

1. $\mathbb{E}|M(t)| < \infty$ *for all* $t \geq 0$.
2. $\mathbb{E}(M(t+s)|\mathcal{F}_t) = M(t)$ *for all* $t, s \geq 0$.

The independence of the increments can be used to easily identify basic martingales which will play a role in the following.

Proposition 2.1.3. *Let W be a Wiener-process w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then the following process are martingales.*

1. $W(t)_{t\geq 0}$

2. $(W(t)^2 - t)_{t\geq 0}$

3. $(\exp(\theta W(t) - \frac{1}{2}\theta^2 t))_{t\geq 0}$ *for each* $\theta \in \mathbb{R}$.

The proof of this assertion is very easy and can be done by carefully exploiting the independence of increments property. The last martingale is an example of a so called exponential martingale and can be used to define further probability measures. The therefore needed tool is a generalisation of Bayes-Theorem.

Theorem 2.1.4 (Change of measure)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$. Let L be a positive martingale w.r.t. $\mathbb{P}$ and $\bar{\mathbb{P}}$ a further probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = L(t) \quad \text{for all} \quad t \geq 0.$$

Then

(i) The conditional expectation w.r.t. $\bar{\mathbb{P}}$ can be calculated by computing the conditional expectation w.r.t. $\mathbb{P}$. More precisely, if Y is measurable w.r.t. $\mathcal{F}_t$ and integrable w.r.t. $\bar{\mathbb{P}}$, then for $s < t$

$$\bar{\mathbb{E}}(Y|\mathcal{F}_s) = \frac{\mathbb{E}(YL(t)|\mathcal{F}_s)}{L(s)}.$$

(ii) M is a $\bar{\mathbb{P}}$-martingale if and only if ML is a $\mathbb{P}$-martingale.

(iii) Let R be a positive $\mathbb{P}$-martingale with $\mathbb{E}R(t) = 1$ for all $t \geq 0$. Then a probability measure Q_T can be defined on each $\mathcal{F}_T$ by

$$\frac{dQ_T}{d\mathbb{P}}|_{\mathcal{F}_t} = R(t) \quad \text{for all} \quad t \leq T.$$

Proof. (i): We use the definition of conditional expectation directly. For $A \in \mathcal{F}_s$ we get

$$\int_A Y d\bar{\mathbb{P}} = \int_A YL(t)d\mathbb{P} = \int_A \mathbb{E}(YL(t)|\mathcal{F}_s)d\mathbb{P} = \int_A \frac{\mathbb{E}(YL(t)|\mathcal{F}_s)}{L(s)}d\bar{\mathbb{P}}.$$

This yields the first assertion.

ad (ii): This follows from (i) due to

$$\begin{aligned}
(M_t) \text{ is a } \overline{\mathbb{P}}\text{-martingale} \quad &\Leftrightarrow \quad \overline{\mathbb{E}}(M_t|\mathcal{F}_s) = M_s \qquad \text{for all } s \leq t \\
&\Leftrightarrow \quad \mathbb{E}(M_t L_t|\mathcal{F}_s)\frac{1}{L_s} = M_s \qquad \text{for all } s \leq t \\
&\Leftrightarrow \quad \mathbb{E}(M_t L_t|\mathcal{F}_s) = M_s L_s \qquad \text{for all } s \leq t \\
&\Leftrightarrow \quad ML \text{ is a } \mathbb{P}\text{-martingale}
\end{aligned}$$

ad (iii) Due to $\mathbb{E}R(T) = 1$ an equivalent probability measure on $(\Omega, \mathcal{F}_T)$ is defined by

$$Q_T(A) = \int_A R(T) d\mathbb{P} \quad \text{for all} \quad A \in \mathcal{F}_T.$$

Due to

$$\frac{dQ_T}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathbb{E}(R(T)|\mathcal{F}_t) = R(t)$$

for all $t < T$ the assertion follows.

$\square$

The preceding theorem can be used to state a first simple version of Girsanov's theorem

Theorem 2.1.5 (Girsanov)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration$(\mathcal{F}_t)_{t \geq 0}$ and W a Wiener-process w.r.t. $\mathbb{P}$. Let $\mathbb{P}_\vartheta$ be a further probability measure on $(\Omega, \mathcal{F})$ such that*

$$\frac{d\mathbb{P}_\vartheta}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp(\vartheta W(t) - \frac{1}{2}\theta^2 t) = L(t) \quad \textit{for all} \quad t \geq 0.$$

Then

$$\bar{W}(t) = W(t) - \vartheta t, \quad t \geq 0$$

defines a Wiener-process w.r.t. $\mathbb{P}_\theta$.

Proof. One has to verify that $\bar{W}$ satisfies the defining properties of a Wiener-process w.r.t. $\mathbb{P}_\vartheta$. Clearly $\bar{W}$ starts at 0 and has continuous paths. To show that the increments are independent we consider $g : \mathbb{R} \longrightarrow \mathbb{R}$ measurable and bounded. Then

$$\begin{aligned}
\mathbb{E}_\vartheta(g(\overline{W}(t) - \overline{W}(s))|\mathcal{F}_s) &= \mathbb{E}(g(\overline{W}(t) - \overline{W}(s))L_t|\mathcal{F}_s)\frac{1}{L_s} \\
&\text{with } L_t = \exp(\vartheta W(t) - \frac{1}{2}\vartheta^2 t) \\
&= \mathbb{E}(g(W(t) - W(s) - \vartheta(t-s))\frac{L_t}{L_s}|\mathcal{F}_s) \\
&= \mathbb{E}(g(W(t) - W(s) - \vartheta(t-s))\exp(\vartheta(W(t) - W(s)) - \frac{1}{2}\vartheta^2(t-s))|\mathcal{F}_s) \\
&= \mathbb{E}g(W(t) - W(s) - \vartheta(t-s))\exp(\vartheta(W(t) - W(s)) - \frac{1}{2}\vartheta^2(t-s)) \\
&= \mathbb{E}g(W(t-s) - \vartheta(t-s))\exp(\vartheta W(t-s) - \frac{1}{2}\vartheta^2(t-s)) \\
&= \mathbb{E}_\vartheta g(\overline{W}(t-s))
\end{aligned}$$

Hence, $\bar{W}(t) - \bar{W}(s)$ is independent of $\mathcal{F}_s$ and equally distributed as $\bar{W}(t-s)$ with a $\mathbb{N}(0, t-s)$-distribution due to

$$
\begin{aligned}
\mathbb{E}_\vartheta g(\overline{W}(t)) &= \mathbb{E} g(W(t)-\vartheta t)\exp(\vartheta W(t)-\frac{1}{2}\vartheta^2 t) \\
&= \mathbb{E} g(W(t)-\vartheta t)\exp(\vartheta (W(t)-\vartheta t)+\frac{1}{2}\vartheta^2 t) \\
&= e^{\frac{1}{2}\vartheta^2 t}\int g(x)e^{\vartheta x}N(-\vartheta t,t)(dx) \\
&= e^{\frac{1}{2}\vartheta^2 t}\frac{1}{\sqrt{2\pi t}}\int g(x)e^{\vartheta x}\exp(-\frac{1}{2t}(x+\vartheta t)^2)dx \\
&= \frac{1}{\sqrt{2\pi t}}\int g(x)e^{-\frac{1}{2t}x^2}dx \\
&= \int g(x)N(0,t)(dx)
\end{aligned}
$$

□

The Wiener-process W fulfills a further property which is of interest in finance for pricing barrier options. This is the so called reflection principle.
Let τ be a stopping time with $\mathbb{P}(\tau < \infty) = 1$. Then the reflected process w.r.t. τ is defined by

$$\hat{W}(t) = \begin{cases} W(t) & \text{for } t \leq \tau \\ W(\tau) - (W(t) - W(\tau)) & \text{for } t \geq \tau \end{cases} \tag{2.1}$$

The reflection principle states that $\hat{W}$ is also a Wiener-process. This can be proven by exploiting the strong Markov property of the Wiener-process. A useful application is the computation of the joint distribution of $W(T)$ and $M(T) = \sup_{t\leq T} W(s)$.

Theorem 2.1.6. *Let W denote a Wiener-process and M its running maximum. Then for $x \in \mathbb{R}$ and $z \geq x$ it holds*

$$\mathbb{P}(W(T) \leq x, M(T) \leq z) = \Phi(\frac{x}{\sqrt{T}}) - \Phi(\frac{x-2z}{\sqrt{T}})$$

with Φ denoting the distribution function of the $\mathbb{N}(0,1)$-distribution.
For the process X defined by $X(t) = W(t) + at$ it follows

$$\mathbb{P}(X(T) \leq x, \sup_{t\leq T} X(t) \leq z) = \Phi(\frac{x-aT}{\sqrt{T}}) - e^{2az}\Phi(\frac{x-2z-aT}{\sqrt{T}}).$$

Proof. For $x \in \mathbb{R}$ and $z > x$ we consider the first time that W reaches z, i.e.

$$\tau = \inf\{t \geq 0 : W(t) = z\}$$

and denote by $\hat{W}$ the reflected process w.r.t. τ. Then $\hat{W}$ is a Wiener-process and

$$\mathbb{P}(W(T) \leq x, M(T) \geq z) \quad = \quad \mathbb{P}(\hat{W}(T) \geq z + z - x, M(T) \geq z)$$

$$
\begin{aligned}
&= \mathbb{P}(\hat{W}(T) \geq 2z - x, \sup_{t\leq T} \hat{W}(t) \geq z) = \mathbb{P}(\hat{W}(T) \geq 2z - x) \\
&= \Phi(\frac{x-2z}{\sqrt{T}})
\end{aligned}
$$

But this implies

$$
\begin{aligned}
\mathbb{P}(W(T) \leq x, M(T) \leq z) &= \mathbb{P}(W(T) \leq x) - \mathbb{P}(W(T) \leq x, M(T) \geq z) \\
&= \Phi(\frac{x}{\sqrt{T}}) - \Phi(\frac{x-2z}{\sqrt{T}}).
\end{aligned}
$$

which yields the first formula.
To prove the second formula we change the measure by applying Girsanov's theorem and introducing

$$
\frac{d\mathbb{P}_a}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp(aW(t) - \frac{1}{2}a^2 t) \quad \text{for all} \quad t \leq T.
$$

Then W has the same distribution w.r.t. $\mathbb{P}_a$ as X w.r.t. $\mathbb{P}$. Hence

$$
\begin{aligned}
\mathbb{P}(X(T) \leq x, \sup_{t\leq T} X(t) \leq z) &= \mathbb{P}_a(W(T) \leq x, M(T) \leq z) \\
&= \int_{\{W(T)\leq x, M(T)\leq z\}} \exp(aW(T) - \frac{1}{2}a^2 T) d\mathbb{P} \\
&= \mathbb{E} g(W(T)) 1_{\{M(T)\leq z\}}
\end{aligned}
$$

with $g(y) = \exp(ay - \frac{1}{2}a^2 T) 1_{(-\infty,x]}(y)$.
Due to the first formula the condition distribution function fulfills

$$
\mathbb{P}(W(T) \leq x | M(T) \leq z) = \begin{cases} 1 & \text{if } x \geq z \\ \frac{\Phi(\frac{x}{\sqrt{T}}) - \Phi(\frac{x-2z}{\sqrt{T}})}{\mathbb{P}(M(T)\leq z)} & \text{if } x \leq z. \end{cases} \tag{2.2}
$$

Taking derivative w.r.t. x yields the conditional density

$$
h(y) = \frac{1}{\sqrt{T}\mathbb{P}(M(T) \leq z)} (\varphi(\frac{y}{\sqrt{T}}) - \varphi(\frac{y-2z}{\sqrt{T}}))
$$

for all $y \leq z$. This implies

$$
\begin{aligned}
\mathbb{E} g(W(T)) 1_{\{M(T)\leq z\}} &= \mathbb{P}(M(T) \leq z) \int_{-\infty}^{\infty} g(y) h(y) dy \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{T}} (\varphi(\frac{y}{\sqrt{T}}) - \varphi(\frac{y-2z}{\sqrt{T}})) \exp(ay - \frac{1}{2}a^2 T) dy \\
&= \Phi(\frac{x-aT}{\sqrt{T}}) - e^{2az} \Phi(\frac{x-2z-aT}{\sqrt{T}}),
\end{aligned}
$$

since

$$
\int_{-\infty}^{x} \frac{1}{\sqrt{T}} \varphi(\frac{y}{\sqrt{T}}) \exp(ay - \frac{1}{2}a^2 T) dy
$$

$$
\begin{aligned}
&= \mathbb{E}1_{\{W(T)\le x\}}\exp(aW(T)-\frac{1}{2}a^2T)\\
&= \mathbb{P}_a(W(T)\le x)=\mathbb{P}_a(W(T)-aT\le x-aT)=\Phi(\frac{x-aT}{\sqrt{T}})
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{x}\frac{1}{\sqrt{T}}\varphi(\frac{y-2z}{\sqrt{T}})\exp(ay-\frac{1}{2}a^2T)dy\\
&= \mathbb{E}1_{\{W(T)+2z\le x\}}\exp(a(W(T)+2z)-\frac{1}{2}a^2T)\\
&= \exp(2az)\mathbb{P}_a(W(T)+2z\le x)\\
&= \exp(2az)\Phi(\frac{x-2z-aT}{\sqrt{T}})
\end{aligned}
$$

□

2.2 Pricing in the Black-Scholes Model

The Black-Scholes model is a continuous time model for a financial market that consists of

- a money market account with price process $\beta(t)=e^{rt}$ for all $t\ge 0$ and
- a risky asset with price-process

$$S(t)=S(0)e^{\mu t}\exp(\sigma W(t)-\frac{1}{2}\sigma^2 t).$$

The process W denotes a Wiener-process and μ,σ,r can be seen as parameters that fix the distribution.

- The number $\mu\in\mathbb{R}$ denotes the so called rate of return and affects the expected evolution of the risky asset, since

$$\mathbb{E}S(t)=e^{\mu t}\quad\text{for all}\quad t\ge 0.$$

- The value $\sigma>0$ affects the fluctuation of the risky asset due to

$$\operatorname{Var}\log(\frac{S(t)}{S(0)})=\operatorname{Var}\sigma W(t)=\sigma^2 t$$

  and is often called volatility.
- The interest rate $r\in\mathbb{R}$ represents the evolution of a risk-free money market account.

To take into account the time-value of money the so called discounted price-process is introduced by

$$S^*(t) = \frac{S(t)}{\beta(t)} = S(0)e^{(\mu-r)t}\exp(\sigma W(t) - \frac{1}{2}\sigma^2 t) \quad \text{for all} \quad t \geq 0.$$

We observe that only for $\mu = r$ the discounted price-process is a martingale. In this case the risky asset has the same expected return as the money market account and we say that the market is risk-neutral. The question arises whether starting from μ a change of the stand-point, a change of measure, can be done such that the market is risk-neutral. This consideration is incorporated in the term equivalent martingale measure.

Definition 2.2.1. *We consider a Black-Scholes model along the running-time T. An equivalent martingale measure $\mathbb{P}^*$ is a probability measure on $(\Omega, \mathcal{F}_T)$ with the following properties*

1. *$\mathbb{P}$ and $\mathbb{P}^*$ are equivalent probability measures on $(\Omega, \mathcal{F}_T)$.*
2. *The discounted price process $S^*(t) = \frac{S(t)}{\beta(t)}$, $0 \leq t \leq T$ is a martingale w.r.t. $\mathbb{P}^*$.*

An application of Girsanov's theorem provides the existence of an equivalent martingale measure.

Theorem 2.2.2. *In a Black-Scholes model with running-time $T > 0$ an equivalent martingale measure exists.*

Proof. We know form 2.1.5 that an equivalent probability measure $\mathbb{P}^*$ can be defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} = L(t) = \exp(\vartheta W(t) - \frac{1}{2}\vartheta^2 t) \quad \text{for all} \quad t \leq T.$$

The parameter ϑ has to be chosen such that the discounted price process becomes a martingale. Since $W^*(t) = W(t) - \vartheta t$ is a Wiener-process w.r.t. $\mathbb{P}^*$ we obtain

$$\begin{aligned} S^*(t) &= S(0)e^{(\mu-r)t}\exp(\sigma W(t) - \frac{1}{2}\sigma^2 t) \\ &= S(0)e^{(\mu-r)t}\exp(\sigma(W^*(t) + \vartheta t) - \frac{1}{2}\sigma^2 t) \\ &= S(0)e^{(\mu-r+\sigma\vartheta)t}\exp(\sigma(W^*(t) - \frac{1}{2}\sigma^2 t) \end{aligned}$$

Thus S^* is a martingale if and only if

$$\mu - r + \sigma\vartheta = 0 \quad \Longleftrightarrow \quad \vartheta = -\frac{\mu - r}{\sigma}$$

and the theorem is proven. $\square$

The existence of an equivalent martingale measure is an important property of the Black-Scholes model. It opens a probabilistic way to pricing derivatives. As we will see later in the course all financial derivatives with an integrable payoff C at T can be replicated and the initial value of each replicating strategy coincides with the expected discounted payoff under the equivalent martingale measure. Therefore the following definition is justified.

Definition 2.2.3. *Let $\mathbb{P}^*$ be the equivalent martingale measure in the Black-Scholes model and C be an $\mathcal{F}_T$-measurable payoff at T with $\mathbb{E}^* \frac{C}{\beta(T)} < \infty$. Then the initial arbitrage-free price of C is defined by*

$$p_0(C) = \mathbb{E}^* \frac{C}{\beta(T)} = \mathbb{E}^* C^*.$$

Later we will clarify why this definition is reasonable. Simply speaking, pricing of a derivative with payoff C at T means

- determine a reasonable equivalent martingale measure $\mathbb{P}^*$
- compute $\mathbb{E}^* C^*$

This pricing mechanism is reasonable in more or less all financial market models and mainly used in practise.
Before we give some applications we note that S^* is a positive martingale under $\mathbb{P}^*$. Thus a further equivalent change of measure can be done by defining a probability measure $\mathbb{P}^*_\sigma$ via

$$\frac{d\mathbb{P}^*_\sigma}{d\mathbb{P}^*}|_{\mathcal{F}_t} = \frac{S^*(t)}{S(0)} = \exp(\sigma W^*(t) - \frac{1}{2}\sigma^2(t))$$

for all $t \leq T$. Girsanov provides that

$$W^{**}(t) = W^*(t) - \sigma t$$

is a Wiener-process w.r.t. $\mathbb{P}^*_\sigma$.
To give a full picture we have the following evolution of the stock price process under the different measure

- The subjective probability measure $\mathbb{P}$:

$$S(t) = S(0)e^{\mu t} \exp(\sigma W(t) - \frac{1}{2}\sigma^2 t)$$

- The equivalent martingale measure $\mathbb{P}^*$:

$$S(t) = S(0)e^{rt} \exp(\sigma W^*(t) - \frac{1}{2}\sigma^2 t)$$

- The further transformed measure $\mathbb{P}^*_\sigma$:

$$S(t) = S(0)e^{(r+\sigma^2)t} \exp(\sigma W^{**}(t) - \frac{1}{2}\sigma^2 t)$$

This means that S is a geometric Wiener-process with

- trend μ and volatility σ w.r.t $\mathbb{P}$,
- trend r and volatility σ w.r.t. $\mathbb{P}^*$,
- trend $r+\sigma^2$ and volatility σ w.r.t. $\mathbb{P}^*_\sigma$.

This can be exploited by deriving the Black-Scholes formula.

Theorem 2.2.4. *We consider a call with maturity T and strike K in a Black-Scholes model with volatility σ and initial stock price $S(0)$. Then the initial arbitrage-free price of the call is given by*

$$c(S(0),T,\sigma,K) = S(0)\Phi(h_1(S(0),T) - Ke^{-rT}\Phi(h_2(S(0),T) \tag{2.3}$$

with

$$\begin{aligned} h_1(S_0,T) &= \frac{\log\left(\frac{S(0)}{K}\right) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ h_2(S_0,T) &= \frac{\log\left(\frac{S(0)}{K}\right) + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \end{aligned}$$

Proof. In the first step we compute

$$\begin{aligned} \mathbb{E}^* e^{-rT}(S(T)-K)^+ &= \mathbb{E}^\star e^{-rT} S_T \mathbb{1}_{\{S_T>K\}} - \mathbb{E}^\star e^{-rT} K \mathbb{1}_{\{S_t>K\}} \\ &= S(0)\mathbb{E}^\star \frac{S_T^\star}{S(0)} \mathbb{1}_{\{S_T>K\}} - e^{-rT} K \mathbb{E}^\star \mathbb{1}_{\{S_T>K\}} \\ &= S(0) \underbrace{\mathbb{P}^\star_\sigma(S_T>K)}_{(1)} - e^{-rT} K \underbrace{\mathbb{P}^\star(S_T>K)}_{(2)} \end{aligned}$$

To compute the probabilities we remind you on the representation of S under $\mathbb{P}^*$ and $\mathbb{P}^*_\sigma$. It follows

$$\begin{aligned} \mathbb{P}^\star_\sigma(S_T>K) &= \mathbb{P}^\star_\sigma\left(\log\left(\frac{S_T}{S(0)}\right) > \log\left(\frac{K}{S(0)}\right)\right) \\ &= \mathbb{P}^\star_\sigma\left(\sigma W_T^{\star\star} - \frac{1}{2}\sigma^2 T + (r+\sigma^2)T > \log\left(\frac{K}{S(0)}S\right)\right) \\ &= \mathbb{P}^\star_\sigma\left(\underbrace{\frac{W_T^{\star\star}}{\sqrt{T}}}_{\sim N(0,1)} > \frac{\log\left(\frac{K}{S(0)}\right) + \frac{1}{2}\sigma^2 T - (r+\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &= \Phi\left(\frac{\log\left(\frac{S(0)}{K}\right) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

and

$$
\begin{aligned}
\mathbb{P}^\star(S_T > K) &= \mathbb{P}^\star\left(\log\left(\frac{S_T}{S(0)}\right) > \log\left(\frac{K}{S(0)}\right)\right) \\
&= \mathbb{P}^\star\left(\sigma W_T^\star - \frac{1}{2}\sigma^2 T + rT > \log\left(\frac{K}{S(0)}\right)\right) \\
&= \mathbb{P}^\star\left(\underbrace{\frac{W_T^\star}{\sqrt{T}}}_{\sim N(0,1)} > \frac{\log\left(\frac{K}{S(0)}\right) + \frac{1}{2}\sigma^2 T - rT}{\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{\log\left(\frac{S(0)}{K}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

Hence (2.3) follows.

□

Note that due to the put-call parity the price of a put can be easily calculated too. Put and call are examples of so called path independent options since the payoff at maturity is only a function of the terminal stock-price. More delicate is the problem of finding pricing formulas for path dependent options. This can be done for so called one-sided barrier options.

Theorem 2.2.5. *A down and out call with maturity T, strike K and barrier $B < S(0)$ is an option with payoff $(S(T) - K)^+$ at maturity T if the barrier B is not hit during the running-time. This corresponds to a derivative with payoff*

$$C = (S(T) - K)^* \mathbb{1}_{\{\inf_{t \le T} S(t) > B\}}.$$

The initial price of a down and out call is given by

$$p_0(C) = c(S_0, T, K) - (\frac{S_0}{B})^{\frac{2b}{\sigma}} c(S_0, T, K\frac{S_0^2}{B^2}) \tag{2.4}$$

with $b = -\frac{r}{\sigma} - \frac{1}{2}\sigma$.

This means that the price of a down and out call can be expressed by call prices w.r.t. different strikes.

Proof. Nearly the same calculations as in the ordinary call can be done here.

$$
\begin{aligned}
p_0(C) &= \mathbb{E}^* e^{-rT}(S(T) - K)^+ 1_{\{\inf_{t \le T} S(t) > B\}} \\
&= \mathbb{E}^* e^{-rT} S(T) 1_{\{S(T) > K, \inf_{t \le T} S(t) > B\}} - K e^{-rT}\mathbb{P}^*(S(T) > K, \inf_{t \le T} S(t) > B) \\
&= S_0 \mathbb{P}^*_\sigma(S(T) > K, \inf_{t \le T} S(t) > B) - K e^{-rT}\mathbb{P}^*(S(T) > K, \inf_{t \le T} S(t) > B),
\end{aligned}
$$

Further elementary calculations yield

$$\begin{aligned}\mathbb{P}^*(S(T)>K, \inf_{t\le T} S(t)>B) &= \mathbb{P}^*(-\log\frac{S(T)}{S_0}<\log\frac{S_0}{K}, -\inf_{t\le T}\log\frac{S(t)}{S_0}<\log\frac{S_0}{B})\\ &= \mathbb{P}^*(X(T)<\frac{1}{\sigma}\log\frac{S_0}{K}, \sup_{t\le T} X(t)<\frac{1}{\sigma}\log\frac{S_0}{B})\end{aligned}$$

with

$$X(t) = -\frac{1}{\sigma}\log\frac{S(t)}{S_0} = -W^*(t) + (\frac{1}{2}\sigma - \frac{r}{\sigma})t.$$

The process X is a Wiener-process with drift $a = \frac{1}{2}\sigma - \frac{r}{\sigma}$ and therefore the probability is given by

$$\mathbb{P}(X(T)\le x, \sup_{t\le T} X(t)\le z) = \Phi(\frac{x-aT}{\sqrt{T}}) - e^{2az}\Phi(\frac{x-2z-aT}{\sqrt{T}}).$$

due to 2.1.6.
W.r.t. $\mathbb{P}^*_\sigma$ the process X is a Wiener-process with drift $b = -\frac{r}{\sigma}-\frac{1}{2}\sigma$. Hence an application of 2.1.6 provides

$$\mathbb{P}(X(T)\le x, \sup_{t\le T} X(t)\le z) = \Phi(\frac{x-bT}{\sqrt{T}}) - e^{2bz}\Phi(\frac{x-2z-bT}{\sqrt{T}}).$$

Due to $\frac{2a}{\sigma} = \frac{2b}{\sigma} + 2$ and collecting all terms (2.4) follows.

$\square$

Example 2.2.6. As application of our pricing framework in the Black-Scholes model we consider an equity-linked bond and as a specific example an equity-linked bond on the Tesla stock. Ingredients are

- Security Identification Number: ISIN DE000HVB50E1
- Underlying: Tesla A1CX3T stock traded at Nasdag in Dollar
- Nominal: 1000 Euro
- reference price: price of the underlying at 22.01.2021 = 846, 64 USD
- strike: 677.312 USD = 80% of the reference price
- interest rate: 16%
- running time: 1 year
- exchange-ratio: 1 Euro = 1.217 USD at 21.01.2021
- subscription ratio: Nominal * exchange-ratio/strike = 1000 ∗ 1.217/677.312 = 1.7968

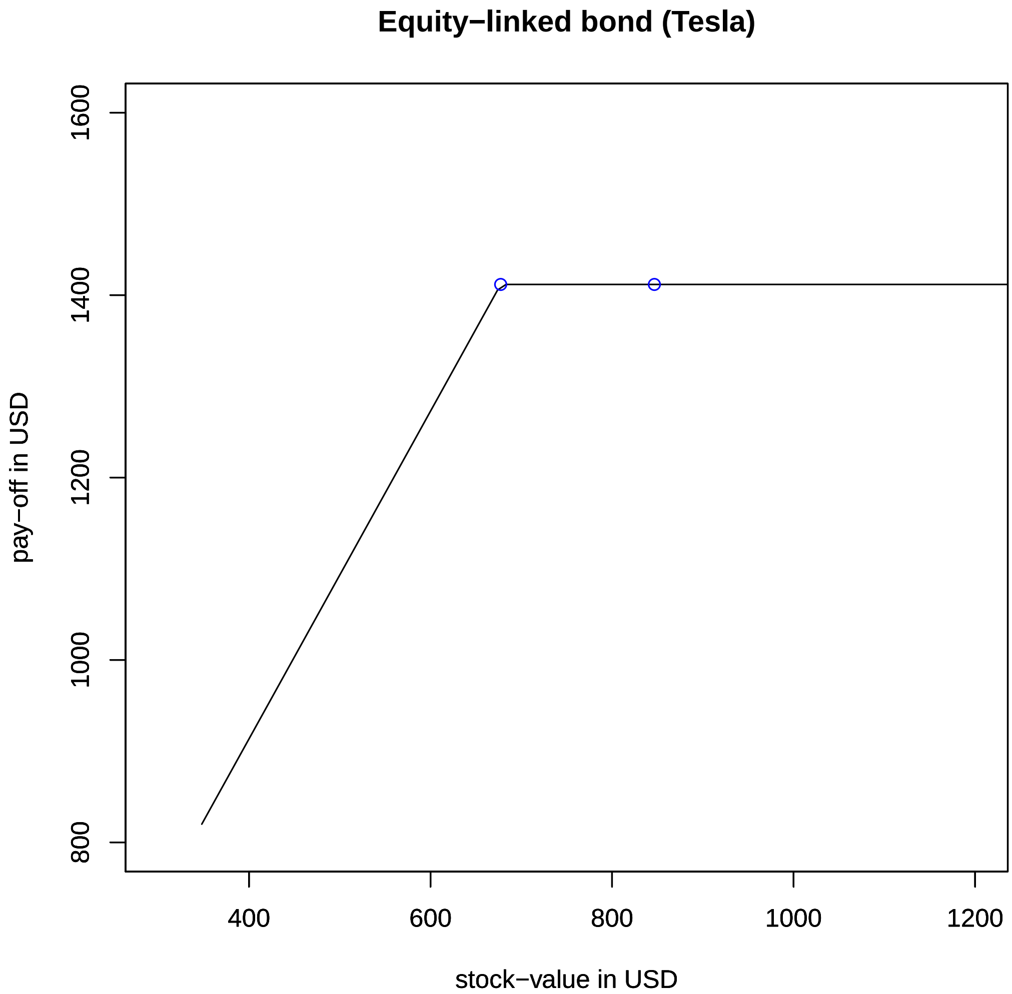


Figure 2.1: payoff equity-linked bond Tesla at maturity T

Pay-off:
Denote by N the Nominal, R the interest-rate, K the strike, T the maturity, w the exchange-ratio Euro to USD and S the price-process of the Tesla stock in USD.
The holder of the equity-linked bond receives at maturity the coupon

$$C = N \cdot R \cdot T \quad \text{Euro} \quad = \quad N \cdot R \cdot T \cdot w \quad \text{Dollar}$$

in any case.
If the Tesla stock price is at maturity above the strike, he receives the nominal. Otherwise Tesla-stocks will be delivered corresponding to the subscription ratio 1.7968. This corresponds to a pay-off in USD at maturity

$$A = \begin{cases} Nw + Cw & , S(T) > K \\ \frac{Nw}{K} S(T) + Cw & , S(T) \leq K \end{cases}.$$

In the case $S(T) \leq K$ there is a loss in comparison with the nominal

$$(Nw - \frac{Nw}{K} S(T)) = \frac{Nw}{K}(K - S(T))$$

in comparison with the nominal. By buying of $\frac{Nw}{K}$ put-options according to the strike K one can eliminate the down-side risk and the portfolio of

- equity-linked bond
- $\frac{Nw}{K}$ put-options on the Tesla stock with strike K and maturity T

replicate the risk-free pay-off $Nw + Cw$ Dollar at T. The replication-principle implies that the initial USD price of the equity-linked bond can be expressed by

$$p_0(A) = (N + C)wB(0,T) - \frac{Nw}{K} p(S(0), T, K).$$

By division with the exchange-ratio w one would receive an initial price in Euro. We are able to calculate the put-price in a Black-Scholes Model with the Black-Scholes call formula and applying the put-call parity. Therefore we are able to compute the today's model-price of the equity-linked bond, if the parameters in the BS-model are fixed. As values of the volatility σ for Tesla and the interest-rate r of a money-market account it is reasonable to take

$$r = 0,043\% \quad \text{and} \quad \sigma = 60\%.$$

Then we end up at a price

$$949,123 \quad \text{Euro}.$$

3 Preliminaries from Stochastic Analysis

In this chapter we will give a short overview of results from stochastic analysis that are used in mathematical finance. The following books can be recommended for a further reading:

1. Rogers, Williams [1]
2. Karatzas, Shreve [2]
3. Revuz, Yor [3]

The presented contents can be found with more detailed proofs in the lecture notes (in German) to the course Stochastic Analysis.

3.1 Martingales

We start by repeating some basic facts on martingales.

Setup

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and want to confirm some basic definitions.

- a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a family of increasing sub σ-fields, i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s \leq t$. Let $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$ denote all the information gained from the whole filtration.
- a stochastic process X is a family $(X_t)_{t\geq 0}$ of random variables.
- the process X is adapted w.r.t. $\mathbb{F}$, if each $X(t)$ is measurable w.r.t. $\mathcal{F}_t$.
- the canonical filtration of X is defined by $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$ for all $t \geq 0$. It is the smallest filtration that covers the information obtained by observing X.
- We say that X has continuous paths if

  $$t \to X_t(\omega)$$

  is a.s. continuous. The process X is right continuous resp. left continuous if its paths are a.s. right resp. left continuous.

- We say that X is a cadlag process if its paths are a.s. right continuous with existing limits from the left.
- A subset $A \subset \Omega$ is called negligible, if there is a nullset N such that $A \subset N$
- We say that a path-property E is a.s. fulfilled if

$$\{\omega \in \Omega : \omega \text{ does not fulfill } E \ \}$$

is negligible.

For processes in continuous time there exist two equivalence terms.

Definition 3.1.1. *Two stochastic processes X and Y are called indistinguishable if*

$$\{\omega : \exists t \geq 0 : X_t(\omega) \neq Y_t(\omega)\} = \bigcup_{t \geq 0} \{\omega : X_t(\omega) \neq Y_t(\omega)\}$$

is negligible.

If the paths of two processes coincide almost surely they are undistinguishable. A weaker property is the following.

Definition 3.1.2. *Two stochastic processes X and Y are modifications if*

$$\{\omega : X_t(\omega) \neq Y_t(\omega)\}$$

is negligible for all $t \geq 0$.

It two process are modifications they coincide on countably infinite time values a.s.. If they are indistinguishable they coincide for all time points a.s.
The following remark can be easily proven.

Remark 3.1.3. *The following assertions hold:*

(i) Are X and Y indistinguishable, they are modifications.

(ii) Are X and Y modifications and do they have right-continuous paths they are indistinguishable.

Definition 3.1.4. *Some $(\mathcal{F}_t)_{t \geq 0}$ adapted process X is called martingale w.r.t. $(\mathcal{F}_t)_{t \geq 0}$, if:*

(i) $\mathbb{E}|X_t| < \infty$ *for all* $t \geq 0$

(ii) $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ *for all* $0 \leq s \leq t$

respectively submartingale, if

(i) $\mathbb{E}|X_t| < \infty$ *for all* $t \geq 0$

(ii) $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$ *for all* $0 \leq s \leq t$

and supermartingal, if

(i) $\mathbb{E}|X_t| < \infty$ *for all* $t \geq 0$

(ii) $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$ *for all* $0 \leq s \leq t$

Important examples for martingales are given by the Wiener-process W, the exponential martingale $(\exp(\vartheta W(t) - \frac{1}{2}\vartheta t))_{t\geq 0}$ and the compensated squared Wiener-process $(W^2(t) - t)$. One benefit of stochastic analysis is the fact that martingales can be easily defined by stochastic integral processes.
At infinity three phenomena may occur.

Theorem 3.1.5 (Martingale convergence)**.** *Let* $(X_t)_{t\geq 0}$ *be a right continuous submartingale with*

$$\sup_{t\geq 0} \mathbb{E}X_t^+ < \infty.$$

Then there exists some $\mathcal{F}_\infty$*-measurable random variable* X_∞ *such that*

$$X_t \longrightarrow X_\infty \qquad \mathbb{P}\text{-}a.s.$$

and

$$\mathbb{E}|X_\infty| < \infty$$

A proof can be found in [3].
Some corollaries can be drawn from the martingale convergence theorem.

Corollary 3.1.6. *The following assertions hold.*

(i) Each positive right continuous martingale converges.

(ii) For each $\vartheta \in \mathbb{R}$ *the process* $\exp(\vartheta W(t) - \frac{1}{2}\vartheta^2 t) \xrightarrow[t\to\infty]{a.s.} 0$.

(iii) Each L_p *bounded martingale does converge*

(iv) The Wiener-process W *does not converge.*

Proof. A positive martingale X is also a supermartingale and therefore $-X$ a negative submartingale which converges due to $\mathbb{E}(-X_t)^+ < \infty$. The process $\exp(\vartheta W(t) - \frac{1}{2}\vartheta^2 t)$ converges since it is a positive martingale. The strong law of large numbers state for the Wiener-process that

$$\frac{W(t)}{t} \xrightarrow[t\to\infty]{a.s.} 0.$$

Hence

$$\exp(\vartheta W(t) - \frac{1}{2}\vartheta^2 t) = \exp(t(\frac{\vartheta W(t)}{t} - \frac{1}{2}\vartheta^2) \xrightarrow[t\to\infty]{a.s.} 0.$$

The process X is called L_p-bounded if $\sup_{t\geq 0} \mathbb{E}|X(t)|^p < \infty$. From this condition an application of Hölders inequality implies $\sup_{t\geq 0} \mathbb{E}|X(t)| < \infty$ and we have convergence due to 3.1.5.

The Wiener-process does not fulfill the assumptions on 3.1.5. Since every $a \in \mathbb{R}$ can be reached by the Wiener-process a.s.

$$\sup_{t\geq 0} W(t) = \infty = -\inf_{t\geq 0} W(t).$$

Hence a.s. every path of W does not converge. $\square$

Besides a.s. convergence L_p convergence is of interest, in particular in L_1. The martingale convergence theorem only establishes a.s. convergence, a slight stronger condition must be satisfied to ensure L_1-convergence. This leads to the term uniformly integrability.

Definition 3.1.7. *Let I be an index set and $(X(t))_{t\in I}$ be a family of real-valued random variables. This family is called uniformly integrable if*

$$\sup_{t\in I} \mathbb{E}|X_t|\mathbb{1}_{\{|X_t|>a\}} \overset{a\to\infty}{\longrightarrow} 0.$$

A random variable Y is integrable iff $\mathbb{E}|Y|\mathbb{1}_{\{|Y|>a\}} \overset{a\to\infty}{\to} 0$. Uniformly integrability means that his convergence takes place uniformly in $i \in I$.
To check the definition can be tedious. The following proposition can be helpful.

Proposition 3.1.8. *Let $(X(t))_{t\in I}$ be a family of real-valued random variables. Then the following statements are equivalent..*

(i) $(X(t))_{t\in I}$ *is uniformly integrable,*

(ii) The following conditions hold
  - *a)* $\sup_{t\in I} \mathbb{E}|X_t| < \infty$,
  - *b)* $\forall \varepsilon > 0 \exists \delta > 0 : \forall A \in \mathcal{F} : \mathbb{P}(A) < \delta \Longrightarrow \sup_{t\in I} \mathbb{E}|X_t|\mathbb{1}_A < \varepsilon$.

(iii) There exists some non-negative, increasing, convex function $G : [0,\infty) \to [0,\infty)$ such that

$$\lim_{x\to\infty} \frac{G(x)}{x} = +\infty \textit{ and } \sup_{t\in I} \mathbb{E}G(|X_t|) < \infty.$$

(iv) There exists some non-negative, increasing, function $G : [0,\infty) \to [0,\infty)$ such that

$$\lim_{x\to\infty} \frac{G(x)}{x} = +\infty \textit{ and } \sup_{t\in I} \mathbb{E}G(|X_t|) < \infty.$$

A proof can be found in the book of Klenke [4].
The proposition can be used to prove the following useful results.

Corollary 3.1.9. *The following statements hold true*

(i) Each finite family of integrable random variable is uniformly integrable.

(ii) *Let* $(X(t))_{t\in I}$ *and* $(X(t))_{t\in J}$ *uniformly integrable. Then* $(X(t))_{t\in I\cup J}$ *is also uniformly integrable.*

(iii) *If there exists some integrable random variable* Y *such that* $|X_t| < Y$ *for all* $t \in I$, *then* $(X(t))_{t\in I}$ *is uniformly integrable.*

(iv) *From* $\sup_{t\in I} \mathbb{E}|X_t| < \infty$ *the uniformly integrability can not be deduced in general.*

(v) *If* $\sup_{t\in I} \mathbb{E}|X_t|^p < \infty$ *for some* $p > 1$, *then* $(X(t))_{t\in I}$ *is uniformly integrable.*

By applying the corollary we can deduce that each L_p bounded martingale with $p > 1$ is uniformly integrable. The L_1 boundedness is not sufficient for L_1 convergence and uniform integrability.
The next important example has applications in stochastic analysis.

Example 3.1.10. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and I be a set of sub-σ-fields of $\mathcal{F}$. Then, the family $(\mathbb{E}(Y|\mathcal{G}))_{\mathcal{G}\in I}$ is uniformly integrable if Y is integrable.

This means that w.r.t. $Y \in L_1$ a uniformly integrable martingale can be defined by

$$X(t) = \mathbb{E}(Y|\mathcal{F}_t) \quad \text{for all} \;\; t \geq 0.$$

Since a uniformly integrable process that converges in probability also converges in L_1 the above defined martingale converges in L_1. More precise we have the following theorem.

Theorem 3.1.11. *Let* $(X_t)_{t\geq 0}$ *be some stochastic process with* $\mathbb{E}|X_t| < \infty$ *for all* $t \geq 0$ *and let* X_∞ *be some further random variable. Let the following conditions be held true*

a) $(X_t)_{t\geq 0}$ *is uniformly integrable,*

b) $X_t \longrightarrow X_\infty$ *converges in probability, i.e.*

$$\mathbb{P}(|X_t - X_\infty| > \epsilon) \stackrel{t\to\infty}{\longrightarrow} 0$$

Then X_t *converges in* L_1 *to* X_∞, *i.e.*

$$\lim_{t\to\infty} \mathbb{E}|X_t - X_\infty| = 0$$

Proof. At first we have to clarify that $X_\infty \in L_1$.
There exists a subsequence (t_n) such that $(X(t_n)) \to X_\infty$ a.s. Hence,

$$\begin{aligned}
\mathbb{E}|X_\infty| = \mathbb{E} \liminf_{n\to\infty} |X_{t_n}| &\leq \liminf_{n\to\infty} \mathbb{E}|X_{t_n}| \\
&\leq \sup_{n\in\mathbb{N}} \mathbb{E}|X_{t_n}| \\
&\leq \sup_{t\geq 0} \mathbb{E}|X_t| \underset{a)}{<} \infty
\end{aligned}$$

Prove next the L_1 convergence. Let $\varepsilon > 0$.

Due to uniformly integrability there exists some $\delta > 0$ such that

$$\mathbb{P}(A) < \delta \Rightarrow \mathbb{E}|X_t|\mathbb{1}_A < \frac{\varepsilon}{3} \text{ for all } t \geq 0$$

and

$$\mathbb{P}(A) < \delta \Rightarrow \mathbb{E}|X_\infty|\mathbb{1}_A < \frac{\varepsilon}{3}.$$

Due to convergence in probability there exists some $T \geq 0$ such that

$$\mathbb{P}(|X_t - X_\infty| > \frac{\varepsilon}{3}) < \delta \text{ for all} t \geq T.$$

Hence, for all $t \geq T$

$$\begin{aligned}
\mathbb{E}|X_t - X_\infty| &= \mathbb{E}|X_t - X_\infty|\mathbb{1}_{\{|X_t - X_\infty| \leq \frac{\varepsilon}{3}\}} + \mathcal{E}|X_t - X_\infty|\mathbb{1}_{\{|X_t - X_\infty| > \frac{\varepsilon}{3}\}} \\
&\leq \frac{\varepsilon}{3}\mathbb{P}(|X_t - X_\infty| \leq \frac{\varepsilon}{3}) + \mathbb{E}|X_t|\mathbb{1}_{\{|X_t - X_\infty| > \frac{\varepsilon}{3}\}} + \mathbb{E}|X_\infty|\mathbb{1}_{\{|X_t - X_\infty| > \frac{\varepsilon}{3}\}} \\
&\leq 3\frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

□

An L_p version is the following

Theorem 3.1.12. *Let $p > 1$ and $(X_t)_{t\geq 0}$ be some stochastic process with $\mathbb{E}|X_t|^p < \infty$ for all $t \geq 0$. Let X_∞ be some further random variable. If*

a) $(|X_t|^p)_{t\geq 0}$ is uniformly integrable and

b) $X_t \longrightarrow X_\infty$ in probability,

then X_t converges in L_p to X_∞, i.e.

$$\lim_{t\to\infty} \mathbb{E}|X_t - X_\infty|^p = 0$$

The main result of this section is that the set of uniformly integrable martingales can be identified with an L_1-space.

Theorem 3.1.13 (Isomorphism I)**.** *Let $(\mathcal{F}_t)_{t\geq 0}$ be some Filtration and $\mathfrak{M}$ the set of uniformly integrable $(\mathcal{F}_t)_{t\geq 0}$-martingales.*
Then,

$$\begin{aligned}
J: \quad \mathfrak{M} &\longrightarrow L_1(\Omega, \mathcal{F}_\infty, \mathbb{P}) \\
X &\mapsto X_\infty := \lim_{t\to\infty} X_t
\end{aligned}$$

is an isomorphism with inverse

$$\begin{aligned} I: \quad L_1(\Omega, \mathcal{F}_\infty, \mathbb{P}) &\longrightarrow \mathfrak{M} \\ Y &\mapsto (\mathbb{E}(Y|\mathcal{F}_t))_{t\geq 0} \end{aligned}$$

3.1.1 Optional Sampling

A martingale can be seen as as a fair game of luck. To specify this statement we have to introduce stopping times.

Definition 3.1.14. *Let $(\mathcal{F}_t)_{t\geq 0}$ be some Filtration. A stopping time τ is a mapping*

$$\tau : \Omega \longrightarrow [0, \infty) \cup \{+\infty\}$$

such that

$$\{\tau \leq t\} \in \mathcal{F}_t \qquad \textit{for all } t \geq 0$$

The decision of stopping before t may only depend on the information up to time t. Hence, a stopping time cannot look in the future, it is non-anticipative.
The σ-field of those events that are observable by τ is defined by

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Note, that a definition of the form

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for alle } t \geq 0\}$$

yields the same set of events.
Some basic facts on stopping times are the following.

Proposition 3.1.15. *For stopping times σ, τ the following statements hold true:*

(i) $\sigma \wedge \tau (= \min(\sigma, \tau)), \sigma \vee \tau (= \max(\sigma, \tau)), \sigma + \tau$ *are stopping times.*

(ii) If $\sigma \leq \tau$, then $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.

(iii) If X is a cadlag-process, then $X_\tau \mathbb{1}_{\{\tau < \infty\}}$ is $\mathcal{F}_\tau-$measurable.

(iv) $\mathcal{F}_{\tau\wedge\sigma} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.

(v) If $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of stopping times, then

$$\sup_n \tau_n$$

is a $(\mathcal{F}_t)_{t\geq 0}$ stopping time and

$$\inf_n \tau_n$$

a $(\mathcal{F}_{t+})_{t\geq 0}$ stopping time.

(vi) A random variable X is $\mathcal{F}_\tau$-measurable, if and only if $X\mathbb{1}_{\{\tau\leq t\}}$ is measurable w.r.t. $\mathcal{F}_t$ for all $t \geq 0$.

A proof can be found in most text-books of stochastic processes.
One can see a stochastic-process as the evolution of a payoff from a game of luck. A stopping time can be seen as that random time point τ that a player finishes playing the game and receiving the payoff $X(\tau)$. If this time is bounded a player cannot make a gain on average. This is the result of the first version of optional stopping.

Theorem 3.1.16 (Optional Sampling I)**.** *Let X be some $(\mathcal{F}_t)_{t\geq 0}$-martingale with cadlag paths and τ be a bounded stopping time, i.e. there exists some $T > 0$ with $\tau \leq T$ $\mathbb{P}$-a.s. Then:*

(i) $\mathbb{E}(X_T|\mathcal{F}_\tau) = X_\tau$ $\mathbb{P}$-*a.s.*

(ii) $\mathbb{E}X_\tau = \mathbb{E}X_0$

The benefit of this theorem is twofold. First it clarifies the origin of the name martingale in the sense that the value of the process at a time-point τ can be deduced from the values at a future time-point by taking conditional expectation. This will be applied in finance several time. Secondly a gambler cannot find a bounded stopping strategy that improves on average his payoff. This fairness property is also sufficient for a stochastic process to become a martingale. We can state the following theorem.

Theorem 3.1.17. *Let X be some $(\mathcal{F}_t)_{t\geq 0}$ adapted process with cadlag paths and*

$$\mathbb{E}|X_t| < \infty \quad \text{for all } t \geq 0.$$

Then the following statements are equivalent:

(i) X is a martingale

(ii) For each bounded stopping time τ

$$\mathbb{E}X_\tau = \mathbb{E}X_0.$$

The disadvantage so far is that only bounded stopping times are treated. This is necessary since counter examples exist. If we consider a Wiener-process and the first hitting time of an $a > 0$. Then $\mathbb{E}W(\tau) = a \neq 0$. Thus further conditions have to be required to ensure optional sampling for unbounded stopping times.
One version is the following.

Theorem 3.1.18 (Optional Sampling II)**.** *Let X be some $(\mathcal{F}_t)_{t\geq 0}$ adapted uniformly integrable martingale with cadlag paths.
Then:*

(i) There exists some $\mathcal{F}_\infty$-measurable mapping X_∞ such that

$$\mathbb{E}(X_\infty | \mathcal{F}_\tau) = X_\tau \qquad \mathbb{P}\text{-almost sure}$$

for each stopping time τ.

(ii) $\mathbb{E}X_\tau = \mathbb{E}X_0$ for each stopping time τ.

And also a characterisation of a uniformly integrable martingale can be given.

Theorem 3.1.19. *Let $(X_t)_{t\geq 0}$ be some $(\mathcal{F}_t)_{t\geq 0}$ adapted process with cadlag paths and let X_∞ be some $\mathcal{F}_\infty$-measurable random variable.*
The process $(X_t)_{t\geq 0}$ is a uniformly integrable martingale with

$$\lim_{t\to\infty} X_t = X_\infty,$$

if and only if X_τ is integrable for each stopping time τ with

$$\mathbb{E}X_\tau = \mathbb{E}X_0.$$

The preceding theorems can often be useful in proving results in stochastic analysis. Here we can give an easy application.

Corollary 3.1.20. *Let X be some $(\mathcal{F}_t)_{t\geq 0}$-martingale with cadlag paths and τ be a stopping time.*
Then, the stopped process X^τ, defined by

$$X_t^\tau := X_{t\wedge\tau} = X_t \mathbb{1}_{\{t\leq\tau\}} + X_\tau \mathbb{1}_{\{t>\tau\}}$$

is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.

Proof. The process X^τ is adapted with cadlag paths and for each bounded stopping time σ it holds

$$\mathbb{E}X^\tau(\sigma) = \mathbb{E}X(\tau\wedge\sigma) = \mathbb{E}X(0).$$

Hence 3.1.17 yields the assertion. □

At the end of this section we will give some applications on optional sampling.

Corollary 3.1.21. *Let W be a Wiener-process and τ_a denote the hitting time of $a \in \mathbb{R}$. Then*

(i) $\mathbb{P}(\tau_a < \infty) = 1,$

(ii) $\mathbb{E}\,\tau_a = \infty.$,

(iii) if $a, b > 0$ then

$$\mathbb{P}(\tau_b < \tau_{-a}) = \frac{a}{a+b} \quad \textit{and} \quad \mathbb{E}\tau_{ab} = ab$$

with $\tau_{ab} = \tau_{-a} \wedge \tau_b$ denoting the exit time of the interval $(-a, b)$.

Proof. To prove the first and second assertion one has to consider for each $\lambda > 0$ the martingale

$$M_\lambda(t) = \exp(\lambda W(t) - \frac{1}{2}\lambda^2 t).$$

Then $M_\lambda^{\tau_a}$ is a bounded martingale, hence uniformly integrable and converges to

$$\exp(\lambda a - \frac{1}{2}\lambda^2 \tau_a)\mathbb{1}_{\{\tau_a < \infty\}}.$$

Optional sampling gives

$$1 = \mathbb{E}M_\lambda^{\tau_a}(0) = \mathbb{E}M_\lambda^{\tau_a}(\infty) = \mathbb{E}\exp(\lambda a - \frac{1}{2}\lambda^2\tau_a)\mathbb{1}_{\{\tau_a<\infty\}} = e^{\lambda a}\mathbb{E}e^{-\frac{1}{2}\lambda^2\tau_a}\mathbb{1}_{\{\tau_a<\infty\}}.$$

Hence

$$\mathbb{E}e^{-\frac{1}{2}\lambda^2\tau_a}\mathbb{1}_{\{\tau_a<\infty\}} = e^{-\lambda a} \qquad \text{for all } \lambda > 0$$

and monotone convergence implies

$$\mathbb{P}(\tau_a < \infty) = \lim_{\lambda\to 0}\mathbb{E}e^{-\frac{1}{2}\lambda^2\tau_a}\mathbb{1}_{\{\tau_a<\infty\}} = \lim_{\lambda\to 0} e^{-\lambda a} = 1$$

With the arguments above we have computed the Laplace-transform of τ_a, since

$$L_{\tau_a}(\nu) = \mathbb{E}\, e^{-\nu\tau_a} \underset{\substack{\nu=\frac{1}{2}\lambda^2 \\ \Leftrightarrow\sqrt{2\nu}=\lambda}}{=} e^{-\sqrt{2\nu}a}.$$

The Laplace-transform determines the distribution of τ_a and it follows that τ_a has the density

$$g_a(t) = \frac{1}{\sqrt{2\pi t^3}}\exp(-\frac{1}{2}\frac{a^2}{t})\mathbb{1}_{(0,\infty)}(t)$$

due to

$$\mathbb{E}\, e^{-\nu\tau_a} = \int_0^\infty e^{-\nu t} g_a(t)dt = e^{-\sqrt{2\nu}a} \qquad \text{for all } \nu > 0.$$

Amongst others

$$\mathbb{E}\,\tau_a = \int_0^\infty t g_a(t)dt = \infty$$

follows.

For the third assertion we can consider the Wiener-process itself as martingale. Then optional sampling implies

$$0 = \mathbb{E}W_{\tau_{ab}} = -a\mathbb{P}(\tau_{-a} < \tau_b) + b\mathbb{P}(\tau_b < \tau_{-a})$$

and together with

$$1 = \mathbb{P}(\tau_{-a} < \tau_b) + \mathbb{P}(\tau_b < \tau_{-a})$$

the first part of (iii) follows. For the second part we consider $W(t)^2 - t$. Then, optional sampling gives

$$\mathbb{E}\,\tau_{ab} = \mathbb{E}W(\tau_{ab})^2 = a^2 \frac{b}{a+b} + b^2 \frac{a}{a+b} = ab.$$

$\square$

3.1.2 Doob's martingale inequalities

In stochastic analysis the so called $\mathcal{H}_p$-space is of importance. This is the content of this section. First we apply Jensen's inequality to obtain certain submartingales.

Proposition 3.1.22. *Let E be a convex subset of $\mathbb{R}$.*

(i) If X is a martingale with values in E and if

$$f : E \longrightarrow \mathbb{R}$$

is convex, such that $\mathbb{E}|f(X_t)| < \infty$ for all $t \geq 0$, then $(f(X_t))_{t\geq 0}$ is a submartingale.

(ii) If X is a submartingale in E and

$$f : E \longrightarrow \mathbb{R}$$

convex and increasing with $\mathbb{E}|f(X_t)| < \infty$ for all $t \geq 0$, then $(f(X_t))_{t\geq 0}$ is a submartingale.

Typical situations where we can apply this are

Corollary 3.1.23. *The following assertions are true.*

(i) X martingale $\Rightarrow |X|$ submartingale,

(ii) X martingale and $\mathbb{E}|X_t|^p < \infty$ for all $t \geq 0 \Rightarrow (|X_t|^p)_{t\geq 0}$ submartingale,

(iii) X martingale $\Rightarrow X^+$ submartingale

For a process X we define the running maximum by

$$X^*(T) = \sup_{0\leq t\leq T} |X(t)| \quad \text{for all} \;\; T > 0.$$

Doob's maximal inequality give bounds of the running maximum in terms of the terminating random variable. The L_p-space is a Banach-space with the norm defined by

$$||X||_p = (\mathbb{E}|X|^p)^{\frac{1}{p}} \quad \text{for all} \;\; X \in L_p.$$

Theorem 3.1.24 (Doob's Maximal Inequalities)**.** *Let $(X_t)_{t\geq 0}$ be some right continuous martingale* ***or*** *some positive submartingal w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then the following holds for $X_T^\star := \sup_{0\leq t\leq T} |X_t|$:*

(i) $\lambda^p \mathbb{P}(X_T^\star \geq \lambda) \leq \mathbb{E}|X_T|^p \mathbb{1}_{\{X_t^\star \geq \lambda\}} \leq \mathbb{E}|X_T|^p$ *for all* $p \geq 1$,

(ii) $\lambda^p \mathbb{P}(X_\infty^\star > \lambda) \leq \sup_{t\geq 0} \mathbb{E}|X_t|^p$ *for all* $p \geq 1$,

(iii) $||X_T^\star||_p \leq \frac{p}{p-1} \sup_{0\leq t\leq T} ||X_t||_p$ *for all* $p > 1$,

(iv) $||X_\infty^\star||_p \leq \frac{p}{p-1} \sup_{t\geq 0} ||X_t||_p$ *for all* $p > 1$,

Remark 3.1.25. The assertions (iii) and (iv) are called Doob's L_p-inequalities. These are equivalent with

$$\mathbb{E}|X_T^\star|^p \leq \left(\frac{p}{p-1}\right)^p \sup_{0\leq t\leq T} \mathbb{E}|X_t|^p \quad \text{for all } p > 1, 0 < T \leq \infty$$

Usually the canonical filtration of a stochastic-process X is not sufficient to satisfy the necessary technical purposes. It has to be slightly enlarged which leads to the so called usual conditions.

Definition 3.1.26. *A filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ fulfills the usual conditions if it fulfills*

(i) right continuity, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$ *for all* $t \geq 0$.

(ii) completeness, i.e. $\mathcal{F}_0$ *contains all negligible sets.*

Technical advantages of a filtration that satisfies the usual conditions are.

1. modifications of adapted processes are again adapted,
2. entrance times into Borel sets are stopping times,
3. paths of martingales can be regulated.

More precise:
Each martingale w.r.t. a filtration that satisfies the usual conditions has a modification with right continuous paths. This is the reason why one can always assume that a martingale has right continuous paths.
In the following we will show how a filtration $\left(\mathcal{F}_t^{(0)}\right)_{t\geq 0}$ can be enlarged such that the usual conditions are fulfilled. In a first step we add the negligible sets $\mathcal{N}$ by defining

$$\mathcal{F}_t^{(1)} = \sigma(\mathcal{F}_t \cup \mathcal{N}) \quad \forall t \geq 0.$$

In the second step we ensure the right continuity by defining

$$\mathcal{F}_t = \mathcal{F}_{t+}^{(1)} \quad \forall t \geq 0.$$

Then $\mathbb{F}^{(1)}$ is complete and $\mathbb{F}$ fulfills the usual conditions. We see that an assumption of completeness is not very restrictive.
At the end of this section the $\mathcal{H}_p$-spaces are introduced. These enable a characterization of L_p bounded martingales.

Definition 3.1.27. *For $p > 1$ the space $\mathcal{H}_p$ is defined by*

$$\mathcal{H}_p := \{X : X \textit{ is a cadlag martingale w.r.t. } (\mathcal{F}_t)_{t\geq 0} \textit{ with } \sup_{t\geq 0} \mathbb{E}|X_t|^p < \infty\}$$

The space $\mathcal{H}_p$ consists of all those martingales that are bounded in L_p. Due to this boundedness a norm on $\mathcal{H}_p$ can be defined by

$$||X||_{\mathcal{H}_p} := \sup_{t\geq 0}(\mathbb{E}|X_t|^p)^{\frac{1}{p}} = \sup_{t\geq 0} ||X_t||_p = \left(\sup_{t\geq 0} \mathbb{E}|X_t|^p\right)^{\frac{1}{p}}.$$

Doob's L_p-inequalities ensure that the space $\mathcal{H}_p$ is isometric isomorph to an L_p space.

Theorem 3.1.28 (Isometry II)**.** *The mapping*

$$\begin{aligned} J: \quad \mathcal{H}_p &\longrightarrow L_p(\Omega, \mathcal{F}_\infty, \mathbb{P}) \\ X &\mapsto \quad X_\infty := \lim_{t\to\infty} X_t \end{aligned}$$

defines an isometry between Banach-spaces and its inverse is given by

$$\begin{aligned} I: \quad L_p(\Omega, \mathcal{F}_\infty, \mathbb{P}) &\longrightarrow \quad \mathcal{H}_p \\ X \quad &\mapsto (\mathbb{E}(X_\infty|\mathcal{F}_t))_{t\geq 0} \end{aligned}$$

Isometry means that

$$||J(X)||_p = ||X_\infty||_p = ||X||_{\mathcal{H}_p} \qquad \textit{for all } X \in \mathcal{H}_p$$

Proof. Note that $\mathcal{H}_p$ is a subspace of $\mathfrak{M}$ and L_p a subspace of L_1. Due to 3.1.9 the mappings I and J are inverse isomorphisms of $\mathfrak{M}$ and Ł$_1$. It remains to show that the subspaces are mapped together and that the isometry property holds. If we take $X \in \mathcal{H}_p$ then Doob's L_p inequality implies

$$||X_\infty||_p \leq ||X_\infty^\star||_p \leq \frac{p}{p-1} \sup_{t\geq 0} ||X_t||_p = \frac{p}{p-1} ||X||_{\mathcal{H}_p} < \infty.$$

Since

$$X^{\star p}_\infty = (\sup_{t\geq 0} |X_t|^p)$$

is an integrable upper bound the dominated convergence theorem yields

$$\mathbb{E}|X_\infty|^p = \lim_{t\to\infty} \mathbb{E}|X_t|^p = \sup_{t\geq 0} \mathbb{E}|X_t|^p = ||X||^p_{\mathcal{H}_p}.$$

Hence

$$||X_\infty||_p = ||X||_{\mathcal{H}_p}.$$

Contrary let $X_\infty \in L_p$. Then $I(X_\infty) = (\mathbb{E}(X_\infty|\mathcal{F}_t))_{t\geq 0}$ and

$$\mathbb{E}(|\mathbb{E}(X_\infty|\mathcal{F}_t)|^p) \leq \mathbb{E}\mathbb{E}(|X_\infty|^p|\mathcal{F}_t) = \mathbb{E}|X_\infty|^p < \infty,$$

which implies $\Rightarrow I(X_\infty) \in \mathcal{H}_p$. □

In stochastic analysis the Hilbert-space $\mathcal{H}_2$ is of particular importance. We define

$$\mathcal{H}_{2,c} := \{X \in \mathcal{H}_2 : X \text{ has continuous paths}\}.$$

Then $\mathcal{H}_{2,c}$ is a closed subspace of $\mathcal{H}_2$.

Proposition 3.1.29. *The space $\mathcal{H}_{2,c}$ is a closed subspace of $\mathcal{H}_2$.*

Proof. Clearly $\mathcal{H}_{2,c}$ is a subspace of $\mathcal{H}_2$. To prove closedness we consider a sequence $X^{(n)} \in \mathcal{H}_{2,c}$ that converges to $X \in \mathcal{H}_2$ and have to show that X has continuous paths. Due to Doob's inequality

$$\mathbb{E}(\sup_{t\geq 0}|X_t^{(n)} - X_t|)^2 \leq 4\sup_{t\geq 0}\mathbb{E}|X_t^{(n)} - X_t|^2 \overset{n\to\infty}{\longrightarrow} 0$$

and therefore $\sup_{t\geq 0}|X_t^{(n)} - X_t|^2 \longrightarrow 0$ in L_1. Thus there exists a subsequence (n_k) such that

$$\sup_{t\geq 0}|X_t^{(n_k)} - X_t|^2 \longrightarrow 0 \qquad \mathbb{P}\text{-almost sure.}$$

Hence also

$$\sup_{t\geq 0}|X_t^{(n_k)} - X_t| \longrightarrow 0 \qquad \mathbb{P}\text{-almost sure.}$$

Due to the continuity of $X^{(n_k)}$ the process X as uniform limit has continuous paths. □

3.2 Stochastic Integration

In this section the mathematical techniques will be developed that are needed to explain trading in a continuous time market model. It turns out that the so far known analytical tools of integration are not sufficient since the processes of interest have no paths of finite variation.

3.2.1 Motivation

Let S denote the price process of a risky asset, a stock for example. A trader takes at each time point t a position $H(t)$ in the stock. If $H(t)$ is positive he has a long position of $H(t)$ stocks. If negative he is short with $|H(t)|$ stocks. The process $(H(t))_{0\leq t\leq T}$ defines his trading strategy during the trading time $[0,T]$. The gain of his strategy from 0 to T can be seen as the integral

$$\int_0^T H(t)dS(t).$$

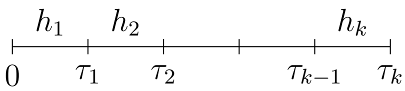
The following question arises.

- How can we mathematically define such an integral?
- Which processes S can be used as integrators?
- Which processes H can be used as integrands?

In a first step we take the standpoint of a real trader. Of course he can't trade continuously in time. But he can choose dependent on the evolution of information a finite number of stopping-times

$$0=\tau_0<\tau_1<\tau_2<\cdots<\tau_N\leq T$$

such that he may change his position at these finite trading points. Let's say that he takes $h(1)$ stocks in the first trading period $(0,\tau_1]$, then $h(2)$ in the period $(\tau_1,\tau_2]$ etc.. This means he has to choose a process $(h(k))_{\{k=1,\cdots N\}}$ that indicates his stock position in the trading periods. Since he cannot look in the future we have to assume that $h(k)$ is $\mathcal{F}_{\tau_{k-1}}$ measurable. Such a strategy we will call performable strategy. Note that N can be random and formally we can achieve this by taking an infinite sequence of stopping times such that a.s. only a finite number terminates before T.



Definition 3.2.1. *Let $(\mathcal{F}_t)_{0\leq t\leq T}$ be a filtration. Then a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}_0}$ together with a stochastic process $(h(n))_{n\in\mathbb{N}}$ is called a performable strategy if the following holds.*

(i) $0=\tau_0<\tau_1<\tau_2\cdots$

(ii) Almost sure there are only a finite number of stopping times that stop before T.

(iii) $h(n)$ is $\mathcal{F}_{\tau_{n-1}}$ measurable.

Note that

$$N = \sup\{n : \tau_n < T\}$$

is random but finite and $N+1$ can be seen as number of trading periods. In each trading period $(\tau_{n-1}, \tau_n]$ the trader receives a gain $h(n)(S(\tau_n) - S(\tau_{n-1}))$. Thus his total gain which can be seen as stochastic integral is

$$\sum_{n=1}^{N} h(n)(S(\tau_n) - S(\tau_{n-1})) + h(N+1)(S(T) - S(\tau_N)) = \int_0^T H(t)dS(t)$$

with H defined by

$$H(t) = \sum_{n=1}^{N} h(n)\mathbb{1}_{(\tau_{n-1},\tau_n]}(t) + h(N+1)\mathbb{1}_{(\tau_N,T]}(t).$$

One can imagine that the set of performable strategies is very rich and that for a thoroughly mathematical treatment also limits of these strategies have to be taken into account. Hence, it has to be clarified from a mathematical standpoint when such limiting procedures could be drawn. This is the objective of stochastic integration.
A path-wise approach is justified if the process S has paths of bounded variation. Then all progressively-measurable processes H with paths that are Lebesgue-Stieltjes integrable can be integrated. More precise:

Definition 3.2.2. *Let $\mathcal{B}_{[0,t]}$ be the Borel-σ-field on $[0,t]$ for all $t \geq 0$. Let furthermore $(\mathcal{F}_t)_{t\geq 0}$ be a Filtration. A stochastic process*

$$X : [0,\infty] \times \Omega \longrightarrow \mathbb{R}$$

is called progressively-measurable, if for each $t > 0$

$$X : [0,t] \times \Omega \longrightarrow \mathbb{R}$$

is measurable w.r.t. $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$. This means, that for all $B \in \mathcal{B}$

$$\{(s,\omega) \in [0,t] \times \Omega : X_s(\omega) \in B\} \in \mathcal{B}_{[0,t]} \otimes \mathcal{F}_t.$$

For processes in continuous time there are many technical difficulties concerning questions of measurability. The progressively-measurable assumption solves those problems many times.

Remark 3.2.3.
1. *Each process with right continuous paths is progressively measurable.*
2. *If X is progressively-measurable and τ a stopping time, then $X(\tau)\mathbb{1}_{\{\tau<\infty\}}$ is measurable w.r.t. $\mathcal{F}_\tau$.*

3. *Let $\mathcal{F}_0$ contain all negligible sets and let X be progressively-measurable with*

$$\int_0^t |X_s(\omega)|ds < \infty \quad \mathbb{P}- \quad a.s. \quad \text{for all} \quad t \geq 0.$$

Then an adapted process Y with $\mathbb{P}$-a.s. paths is defined by

$$Y(t) = \int_0^t X(s)ds \quad \text{for all} \quad t \geq 0.$$

3.2.2 The Doléans-measure

Processes S of interest in finance are the Wiener-process, geometric Wiener-process and in general semi-martingales. These processes have no paths of finite variation. Therefore a new approach has to be developed such that a stochastic integral can be reasonably defined. We have seen that performable strategies are in some sense previsible since at a trading point we have to hold our position for a short time into the future. This concept will be transferred to continuous time.

Definition 3.2.4. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration that satisfies the usual conditions.*

1. *A previsible rectangle R is a set of the form*

$$R = \mathbb{1}_{(s,t]\times F_s} \quad \text{with} \quad 0 \leq s < t, F_s \in \mathcal{F}_s.$$

   We denote by $\mathcal{R}$ the set of all previsible rectangles.

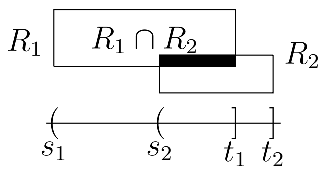
2. *The σ-field $\mathcal{P}$ of previsible sets on $(0,\infty)\times\Omega$ is defined by*

$$\mathcal{P} = \sigma(\mathcal{R}).$$

3. *A stochastic process X is called previsible if it is measurable w.r.t. $\mathcal{P}$.*

There are some parallels between the definition of the previsible σ-field and the definition of the Borel sets of $(0,\infty)$. And also nearly in the same way as the Lebesgue-measure is defined the Doléans-measure can be constructed.

Note that $\mathcal{R}$ is a semi-ring:



(i) $\emptyset \in \mathcal{R}$

(ii) $R_1, R_2 \in \mathcal{R} \Rightarrow R_1 \cap R_2 \in \mathbb{R}$ ($\cap$-stable)

(iii) To $R_1, R_2 \in \mathcal{R}$ there exists disjoint sets $H_1, ..., H_m \in \mathcal{R}$ such that

$$R_1 \backslash R_2 = \bigcup_{i=1}^m H_i$$

The semi-ring $\mathcal{R}$ can be extended to a field of sets $\mathcal{A}$ by adding all finite unions of elements of $\mathcal{R}$. Thus, we define

$$\mathcal{A} = \{A \subset (0,\infty) \times \Omega : \text{there exists } \; R_1, \cdots, R_n \text{with } \; A = \bigcup_{i=1}^{n} R_i\}.$$

This field of sets satisfies the following properties:

(i) $\emptyset \in \mathcal{A}$,

(ii) $A, B \in \mathcal{A} \Rightarrow B \setminus A \in \mathcal{A}$,

(iii) $A, B \in \mathcal{A} \to A \cup B \in \mathcal{A}$.

Note, that each $A \in \mathcal{A}$ is a union of disjoint sets of $\mathcal{R}$, i.e. there exists disjoint sets $R_1, \cdots, R_n$ such that $A = \bigcup\limits_{i=1}^{n} R_i$.
The most important examples of previsible processes are listed below.

1. For each $\mathcal{F}_s$ measurable random variable Y the process

$$H = Y \mathbb{1}_{(s,t]}$$

is previsible for all $0 \leq s < t$.

2. Let $((s_n, t_n])_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint intervals and let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that Y_n is $\mathcal{F}_{s_n}$-measurable for all n. Then

$$H = \sum_{n=1}^{\infty} Y_n \mathbb{1}_{(s_n,t_n]}$$

is a previsible process.

3. Each adapted process with left-continuous paths is previsible.

4. Let (τ_n) be an increasing sequence of stopping times and let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that Y_n is $\mathcal{F}_{\tau_n}$-measurable for all n. Then

$$H = \sum_{n=1}^{\infty} Y_n \mathbb{1}_{(\tau_n,\tau_{n+1}]}$$

is a previsible process.

Next we will give some alternative definitions of the previsible σ-field. Note, that for stopping times $\sigma \leq \tau$ the stochastic interval $(\sigma, \tau]$ is defined by

$$(\sigma, \tau] = \{(t, \omega) : \sigma(\omega) < t \leq \tau(\omega)\}.$$

Proposition 3.2.5. *The following statements are true.*

1. *The previsible σ-field is the smallest σ-field that makes all adapted, left continuous processes measurable.*

2. *The previsible σ-field is generated by all stochastic intervals $(\sigma, \tau]$.*

Next we want to construct the Doléans-measure on the previsible σ-field. Let M be some L_2 martingale with right continuous paths, i.e. $\mathbb{E}M(t)^2 < \infty$ for all $t \geq 0$. Then we may define an additive set function μ on $\mathcal{R}$ by

$$\mu_M((s,t] \times \mathcal{F}_s) = \mathbb{E}\mathbb{1}_{F_s}(M(t) - M(s))^2 = \mathbb{E}\mathbb{1}_{F_s}(M(t)^2 - M(s)^2).$$

This set function $\mu_M : \mathcal{R} \longrightarrow [0, \infty)$ has the following properties

(i) $\mu_M(\emptyset) = 0$,

(ii) If $R_1, ..., R_n \in \mathbb{R}$ are pairwise disjoint such that $\bigcup\limits_{i=1}^{n} R_i \in \mathcal{R}$, then

$$\mu_M\left(\bigcup_{i=1}^{n} R_i\right) = \sum_{i=1}^{n} \mu_M(R_i).$$

Such an additive set function μ_M can always be extended to an additive set function on $\mathcal{A}$ by

$$\begin{aligned} \mu_M : \quad & \mathcal{A} \longrightarrow [0, \infty) \\ & A \mapsto \quad \sum_{i=1}^{n} \mu_M(R_i) \end{aligned}$$

with $A = \bigcup\limits_{i=1}^{n} R_i$, $R_i \in \mathcal{R}$
In the last step one can show that due to the martingale property the set function μ_M is a pre-measure on $\mathcal{A}$, hence σ-additive.
But then the extension theorem of Carathéodory applies and there exists a unique extension of μ_M to a measure on the previsible σ-field which is generated from $\mathcal{A}$.

Definition 3.2.6. *The Doléans-measure of an L_2 martingale M is defined by this unique extension and will be denoted by μ_M.*

As example we will determine the Doléans-measure of a Wiener-process.

Proposition 3.2.7. *The Doléan-measure μ_W of a Wiener-process W is given by*

$$\mu_W = \lambda \times \mathbb{P}$$

with λ denoting the Lebesgue-measure.

Proof. It suffices to show the assertion for $R = (s,t] \times \mathcal{F}_s \in \mathcal{R}$ since $\mathbb{R}$ generates the previsible σ-field and is closed w.r.t. intersection. Since the Wiener-process has independent increments it follows

$$\mu_W((s,t] \times \mathcal{F}_s) = \mathbb{E}\mathbb{1}_{F_s}(W(t) - W(s)^2 = \mathbb{P}(F_s)(t-s).$$

$\square$

3.2.3 The Stochastic Integral

We now introduce the stochastic integral stepwise.
For $H = \mathbb{1}_R$ with $R = (s,t] \times F \in \mathcal{R}$ we set

$$I(H) = \mathbb{1}_F(M(t) - M(s)).$$

This can be linearly extended to the so called space of elementary processes $\mathcal{E}$.

Definition 3.2.8. *The vector-space $\mathcal{E}$ is defined as the span of the indicator function of sets from $\mathcal{R}$. This means that each $H \in \mathcal{E}$ has a representation of the form*

$$H = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{R_i}$$

with $\alpha_1 \cdots \alpha_n \in \mathbb{R}$ and $R_1 \times R_n \in \mathcal{R}$. Such an H is called elementary process.

For $H = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{R_i} \in \mathcal{E}$ the stochastic integral is defined by

$$I(H) = \sum_{i=1}^{n} \alpha_i I(H_i).$$

The space of elementary processes $\mathcal{E}$ is a subspace of $L_2(\mu)$ and the so far defined integral I is a norm-preserving map between $\mathcal{E}$ and $L_2(\mathbb{P})$.

Theorem 3.2.9. *The mapping*

$$I : \mathcal{E} \longrightarrow L_2(\mathbb{P})$$

is norm-preserving, i.e.

$$||H||_{L_2(\mu_M)} = ||I(H)||_{L_2(\mathbb{P})}$$

respectively

$$\int H^2 d\mu_M = \mathbb{E}I(H)^2$$

for all $H \in \mathcal{E}$.

Proof. For $H \in \mathcal{E}$ there exists a representation of the form

$$H = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{R_i}$$

with $\alpha_1, .., \alpha_n \in \mathbb{R}$ and pairwise disjoint sets $R_1, ..., R_n \in \mathcal{R}$. These have the form $R_i = (s_i, t_i] \times F_{s_i}$ for $i = 1, ..., n$.
Hence

$$\begin{aligned} I(H)^2 &= \left(\sum_{i=1}^{n} \alpha_i \mathbb{1}_{F_{s_i}} (M_{t_i} - M_{s_i}) \right)^2 \\ &= \sum_{i=1}^{n} \alpha_i^2 \mathbb{1}_{F_{s_i}} (M_{t_i} - M_{s_i})^2 + \sum_{i \neq k} \alpha_i \alpha_k \mathbb{1}_{F_{s_i}} \mathbb{1}_{F_{s_k}} (M_{t_i} - M_{s_i})(M_{t_k} - M_{s_k}) \end{aligned}$$

The mixed terms vanish by taking expectations. Since

$$R_i \cap R_k = \emptyset$$

it follows

$$F_{s_i} \cap F_{s_k} = \emptyset \text{ or } (s_i, t_i] \cap (s_k, t_k] = \emptyset$$

If $F_{s_i} \cap F_{s_k} = \emptyset$ then $\mathbb{1}_{F_{s_i}} \mathbb{1}_{F_{s_k}} = \mathbb{1}_{F_{s_i} \cap F_{s_k}} = 0$.
If $(s_i, t_i] \cap (s_k, t_k] = \emptyset$ we may assume w.l.o.g. $t_i \leq s_k$. Then

$$\begin{aligned} \mathbb{E} \mathbb{1}_{F_{s_i}} \mathbb{1}_{F_{s_k}} (M_{t_i} - M_{s_i})(M_{t_k} - M_{s_k}) &= \mathbb{E}\mathbb{E} \left[\mathbb{1}_{F_{s_i}} \mathbb{1}_{F_{s_k}} (M_{t_i} - M_{s_i})(M_{t_k} - M_{s_k}) \middle| \mathcal{F}_{s_k} \right] \\ &= \mathbb{E} \mathbb{1}_{F_{s_i}} \mathbb{1}_{F_{s_k}} (M_{t_i} - M_{s_i}) \underbrace{\mathbb{E} \left[(M_{t_k} - M_{s_k}) \middle| \mathcal{F}_{s_k} \right]}_{=0, \text{ since } M \text{ martingale}} \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} ||I(H)||_{L_2(\mathbb{P})} &= \mathbb{E} I(H)^2 \\ &= \mathbb{E} \sum_{i=1}^{n} \alpha_i^2 \mathbb{1}_{F_{s_i}} (M_{t_i} - M_{s_i})^2 \\ &= \sum_{i=1}^{n} \alpha_i^2 \mu_M (\mathbb{1}_{(s_i, t_i] \times F_{s_i}}) \\ &= \sum_{i=1}^{n} \alpha_i^2 \mu_M (R_i) \\ &= ||H||_{L_2(\mu_M)} \end{aligned}$$

□

Since $\mathcal{E}$ is a dense subspace in $L_2(\mu_M)$ we may extend the norm-preserving linear mapping I on $L_2(\mu_M)$.

Theorem 3.2.10. *There exists a unique linear extension of I on $L_2(\mu_M)$ that is norm-preserving, i.e.*

$$||I(H)||_{L_2(\mathbb{P})} = ||H||_{L_2(\mu_M)}$$

for all $H \in L_2(\mu_M)$.

Proof. For each $H \in L_2(\mu_M)$ there exists a sequence $(H^{(n)})_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that

$$||H^{(n)} - H||_{L_2(\mu_M)} \longrightarrow 0.$$

Then $(H^{(n)})_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\overline{\mathcal{E}}$. Since I is norm-preserving, $(I(H^{(n)})_{n\in\mathbb{N}}$ is a Cauchy-sequence in $L_2(\mathbb{P})$ due to

$$||I(H^{(n)}) - I(H^{(m)})||_{L_2(\mathbb{P})} = ||I(H^{(n)} - H^{(m)})||_{L_2(\mathbb{P})} = ||H^{(n)} - H^{(m)}||_{L_2(\mu_M)} \overset{n,m\to\infty}{\longrightarrow} 0.$$

Since $L_2(\mathbb{P})$ is complete, there exists some unique $U \in L_2(\mathbb{P})$ such that

$$||U - I(H^{(n)})||_{L_2(\mathbb{P})} \longrightarrow 0,$$

hence $I(H)$ is defined by U.
That I is norm-preserving, follows from the fact that $||H^{(n)}||_{L_2(\mu_M)} \longrightarrow ||H||_{L_2(\mu_M)}$. Then

$$||I(H^{(n)})||_{L_2(\mathbb{P})} \longrightarrow ||I(H)||_{L_2(\mathbb{P})}$$

and due to $||I(H^{(n)})||_{L_2(\mathbb{P})} = ||H^{(n)}||_{L_2(\mu_M)}$ it follows

$$||H||_{L_2(\mu_M)} = ||I(H)||_{L_2(\mathbb{P})}.$$

□

As notation for the integral the integral sign is common and can be used. We define for $H \in L_2(\mu_M)$

$$\int H dM = I(H).$$

The advantage of this here presented approach relies in the fact that the well-known properties of integration from analysis can be taken over and therefore this procedure is quite simple. But this leads to the disadvantage that so far explicit formulas for a stochastic integral are only given for elementary processes H. It is rather tedious to compute the integral for other processes of interest, for example performable strategies. In general one has to compute the integral for an $H \in L_2(\mu_M)$ by finding a suitable approximating sequence $H^{(n)}$ and calculating the limit of $I(H^{(n)})$ in $L_2(\mathbb{P})$.

Proposition 3.2.11. *The following formulas hold true:*

(i) Let σ, τ be bounded stopping times with $\sigma \leq \tau$ and Y be a bounded $\mathcal{F}_\sigma$-measurable random variable. Then

$$\int Y \mathbb{1}_{(\sigma,\tau]} dM = Y(M(\tau) - M(\sigma)).$$

(ii) If $M \in \mathcal{H}_2$ then

$$\int Y \mathbb{1}_{(\sigma,\tau]} dM = Y(M(\tau) - M(\sigma))$$

for all stopping times $\sigma \leq \tau$ and Y square-integrable $\mathcal{F}_\sigma$-measurable.

(iii) Let M be some L_2-martingale with $M(0) = 0$, then for each bounded stopping time τ

$$\mu_M((0, \tau]) = \mathbb{E}M(\tau)^2.$$

(iv) Let M be some L_2-martingale with $M(0) = 0$, then for each stopping time τ with $\mu_M((0, \tau]) < \infty$ the stopped process M^τ is an $\mathcal{H}_2$-martingale and it holds

$$\mu_M((0, \tau]) = \mathbb{E}M(\tau)^2.$$

(v) For the Wiener-process W, bounded stopping times $\sigma \leq \tau$ and square integrable $\mathcal{F}_\sigma$-measurable random variables Y it holds

$$\int Y \mathbb{1}_{(\sigma,\tau]} dW = Y(W(\tau) - W(\sigma)).$$

(vi) For the Wiener-process W it holds

$$\int \mathbb{1}_{(0,T]} W dW = \frac{1}{2}(W(T)^2 - T).$$

Proof. A proof of these assertion is not really exciting. The last statement is of interest and can be shown in the following way. Note, that $W\mathbb{1}_{(0,T]}$ is previsible as left continuous process and is contained in $L_2(\mu_M)$ due to

$$\begin{aligned}
\int H^2 d\mu_W &= \int \mathbb{1}_{(0,T]} W^2 d(\lambda \otimes \mathbb{P}) \\
&= \int\limits_{[0,\infty)\times\Omega} \mathbb{1}_{(0,T]}(t) W_t^2(\omega)(\lambda \otimes \mathbb{P})(dt, d\omega) \\
&\overset{\text{Fubini}}{=} \int\limits_{[0,\infty)} \mathbb{1}_{(0,T]}(t) \int\limits_{\Omega} W_t^2(\omega)\mathbb{P}(d\omega)\lambda(dt) \\
&= \int_0^T \mathbb{E}W_t^2 dt \\
&= \int_0^T t dt \\
&= \frac{1}{2}T^2 < \infty.
\end{aligned}$$

Approximate H in $L_2(\mu_W)$ by

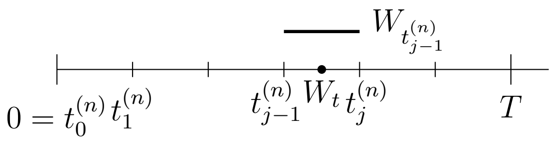
$$H^{(n)} := \sum_{j=1}^{l(n)} W_{t_{j-1}^{(n)}} \mathbb{1}_{\left(t_{j-1}^{(n)}, t_j^{(n)} \right]}$$

with

$$0 = t_0^{(n)} < t_1^{(n)} < ... < t_{l(n)}^{(n)} = T$$

and

$$\max_j t_j^{(n)} - t_{j-1}^{(n)}$$



Then

$$\begin{aligned}
\int (H^{(n)} - H)^2 d\mu_W &= \mathbb{E} \int_0^T (H_t^{(n)} - H_t)^2 dt \\
&= \mathbb{E} \sum_{j=1}^{l(n)} \int\limits_{t_{j-1}^{(n)}}^{t_j^{(n)}} (W_{t_{j-1}^{(n)}} - W_t)^2 dt \\
&= \sum_{j=1}^{l(n)} \int\limits_{t_{j-1}^{(n)}}^{t_j^{(n)}} \mathbb{E}(W_{t_{j-1}^{(n)}} - W_t)^2 dt \\
&= \sum_{j=1}^{l(n)} \int\limits_{t_{j-1}^{(n)}}^{t_j^{(n)}} t_{j-1}^{(n)} - t dt \\
&\leq (\max_j t_j^{(n)} - t_{j-1}^{(n)}) T \overset{n \to \infty}{\longrightarrow} 0
\end{aligned}$$

Hence the continuity of the integral yields

$$\int H^{(n)} dW \longrightarrow \int H dW \qquad \text{in } L_2(\mathbb{P}).$$

Furthermore

$$\int H^n dW = \sum_{j=1}^{l(n)} W_{t_{j-1}^{(n)}} (W_{t_j^{(n)}} - W_{t_{j-1}^{(n)}})$$

and therefore

$$
\begin{aligned}
W_T^2 = \sum_{j=1}^{l(n)} (W^2_{t_j^{(n)}} - W^2_{t_{j-1}^{(n)}}) &= \sum_{j=1}^{l(n)} \Big[(W_{t_j^{(n)}} - W_{t_{j-1}^{(n)}})^2 + 2W_{t_{j-1}^{(n)}}(W_{t_j^{(n)}} - W_{t_{j-1}^{(n)}}) \Big] \\
&= \underbrace{\sum_{j=1}^{l(n)} (W_{t_j^{(n)}} - W_{t_{j-1}^{(n)}})^2}_{\longrightarrow T \text{ in } L_2(\mathbb{P})} + 2 \underbrace{\sum_{j=1}^{l(n)} W_{t_{j-1}^{(n)}}(W_{t_j^{(n)}} - W_{t_{j-1}^{(n)}})}_{\longrightarrow \int H dW \text{ in } L_2(\mathbb{P})}
\end{aligned}
$$

Thus

$$
W_T^2 = T + 2 \int H dW = T + 2 \int \mathbb{1}_{(0,T]} W dW \Leftrightarrow \int \mathbb{1}_{(0,T]} W dW = \frac{1}{2}(W_T^2 - T)
$$

□

3.2.4 The Integral-process

So far the stochastic integral as $L_2(\mathbb{P})$ random variable is constructed and can be seen as the gain a trader would obtain by trading the martingale M w.r.t. a strategy H. Now we clarify how the evolution of its gain-process can be described. We exploit the fact that $L_2(\mathbb{P})$ is isometric isomorph to $\mathcal{H}_2$, see 3.1.13. Hence the integral $I(H)$ can be transformed in a uniquely manner to an $\mathcal{H}_2$-martingale by taking conditional expectation w.r.t. $\mathcal{F}_t$ for all $t \geq 0$. More precisely we use the isometry

$$
\begin{aligned}
J: \quad L_2(\Omega, \mathcal{F}_\infty, \mathbb{P}) &\longrightarrow \quad \mathcal{H}_2 \\
X_\infty &\mapsto (\mathbb{E}(X_\infty | \mathcal{F}_t))_{t \geq 0}.
\end{aligned}
$$

Note that

$$
||X_\infty||_{L_2(\mathbb{P})} = ||J(X)||_{\mathcal{H}_2}
$$

Definition 3.2.12. *For an L_2-martingale M with cadlag-paths and $H \in L_2(\mu_M)$ we define the stochastic integral-process $H \cdot M$ by the mapping*

$$
\begin{aligned}
H \cdot M: \quad L_2(\mu_M) &\longrightarrow L_2(\mathbb{P}) \longrightarrow \mathcal{H}_2 \\
H &\mapsto J(I(H))
\end{aligned}
$$

Hence for $t \geq 0$

$$
(H \cdot M)_t = \mathbb{E}(I(H) | \mathcal{F}_t)
$$

By construction we obtain the martingale property of the integral-process. A disadvantage relies in the fact that basic properties of the integral-process have to be shown. By using the definition explicitly one can calculate the integral-process for elementary-processes.

Theorem 3.2.13. *Let $H \in \mathcal{E}$ be of the form*

$$H = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{R_j}$$

with $\alpha_0, ..., \alpha_n \in \mathbb{R}$ and $R_j = (s_j, t_j] \times F_{s_j} \in \mathcal{R}$.
Then

$$H \cdot M = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{F_{s_j}} (M^{t_j} - M^{s_j}).$$

This means that $H \cdot M$ is indistinguishable from

$$\left(\sum_{j=1}^{n} \alpha_j \mathbb{1}_{F_{s_j}} (M_{t_j \wedge t} - M_{s_j \wedge t}) \right)_{t \geq 0} .$$

In particular

$$(H \cdot M)_0 = 0 \quad \mathbb{P}- \quad a.s. \quad .$$

This means that the integral-process starts from zero.

Proof. It suffices to prove the claim for $H = \mathbb{1}_{R_j}$ with $R_j = (s_j, t_j] \times F_{s_j} \in \mathcal{R}$. Then

$$I(\mathbb{1}_{R_j}) = \mathbb{1}_{F_{s_j}} (M_{t_j} - M_{s_j})$$

and an elementary calculation yields the assertion. □

By exploiting the explicit form for $H \in \mathcal{E}$ and the continuity of the integral operator further properties can easily be deduced.

Theorem 3.2.14. *Let M be an L_2-martingale with cadlag paths and $H \in L_2(\mu_M)$. Then*

(i) $(H \cdot M)_0 = 0$ $\mathbb{P}$*-a.s.*

(ii) $\mathbb{E}(H \cdot M)_t = 0$ *for all* $t \geq 0$,

(iii) $\mathbb{E}(H \cdot M)_\tau = 0$ *for all stopping times* τ,

(iv) If M is continuous, then $H \cdot M$ has continuous paths and $H \cdot M \in \mathcal{H}_{2,c}$.

Proof. One can easily show that the above properties hold true for $\mathcal{H} \in \mathcal{E}$. By using continuity they carry over to $H \in \bar{\mathcal{E}} = L_2(\mu_M)$. □

The integral operator is a mapping of two variables, the martingale M and the previsible process H. In the following we will give further properties.

Proposition 3.2.15. *Let M, N be L_2-martingales with cadlag paths. Then* *(i)* $\mu_{M+N} \le 2(\mu_M + \mu_N)$

(ii) $L_2(\mu_M + \mu_N) = L_2(\mu_M) \cap L_2(\mu_N) \subset L_2(\mu_{M+N})$

(iii) $H \cdot (M + N) = (H \cdot M) + (H \cdot N)$ *for all* $H \in L_2(\mu_M) \cap L_2(\mu_N)$

We would like to extend the set of integrable processes H and the set of integrators M. The main technique which has to be applied is the localisation by stopping and cutting. These operators will be defined next and their properties investigated.

Definition 3.2.16. *For each stochastic-process* $(X_t)_{t\ge 0}$ *and each stopping time* τ *we define the stopped process* X^τ *by*

$$X^\tau := \begin{cases} X_t & t < \tau \\ X_\tau & t \ge \tau \end{cases}$$

Short we write $X^\tau(t) = X(\tau \wedge t)$ *for all* $t \ge 0$*. The cutted process* $X\mathbb{1}_{(0,\tau]}$ *is defined by*

$$X\mathbb{1}_{(0,\tau]} := \begin{cases} X_t & 0 < t \le \tau \\ 0 & t > \tau \end{cases}$$

The following properties are useful for a later localisation.

Theorem 3.2.17. *Let* τ *be a stopping time. Then*

(i) *If* $X \in \mathfrak{M}$*, then* $X^\tau \in \mathfrak{M}$*.*

(ii) *If* $X \in \mathcal{H}_2$*, then* $X^\tau \in \mathcal{H}_2$*.*

(iii) *If* X *has continuous paths, then* X^τ *either.*

(iv) *If* M *is an* L_2*-martingale with cadlag paths and* $H \in L_2(\mu_M)$*, then*

$$H\mathbb{1}_{(0,\tau]} \in L_2(\mu_M) \cap L_2(\mu_{M^\tau})$$

and

$$(H \cdot M)^\tau = H\mathbb{1}_{(0,\tau]} \cdot M = H\mathbb{1}_{(0,\tau]} \cdot M^\tau = H \cdot M^\tau$$

The last property is of main importance since it justifies the notation

$$\int_0^t H(s)dM(s) = \int H\mathbb{1}_{(0,t]}dM.$$

This means that the integral-process evaluated at t coincides with H integrated over the interval $(0, t]$. More precisely we have the following corollary

Corollary 3.2.18. *Let M be an L_2-martingale with cadlag-paths and $H \in L_2(\mu_M)$. Then for each stopping time τ*

$$(H \cdot M)_\tau = (H\mathbb{1}_{(0,\tau]} \cdot M)_\infty = I(H\mathbb{1}_{(0,\tau]}) = \int H\mathbb{1}_{(0,\tau]} dM \qquad \mathbb{P}\text{-almost sure}$$

Proof. This follows from

$$\begin{aligned}(H \cdot M)_\tau &= (H \cdot M)^\tau_\infty \\ &\overset{\text{Theorem 3.2.16}}{=} (H\mathbb{1}_{(0,\tau]} \cdot M)_\infty \\ &= I(H\mathbb{1}_{(0,\tau]}) \\ &= \int H\mathbb{1}_{(0,\tau]} dM\end{aligned}$$

□

Therefore it is shown

$$\mathbb{E}(I(H)|\mathcal{F}_\tau) = (H \cdot M)_\tau = I(H\mathbb{1}_{(0,\tau]})$$

in particular

$$(H \cdot M)_t = \int H\mathbb{1}_{(0,t]} dM = \int_0^t H_s dM_s$$

for all $t \geq 0$.
Next we will list some further properties of the integral-process

Remark 3.2.19. *Let M be an L_2- martingale and τ an arbitrary stopping time. Then*

(i) $\mu_{M^\tau} \leq \mu_M$

(ii) $L_2(\mu_M) \subset L_2(\mu_{M^\tau})$

(iii) If H is previsible, then H^τ either.

(iv) If $H \in L_2(\mu_M)$, then $(H \cdot M)^\tau = H^\tau \cdot M^\tau$

(v) If H is bounded and previsible and $M \in \mathcal{H}_2$, then

$$H^\tau \cdot M = H\mathbb{1}_{(0,\tau]} \cdot M + H_\tau(M - M^\tau)$$

A further useful formula is the following

Proposition 3.2.20. *Let M be an L_2-martingale with cadlag paths and $H \in L_2(\mu_M)$. Let τ be a stopping time and Y a bounded $\mathcal{F}_\tau$-measurable random variable. Then*

$$\int Y\mathbb{1}_{(\tau,\infty)} H dM = Y \int \mathbb{1}_{(\tau,\infty)} H dM$$

respectively

$$(Y\mathbb{1}_{(\tau,\infty)}H)\cdot M) = Y((\mathbb{1}_{(\tau,\infty)}H)\cdot M)$$

respectively

$$((Y\mathbb{1}_{(\tau,\infty)}H)\cdot M)_t = Y((\mathbb{1}_{(\tau,\infty)}H)\cdot M)_t \quad \textit{for all } t \geq 0$$

respectively

$$\int_0^t Y\mathbb{1}_{(\tau,\infty)}(s)H_s dM_s = Y\int_0^t \mathbb{1}_{(\tau,\infty)}(s)H_s dM_s \quad \textit{for all} \;\; t \geq 0$$

Very important is the so called associativity of the integral operator.

Theorem 3.2.21. *Let M be an L_2-martingale with cadlag-paths and H, K previsible processes such that $K \in L_2(\mu_M)$ and $H \in L_2(\mu_{K\cdot M})$.*
Then

$$HK \in L_2(\mu_M)$$

and

$$(HK)\cdot M = H\cdot(K\cdot M)$$

Proof. The main observation is that the Doléans measure of the martingale $K\cdot M$ is absolutely continuous w.r.t. μ_M with density K^2, i.e.

$$\mu_{K\cdot M}(A) = \int_A K^2 d\mu_M \quad \text{for all} \;\; A \in \mathcal{P}.$$

Therefore

$$H \in L_2(\mu_{K\cdot M}) \Longleftrightarrow KH \in L_2(\mu_M).$$

To prove the associativity one has to show this directly for $H \in \mathcal{E}$ and carry this over by continuity to $H \in L_2(\mu_{K\cdot M})$.

□

3.3 Quadratic Variation Process

The objectives of this section are the following

- specification of the path fluctuation of a continuous martingale,
- alternative specifikation of the Doléans-measure,
- Doob-Meyer decomposition of the submartingale M^2 in a martingale N and an increasing, previsible process Λ:

$$M_t^2 = M_0^2 + N_t + \Lambda_t$$

3.3.1 Finite Variation

In real analysis the fluctuation of a function is measured by its variation.

Definition 3.3.1. *Let $f : [0,T] \longrightarrow \mathbb{R}$ be a Borel-measurable function and π some decomposition*

$$\pi : 0 = t_0 < ... < t_n = T$$

of the interval $[0,T]$.
The variation $FV_T(f,\pi)$ of f according to π is defined by

$$FV_T(f,\pi) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

The function f is called of bounded variation on $[0,T]$, if

$$FV_T(f) := \sup_{\substack{\pi \text{ decomposition} \\ \text{of } [0,T]}} FV_T(f,\pi) < \infty$$

Then $FV_T(f)$ gives a measure for the fluctuation of f over $[0,T]$.

Some well known facts are the following

Proposition 3.3.2. *The following statements hold true.*

1. *Each increasing function is of bounded variation.*
2. *The set of bounded variation functions is a vector-space.*
3. *Each Lipschitz-continuous function is of bounded variation.*
4. *Continuous functions need not to be of bounded variation.*
5. *Absolutely-continuous functions functions are of bounded variation.*

If f is continuous and of bounded variation, then we get the variation by taking limits of variation along decompositions whose mesh-size tends to zero. According to a decomposition π

$$\pi : 0 = t_0 < ... < t_n = T$$

of the interval $[0,T]$ we define its mesh-size $|\pi|$ by

$$|\pi| = \max_{i=1,\cdots,n} |t_i - t_{i-1}|.$$

Remark 3.3.3. Let f be a continuous function of bounded variation. Then

$$FV_T(f) = \lim_{|\pi| \to 0} FV_T(f,\pi)$$

Important is the fact that each function of bounded variation can be written as difference of increasing functions.

Theorem 3.3.4. *Let $f : [0,T] \longrightarrow \mathbb{R}$ be a right continuous function. Then it is of bounded variation on $[0,T]$ if and only if there exist increasing functions g_1 and g_2 such that*

$$f = g_1 - g_2$$

Proof. " $\Rightarrow$ " we define for all $t \in [0,T]$

$$\begin{aligned} f^+(t) &:= \frac{1}{2}(FV_t(f) + f(t)) \\ f^-(t) &:= \frac{1}{2}(FV_t(f) - f(t)) \end{aligned}$$

Then

$$\begin{aligned} &f = f^+ - f^- \\ &FV_t(f) = f^+(t) + f^-(t) \qquad \text{for all } t \in [0,T] \end{aligned}$$

and f^+, f^- are increasing functions.

□

Functions of bounded variation can be seen as distribution function of signed measures and this gives a measure theoretic view on functions of bounded variation. To be more precise let f be a right continuous function of bounded variation. Then so f^+ and f^- are right-continuous and the decomposition in f^+ and f^- is uniquely defined by

$$\begin{aligned} &f = f^+ - f^- \\ &FV_t(f) = f^+(t) + f^-(t) \qquad \text{for all } t \in [0,T]. \end{aligned}$$

The increasing functions f^+ und f^- are distribution functions of measures μ^+ resp. μ^- on $[0,T]$:

$$\mu_f^{\pm}((a,b]) := f^{\pm}(b) - f^{\pm}(a) \qquad \text{for all } 0 \leq a < b \leq T$$

By

$$\mu_f(A) := \mu^+(A) - \mu^-(A)$$

for all Borel sets $A \subset [0,T]$ a signed measure μ_f on $[0,T]$ is defined. This is a σ-additive set-function, that can take negative values. We have the follwing relation:

$$\mu_f((a,b]) = f(b) - f(a) \qquad \text{for all } 0 \leq a < b \leq T.$$

The right continuous function

$$t \mapsto FV_t(f)$$

is increasing and defines the so called variation measure $||\mu_f||$ on $[0,T]$ by

$$||\mu_f||((a,b]) := FV_b(f) - FV_a(f) \qquad \text{for all } 0 \leq a < b \leq T$$

It holds:

$$||\mu_f|| = \mu_f^+ + \mu_f^-$$

and μ^+ and μ^- are uniquely determined by

$$\mu_f = \mu_f^+ - \mu_f^-$$
$$||\mu_f|| = \mu_f^+ + \mu_f^-.$$

An integration according to functions of bounded variation can be seen as integration according to signed measures.

Definition 3.3.5. *Let $f : [0,T] \longrightarrow \mathbb{R}$ be a right continuous function of bounded variation with unique decomposition:*

$$\begin{aligned} &f = f^+ - f^- \\ &FV_t(f) = f^+(t) + f^-(t) \qquad \text{for all } t \in [0,T] \end{aligned}$$

A Borel measurable function $g : [0,T] \longrightarrow \mathbb{R}$ is Lebesgue-Stieltjes integrable w.r.t. f, if g is integrable w.r.t. f^+ and f^-.
Then we define

$$\int_0^T gdf := \int_0^T gdf^+ - \int_0^T gdf^-$$

at which

$$\int_0^T gdf^+ := \int_0^T gd\mu_f^+$$

and

$$\int_0^T gdf^- := \int_0^T gd\mu_f^-$$

with

$$\mu_f^{\pm}((a,b]) = f^{\pm}(b) - f^{\pm}(a) \qquad \text{for all } 0 \le a < b \le T$$

To examine integrability one can take the variation measure.

Theorem 3.3.6. *The function g is integrable w.r.t. f if and only if g is integrable w.r.t. the variation measure $||\mu_f||$ and then*

$$\begin{aligned} \left| \int_0^T gdf \right| &\le \int_0^T |g|d||\mu_f|| \\ &= \int_0^T |g(t)|dFV_t(f) \end{aligned}$$

Proof. " $\Rightarrow$ " Let g be integrable w.r.t. f. Then g is integrable w.r.t. f^+ and f^-. Hence

$$\int_0^T |g| df^+ < \infty, \qquad \int_0^T |g| df^- < \infty$$

Then

$$\begin{aligned}
\int_0^T |g| d||\mu_f|| &= \int_0^T |g| df^+ + \int_0^T |g| df^- \qquad (< \infty, \text{ hence } g \text{ is integrable w.r.t. } f) \\
&\geq \left| \int_0^T g df^+ \right| + \left| \int_0^T g df^- \right| \\
&\geq \left| \int_0^T g df^+ - \int_0^T g df^- \right| \\
&= \left| \int_0^T g df \right|
\end{aligned}$$

" $\Leftarrow$ " If $\int_0^T |g| d||\mu_f|| < \infty$ then

$$\int_0^T |g| df^+ + \int_0^T |g| df^- = \int_0^T |g| d||\mu_f|| < \infty$$

$\Rightarrow \int_0^T |g| df^+ < \infty, \qquad \int_0^T |g| df^- < \infty$
$\Rightarrow g$ is integrable w.r.t. f. $\square$

A function $f : [0, \infty) \longrightarrow \mathbb{R}$ is locally of bounded variation, if it is of bounded variation on each bounded interval.

Definition 3.3.7. *A function $f : [0, \infty) \longrightarrow \mathbb{R}$ is called locally of bounded variation, if*

$$FV_T(f) < \infty \qquad \text{for all } T > 0$$

holds.

Remark 3.3.8. Is f locally of bounded variation and right continuous, then their exist unique right continuous increasing functions f^+ and f^- with

$$\begin{aligned}
&f = f^+ - f^- \\
&FV_t(f) = f^+(t) + f^-(t) \qquad \text{for all } t \in [0, \infty).
\end{aligned}$$

The functions f^+ and f^- define measures on $([0,\infty),\mathcal{B})$ by

$$\mu_f^\pm((a,b]) = f^\pm(b) - f^\pm(a) \qquad \text{for all } 0 \le a < b < \infty.$$

Then

$$\mu_f = \mu_f^+ - \mu_f^-$$

denotes the signed measure corresponding to f.

3.3.2 The Quadratic Variation

Martingales with continuous paths have no paths of bounded variation. Their fluctuations are too large. Therefore the quadratic variation instead of the variation will be used to describe the amount of fluctuation.

Definition 3.3.9. *Let $f : [0,T] \longrightarrow \mathbb{R}$ and $\pi : 0 = t_0 < t_1 < ... < t_n = T$.*
The quadratic variation w.r.t. π is defined by

$$V_T^{(2)}(f,\pi) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^2$$

and

$$V_T^{(2)}(f) := \lim_{|\pi|\to 0} V_T^{(2)}(f,\pi)$$

if the limit exists.

Remark 3.3.10. Let $f : [0,T] \longrightarrow \mathbb{R}$ be continuous and of bounded variation on $[0,T]$. Then

$$V_T^{(2)}(f) = 0.$$

The fluctuation is too small. It can't be detected by the quadratic variation.

Proof. The continuous f is uniformly continuous on $[0,T]$. Hence,

$$\begin{aligned} V_T^{(2)}(f,\pi) &= \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^2 \\ &\le \underbrace{\max |f(t_i) - f(t_{i-1})|}_{\overset{|\pi|\to 0}{\underset{f \text{ unif. cont.}}{\longrightarrow}} 0} \underbrace{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|}_{\overset{|\pi|\to 0}{\longrightarrow} FV_T(f)<\infty} \\ &= 0 \end{aligned}$$

□

To introduce the quadratic variation process of a continuous martingale we proceed in the following way.

- First we give an abstract definition by exploiting stochastic integration ,

- Then, we give an extension by localisation,
- At last we show that the defined quadratic variation process coincides with the quadratic variation of paths.

First a continuous bounded martingale M is considered. This means that there exists a constant $C > 0$ such that a.s.

$$\sup_{t,\omega} M_t(\omega) < C.$$

Then

$$M \in L_2(\mu_N)$$

for each $\mathbb{N} \in \mathcal{H}_{2,c}$ due to

$$\int M^2 d\mu_N \le C^2 \mu_N((0,\infty) \times \Omega)) = C^2 \mathbb{E}(N_\infty - N_0)^2 < \infty.$$

In particular

$$M \in L_2(\mu_M).$$

Let us denote by $b\mathfrak{M}_c$ the space of bounded martingales with continuous paths.

Definition 3.3.11. *For $M \in b\mathfrak{M}_c$ with $M_0 = 0$ we define the quadratic variation proess by*

$$\langle M \rangle_t := M_t^2 - 2(M \cdot M)_t \qquad \textit{for all } t \ge 0$$

resp.

$$\langle M \rangle_t = M_t^2 - 2 \int_0^t M_s dM_s \qquad \textit{for all } t \ge 0.$$

We have the following properties

Theorem 3.3.12. *Let $M \in b\mathfrak{M}_c$ with $M_0 = 0$. Then*

(i) $\langle M \rangle_0 = 0$

(ii) $t \mapsto \langle M \rangle_t$ *is adapted with $\mathbb{P}$-a.s. continuous paths.*

(iii) $(M_t^2 - \langle M \rangle_t)_{t \ge 0} \in \mathcal{H}_{2,c}$

(iv) $t \mapsto \langle M \rangle_t$ *is $\mathbb{P}$-a.s. increasing*

Proof. The first three properties follow immediately by definition. The last property cannot be shown easily. It needs a careful approximation by suitable discrete integral processes. We dispense with the details. □

As an application we can give the Doob-Meyer decomposition of M^2 for $M \in b\mathfrak{M}_c$

$$M_t^2 = \underbrace{M_0^2}_{\text{start}} + \underbrace{M_t^2 - \langle M \rangle_t}_{\text{martingale part}} + \underbrace{\langle M \rangle_t}_{\substack{\text{previsible,}\\ \text{increasing}\\ \text{part}}}$$

for all $t \geq 0$.

Definition 3.3.13. *For $M \in b\mathfrak{M}_c$ with $M_0 \neq 0$ we define the quadratic variation process by*

$$\langle M \rangle := \langle M - M_0 \rangle.$$

It holds

$$\begin{aligned}
\langle M \rangle_t = \langle M - M_0 \rangle_t &= (M_t - M_0)^2 - 2 \int_0^t (M_s - M_0) d(M_s - M_0) \\
&= (M_t - M_0)^2 - 2 \int_0^t (M_s - M_0) dM_s - 2 \underbrace{\int_0^t (M_s - M_0) dM_0}_{=0, \text{ due to } M_0 \text{ constant}} \\
&= M_t^2 - 2M_0 M_t + M_0^2 - 2 \int_0^t M_s dM_s + 2 \underbrace{\int_0^t M_0 dM_s}_{\substack{=M_0(M_t - M_0), \text{ due to} \\ M_0 \ \mathcal{F}_0\text{-measurable}}} \\
&= M_t^2 - M_0^2 - 2 \int_0^t M_s dM_s
\end{aligned}$$

and therefore

$$M_t^2 = M_0^2 + 2 \int_0^t M_s dM_s + \langle M \rangle_t \qquad \text{for all } t \geq 0$$

This corresponds to the Doob-Meyer decomposition of the submartingale $(M_t^2)_{t \geq 0}$.
By localisation this definition of $\langle M \rangle$ should be extended to L_2-martingales. A necessary property is the compatibility with stopping.

Theorem 3.3.14. *Let $M \in b\mathfrak{M}_c$ with $M_0 = 0$ $\mathbb{P}$-a.s.. Then for each stopping time τ*

$$\langle M \rangle^\tau = \langle M^\tau \rangle$$

Proof. This follows from the definition of $\langle M \rangle$ and the compatibility of integration with stopping, see 3.2.17.

$$\begin{aligned}\langle M^\tau \rangle &= (M^\tau)^2 - 2(M^\tau \cdot M^\tau)\\ &= (M^2)^\tau - 2(M^\tau \mathbb{1}_{(0,\tau]} \cdot M^\tau)\\ &= (M^2)^\tau - 2(M \mathbb{1}_{(0,\tau]} \cdot M)\\ &= (M^2)^\tau - 2(M \cdot M)^\tau\\ &= (M^2 - 2M \cdot M)^\tau\\ &= \langle M \rangle^\tau\end{aligned}$$

□

The significance relies in the fact that stopping can be interchanged with taking the quadratic variation and this allows to extend its definition.

Theorem 3.3.15. *Let M be a continuous L_2-martingale with $M_0 = 0$ $\mathbb{P}$-almost sure. Then there exists a unique stochastic process $\langle M \rangle$ with the following properties:*

(i) $\langle M \rangle_0 = 0$ *$\mathbb{P}$-almost sure,*

(ii) $(\langle M \rangle_t)_{t \geq 0}$ *is adapted and has $\mathbb{P}$-almost sure continuous and increasing paths,*

(iii) $(M_t^2 - \langle M \rangle_t)_{t \geq 0}$ *is a martingale.*

Definition 3.3.16. *The process $\langle M \rangle$ from Theorem 3.3.15 will be called quadratic variation process of M.*

Proof. We will give the main idea. For each $n \in \mathbb{N}$ a stopping time τ_n is defined by

$$\tau_n = \inf\{t \geq 0 : |M(t)| \geq n\}.$$

Then, the sequence

$$\tau_1 < \tau_2 < \cdots \tau_n$$

is increasing with

$$\sup_n \tau_n = \infty \quad , \quad M^{\tau_n} \in b\mathfrak{M}_c.$$

Thus, for each n the quadratic variation $\langle M^{\tau_n} \rangle$ is well-defined and putting all of these together the quadratic variation of $\langle M \rangle$ can be uniquely defined using the compatibility property 3.3.14. Note that

$$\langle M \rangle \mathbb{1}_{(0,\tau_n]} = \langle M^{\tau_n} \rangle \mathbb{1}_{(0,\tau_n]}.$$

The properties of $\langle M^{\tau_n} \rangle$, $n \in \mathbb{N}$ see 3.3.12 carry over to $\langle M \rangle$.
The uniqueness of the quadratic variation-process follows from the following proposition.

□

Proposition 3.3.17. *Let M be an L_2-martingale with continuous paths, that are locally of bounded variation. Then M is $\mathbb{P}-$ a.s. constant, i.e.*

$$M_t = M_0 \qquad \textit{for all } t \geq 0 \quad \mathbb{P}- \quad a.s.$$

For general L_2-martingales we can define the quadratic variation in the following way.

Definition 3.3.18. *For a continuous L_2-martingale M with $M_0 \neq 0$ we define the quadratic variation process by*

$$\langle M \rangle := \langle M - M_0 \rangle.$$

Then $\langle M \rangle$ is the unique process with the following properties:

(i) $\langle M \rangle_0 = 0$

(ii) $(\langle M \rangle_t)_{t \geq 0}$ is adapted and has $\mathbb{P}$-a.s. increasing and continuous paths.

(iii) $(M_t^2 - M_0^2 - \langle M \rangle_t)_{t \geq 0}$ is a martingale.

Note

$$\begin{aligned}(M_t - M_0)^2 &= M_t^2 - 2M_0M_t + M_0^2 \\ &= M_t^2 - M_0^2 - 2M_0(M_t - M_0)\end{aligned}$$

Hence

$$M_t^2 - M_0^2 - \langle M \rangle_t = \underbrace{(\underbrace{M_t - M_0}_{\text{martingale}})^2 - \langle M \rangle_t}_{\text{martingale}} + \underbrace{2M_0(M_t - M_0)}_{\text{martingale}}$$

is a martingale.
Therefore we obtain the Doob-Meyer decomposition:

$$M_t^2 = M_0^2 + \underbrace{(M_t - M_0)^2 - \langle M \rangle_t + 2M_0(M_t - M_0)}_{\text{martingale}} + \langle M \rangle_t$$

The compatibility property carries over to L_2-martingales.

Theorem 3.3.19. *For a continuous L_2-Martingal M and each stopping time τ*

$$\langle M^\tau \rangle = \langle M \rangle^\tau$$

Proof. We may assume $M_0 = 0$ and verify the defining properties of a quadratic variation process.
The process $\langle M \rangle^\tau$ is adapted, increasing, adapted and starts from zero.
It remains to show that

$$(M^\tau)^2 - \langle M \rangle^\tau$$

is a martingale.
For each bounded stopping time σ it follows

$$\mathbb{E}(M^\tau)_\sigma^2 = \mathbb{E}M_{\tau\wedge\sigma}^2 = \mathbb{E}\langle M\rangle_{\tau\wedge\sigma} = \mathbb{E}\langle M\rangle_\sigma^\tau$$

This implies the martingale-property, see 3.1.17. □

The Doléans-measure can be expressed by the quadratic variation process.

Theorem 3.3.20. *Let M ba a continuous L_2-martingale. Then the Doléans-measure μ_M satisfies*

$$\mu_M(A) = \mathbb{E}\int_0^\infty \mathbb{1}_A(t,\omega)d\langle M\rangle_t(\omega)$$

for each previsible set $A \in \mathcal{P}$. In particular for $H \in L_2(\mu_M)$

$$\int H^2 d\mu_M = \mathbb{E}\int_0^\infty H_t^2(\omega)d\langle M\rangle_t(\omega) \tag{3.1}$$

Proof. It suffices to prove the claim for previsible rectangles $A = (s,t] \times F_s \in \mathcal{R}$.

$$\begin{aligned}
\mu_M(A) &= \mathbb{E}\mathbb{1}_{F_s}(M_t^2 - M_s^2) \\
&= \underbrace{\mathbb{E}\mathbb{1}_{F_s}(M_t^2 - \langle M\rangle_t - (M_s^2 - \langle M\rangle_s))}_{=0,\text{ due to }\left(M_t^2-\langle M\rangle_t\right)_{t\geq 0}\text{martingale}} + \mathbb{E}\mathbb{1}_{F_s}(\langle M\rangle_t - \langle M\rangle_s) \\
&= \mathbb{E}\mathbb{1}_{F_s}(\langle M\rangle_t - \langle M\rangle_s) \\
&= \mathbb{E}\mathbb{1}_{F_s}(\omega)\int_0^\infty \mathbb{1}_{(s,t]}(u)d\langle M\rangle_u(\omega) \\
&= \mathbb{E}\int_0^\infty \mathbb{1}_{F_s}(\omega)\mathbb{1}_{(s,t]}(u)d\langle M\rangle_u(\omega) \\
&= \mathbb{E}\int_0^\infty \mathbb{1}_A(u,\omega)d\langle M\rangle_u(\omega)
\end{aligned}$$

□

This is an alternative way to define the stochastic integral for previsible H that fulfills the right side of (3.1).
In the next step we compute the quadratic variation of a stochastic integral process.

Theorem 3.3.21. *Let M be an L_2-martingale with continuous paths and $H \in L_2(\mu_M)$. Then*

(i) $\langle H \cdot M \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s$ *for all* $t \geq 0$ $\mathbb{P}$ *a.s.* .

(ii) $\langle H \cdot M \rangle^\tau = \langle H \mathbb{1}_{(0,\tau]} \cdot M \rangle$ *for all stopping times* τ.

Proof. $\left(\int_0^t H_s^2 d\langle M \rangle_s \right)_{t \geq 0}$ is adapted, increasing and has continuous paths.
It remains to prove

$$N_t := (H \cdot M)_t^2 - \int_0^t H_s^2 d\langle M \rangle_s \qquad t \geq 0$$

is a martingale. Then the assertion follows with 3.1. For each stopping time τ it holds true

$$(H \cdot M)_\tau = (H \cdot M)_\infty^\tau = (H \mathbb{1}_{(0,\tau]} \cdot M)_\infty$$

Hence

$$\begin{aligned}
\mathbb{E}(H \cdot M)_\tau^2 &= \mathbb{E}(H \mathbb{1}_{(0,\tau]} \cdot M)_\infty^2 \\
&= \mathbb{E}(I(H \mathbb{1}_{(0,\tau]})^2) \\
&\overset{\text{Isometry}}{=} \int H^2 \mathbb{1}_{(0,\tau]} d\mu_M \\
&= \mathbb{E} \int H_s^2 \mathbb{1}_{(0,\tau]}(s) d\langle M \rangle_s \\
&= \mathbb{E} \int_0^\tau H_s^2 d\langle M \rangle_s
\end{aligned}$$

and $\mathbb{E} N_\tau = 0$ for each stopping time τ.
The claim (ii) follows immediately from

$$\langle H \cdot M \rangle^\tau = \langle (H \cdot M)^\tau \rangle = \langle H \mathbb{1}_{(0,\tau]} \cdot M \rangle$$

□

3.3.3 The quadratic covariation

The quadratic variation process can be seen as a quadratic mapping on the set of L_2-martingales. This means that it satisfies the following properties.

Proposition 3.3.22. *The quadratic variation is a quadratic operator.*

(i) $\langle cM \rangle = c^2 \langle M \rangle$ *for all* $c \in \mathbb{R}$ *and each continuous* L_2-*martingale* M,

(ii) $\langle M+N \rangle + \langle M-N \rangle = 2(\langle M \rangle + \langle N \rangle)$ *for all continuous* L_2-*martingales* M, N

Proof. The assertion can be easily shown by exploiting 3.1. □

According to a quadratic mapping a bilinear mapping can be constructed by the so called polarisation technique.

Definition 3.3.23. *The covariation process* $\langle M, N \rangle$ *can be defined by*

$$\langle M, N \rangle := \frac{1}{4}(\langle M+N \rangle - \langle M-N \rangle)$$

for all continuous L_2-*martingales* M, N.

From the properties of the quadratic variation the following properties of the quadratic covariation can be deduced

Theorem 3.3.24. *The quadratic covariation process fulfills:*

(i) $\langle \cdot, \cdot \rangle$ *is a bilinear mapping, i.e.*

$$\begin{aligned} \langle M_1 + M_2, N \rangle &= \langle M_1, N \rangle + \langle M_2, N \rangle \\ \langle cM, N \rangle &= c\langle M, N \rangle \\ \langle M, N_1 + N_2 \rangle &= \langle M, N_1 \rangle + \langle M, N_2 \rangle \\ \langle M, cN \rangle &= c\langle M, N \rangle \end{aligned}$$

(ii) $\langle \cdot, \cdot \rangle$ *is symmetric, i.e.*

$$\langle M, N \rangle = \langle N, M \rangle$$

(iii) $\langle M, N \rangle$ *is uniquely determined by the following properties:*

a) $\langle M, N \rangle_0 = 0$

b) $(\langle M, N \rangle_t)_{t \geq 0}$ *is adapted and has continuous paths of locally bounded variation,*

c) $MN - \langle M, N \rangle$ *is a martingale.*

Due to

$$\langle \int_0^{\cdot} H_s dM_s \rangle = \int_0^{\cdot} H_s^2 d\langle M \rangle_s$$

one may expect

$$\langle \int_0^{\cdot} H_s dM_s, \int_0^{\cdot} K_s dN_s \rangle = \int_0^{\cdot} H_s K_s d\langle M, N \rangle_s.$$

To prove this conjecture the Kunita Watanabe inequality has to be shown. This is well-known in the Lebesgue-Stieltjes integration theory.

Theorem 3.3.25. *Let $f, g, h : [0,\infty) \longrightarrow \mathbb{R}$ be right continuous functions with*

$$f(0) = g(0) = h(0)$$

f is supposed to be of locally bounded variation and g, h are assumed to be increasing. If

$$|f(t) - f(s)|^2 \leq (g(t) - g(s))(h(t) - h(s)) \qquad \text{for all } 0 \leq s < t,$$

then for all mearurable functions $x, y : [0,\infty) \longrightarrow \mathbb{R}$

$$\int_s^t |x(u)|\ |y(u)|\ d||f||_u \leq \left(\int_s^t |x(u)|^2 dg(u)\right)^{\frac{1}{2}} \left(\int_s^t |y(u)|^2 dh(u)\right)^{\frac{1}{2}}$$

Here $||f||$ denotes the variation measure w.r.t. μ_f, hence

$$||f|| := ||\mu_f|| = \mu_{FV.(f)}$$

Proof. see Revuz, Yor [3]. □

By a pathwise application the Kunita Watanabe inequality can be shown.

Theorem 3.3.26 (Kunita Watanabe Inequality)**.** *Let M, N be continuous L_2-martingales and H, K progressively measurable processes.*
Then

$$\int_s^t |H(u)|\ |K(u)|\ d||\langle M, N\rangle||_u \leq \left(\int_s^t H(u)^2 d\langle M\rangle_u\right)^{\frac{1}{2}} \left(\int_s^t K(u)^2 d\langle N\rangle_u\right)^{\frac{1}{2}}$$

Proof. Due to Theorem 3.3.25 it remains to prove

$$|\langle M, N\rangle_t - \langle M, N\rangle_s|^2 \leq (\langle M\rangle_t - \langle M\rangle_s)(\langle N\rangle_t - \langle N\rangle_s) \qquad \text{for all } 0 \leq s < t$$

for $\mathbb{P}$ almost all $\omega \in \Omega$.
Since the quadratic variation is increasing, it follows:

$$\langle M + \lambda N\rangle_t - \langle M + \lambda N\rangle_s \ \geq 0 \qquad \text{for all } \lambda \in \mathbb{R},$$

hence also

$$\langle M\rangle_t + 2\lambda\langle M, N\rangle_t + \lambda^2\langle N\rangle_t - (\langle M\rangle_s + 2\lambda\langle M, N\rangle_s + \lambda^2\langle N\rangle_s) \geq 0 \qquad \text{for all } \lambda \in \mathbb{R},$$

and

$$(\star) \quad \langle M\rangle_t - \langle M\rangle_s + 2\lambda(\langle M, N\rangle_t - \langle M, N\rangle_s) + \lambda^2(\langle N\rangle_t - \langle N\rangle_s) \geq 0 \quad \text{for all } \lambda \in \mathbb{R}.$$

This will be minimised in λ by

$$\lambda^\star = -\frac{\langle M,N\rangle_t - \langle M,N\rangle_s}{\langle N\rangle_t - \langle N\rangle_s}$$

Plugged into $(\star)$ yields:

$$\langle M\rangle_t - \langle M\rangle_s + \frac{(\langle M,N\rangle_t - \langle M,N\rangle_s)^2}{\langle N\rangle_t - \langle N\rangle_s} \geq 2\frac{(\langle M,N\rangle_t - \langle M,N\rangle_s)^2}{\langle N\rangle_t - \langle N\rangle_s}$$

Hence

$$(\langle M,N\rangle_t - \langle M,N\rangle_s)^2 \leq (\langle M\rangle_t - \langle M\rangle_s)(\langle N\rangle_t - \langle N\rangle_s)$$

and the claim follows. □

This can be used to compute the quadratic covariation of stochastic integral-processes.

Theorem 3.3.27. *Let $M, N \in \mathcal{H}_{2,c}$ and $H \in L_2(\mu_M)$. Then*

a) $\mathbb{E}\int\limits_0^\infty |H_s| d||\langle M,N\rangle||_s < \infty$

b) $\mathbb{E}(H\cdot M)_\infty N_\infty = \mathbb{E}\int\limits_0^\infty H_s d\langle M,N\rangle_s$

c) $\langle H\cdot M, N\rangle_t = \int\limits_0^t H_s d\langle M,N\rangle_s \qquad$ *for all* $t \geq 0$ $\mathbb{P}$ *a.s.*

d) Let M and N be continuous L_2-martingales and $H \in L_2(\mu_M)$, then

$$\langle H\cdot M, N\rangle_t = \int\limits_0^t H_s d\langle M,N\rangle_s \qquad \textit{for all } t \geq 0 \ \mathbb{P} \quad \textit{a.s.}$$

In particular for $K \in L_2(\mu_N)$

$$\langle H\cdot M, K\cdot N\rangle_t = \int\limits_0^t H_s K_s d\langle M,N\rangle_s \qquad \textit{for all } t \geq 0 \ \mathbb{P} \quad \textit{a.s.}$$

Proof. a) Kunita Watanabe inequality implies with $K \equiv 1$:

$$\int\limits_0^\infty |H_s| d||\langle M,N\rangle||_s \leq \left(\int\limits_0^\infty H_s^2 d\langle M\rangle_s\right)^{\frac{1}{2}} \underbrace{\left(\int\limits_0^\infty 1 d\langle N\rangle_s\right)^{\frac{1}{2}}}_{=\langle N\rangle_\infty^{\frac{1}{2}}}$$

Due to Cauchy-Schwarz inequality we get

$$\mathbb{E}\int_0^\infty |H_s| d||\langle M,N\rangle||_s \le \left(\mathbb{E}\int_0^\infty H_s^2 d\langle M\rangle_s\right)^{\frac{1}{2}} (\mathbb{E}\langle N\rangle_\infty)^{\frac{1}{2}}$$
$$= \left(\int H^2 d\mu_M\right)^{\frac{1}{2}} (\mathbb{E}\langle N\rangle_\infty)^{\frac{1}{2}}$$
$$\le ||H||_{L_2(\mu_M)}||N||_{\mathcal{H}_2} < \infty$$

since

$$\mathbb{E}\langle N\rangle_\infty = \mathbb{E}(N_\infty^2 - N_0^2) \le \mathbb{E}N_\infty^2 = ||N||_{\mathcal{H}_2}$$

$b)$ For $N \in \mathcal{H}_{2,c}$ is the mapping

$$A: \quad L_2(\mu_M) \longrightarrow \mathbb{R}$$
$$H \mapsto \mathbb{E}\left((H\cdot M)_\infty N_\infty - \int_0^\infty H_s d\langle M,N\rangle_s\right)$$

continuos and linear, since

$$|A(H)| = \left|\mathbb{E}\left((H\cdot M)_\infty N_\infty - \int_0^\infty H_s d\langle M,N\rangle_s\right)\right|$$
$$\le \mathbb{E}|(H\cdot M)_\infty|\ |N_\infty| + \mathbb{E}\int_0^\infty |H_s| d||\langle M,N\rangle||_s$$
$$\underset{\text{C.S.I.}}{\overset{a)}{\le}} \left(\mathbb{E}(H\cdot M)_\infty^2\right)^{\frac{1}{2}} (\mathbb{E}N_\infty^2)^{\frac{1}{2}} + ||H||_{L_2(\mu_M)}||N||_{\mathcal{H}_2}$$
$$= ||H\cdot M||_{\mathcal{H}_2}||N||_{\mathcal{H}_2} + ||H||_{L_2(\mu_M)}||N||_{\mathcal{H}_2}$$
$$\overset{\text{Isometrie}}{=} ||H||_{L_2(\mu_M)}||N||_{\mathcal{H}_2} + ||H||_{L_2(\mu_M)}||N||_{\mathcal{H}_2}$$
$$= 2||H||_{L_2(\mu_M)}||N||_{\mathcal{H}_2}$$

For $H \in \mathcal{E}$ it holds $A(H) = 0$.
Hence $A(H) = 0$ for all $H \in \overline{\mathcal{E}} = L_2(\mu_M)$.
$c)$ $\left(\int_0^t H_s d\langle M,N\rangle_s\right)_{t\ge 0}$ is an adapted process with continuous paths of locally bounded variation.
That

$$\left((H\cdot M)_t N_t - \int_0^t H_s d\langle M,N\rangle_s\right)_{t\ge 0}$$

is a martingale can be shown by applying 3.1.17.

d) as in c) one has to verify, that

$$\left((H \cdot M)_t N_t - \int_0^t H_s d\langle M, N \rangle_s \right)_{t \geq 0}$$

is a martingale and this can be done due to 3.1.17.

$\square$

At the end of this section we conclude that the quadratic variation process gives exactly the quadratic variation of the paths. This results will be provided in two steps.

Theorem 3.3.28. *Let M be a bounded continuous martingale with $M_0 = 0$, such that the quadratic variation process is bounded too.*
Let $\pi^{(n)}$ a sequence of lattices of the form

$$0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < ... \qquad \sup_i t_i^{(n)} = +\infty$$

with

$$|\pi^{(n)}| = \sup_i (t_i^{(n)} - t_{i-1}^{(n)}) \overset{n \to \infty}{\longrightarrow} 0$$

According to a given $t > 0$ this results into a finite lattice until t.

The quadratic variation along such a lattice until t is defined by

$$V_n^{(2)}(t) := \sum_{i \in \mathbb{N}} (M_{t_i^{(n)} \wedge t} - M_{t_{i-1}^{(n)} \wedge t})^2 \qquad \text{for all } t \geq 0.$$

Then

$$\mathbb{E} \sup_{t \geq 0} (V_n^{(2)}(t) - \langle M \rangle_t)^2 \overset{n \to \infty}{\longrightarrow} 0$$

In the second step the claim will be extended to L_2-martingales by localisation.

Theorem 3.3.29. *Let M ba a continuous martingale with $M_0 = 0$. Then for all $T > 0$:*

$$\sup_{0 \leq t \leq T} |V_n^{(2)}(t) - \langle M \rangle_t| \overset{n \to \infty}{\longrightarrow} 0 \qquad \text{in probability}$$

Proof. Let

$$\tau_k = \inf\{t \geq 0 : |M_t| \geq k \text{ or } \langle M \rangle_t \geq k\}$$

Then M^{τ_k} satisfies the assumptions Theorem 3.3.28.

Define

$$V_{n,k}^{(2)}(t) = \sum_{i\in\mathbb{N}} (M^{\tau_k}_{t_i^{(n)}\wedge t} - M^{\tau_k}_{t_{i-1}^{(n)}\wedge t})^2$$
$$= \sum_{i\in\mathbb{N}} (M_{t_i^{(n)}\wedge t\wedge\tau_k} - M_{t_{i-1}^{(n)}\wedge t\wedge\tau_k})^2$$

It follows

$$\mathbb{E}\sup_{t\geq 0}(V_{n,k}^{(2)}(t) - \langle M^{\tau_k}\rangle_t)^2 \overset{n\to\infty}{\longrightarrow} 0$$

and therefore

$$(\star) \quad \mathbb{P}(\sup_{t\geq 0}|V_{n,k}^{(2)}(t) - \langle M^{\tau_k}\rangle_t| > \epsilon) \overset{n\to\infty}{\longrightarrow} 0$$

for each $\epsilon > 0$. Due to

$$\tau_1 \leq \tau_2 \leq \ldots \qquad \sup_{n\in\mathbb{N}} \tau_n = +\infty$$

there exists to $\eta > 0$ some $k_0 \in \mathbb{N}$ with

$$\mathbb{P}(\tau_k \leq T) < \eta \qquad \text{for all } k \geq k_0$$

On $\{\tau_k \geq T\}$

$$V_{n,k}^{(2)}(t) = V_n^{(2)}(t) \qquad \text{for all } t \leq T$$

and

$$\langle M^{\tau_k}\rangle_t = \langle M\rangle_t^{\tau_k} \qquad \text{for all } t \leq T.$$

Hence

$$\begin{aligned}\mathbb{P}(\sup_{t\leq T}|V_n^{(2)}(t) - \langle M\rangle_t| > \epsilon) &\leq \mathbb{P}(\sup_{t\leq T}|V_n^{(2)}(t) - V_{n,k}^{(2)}(t)| > \frac{\epsilon}{3}) \\ &\quad + \mathbb{P}(\sup_{t\leq T}|V_{n,k}^{(2)}(t) - \langle M^{\tau_k}\rangle_t| > \frac{\epsilon}{3}) \\ &\quad + \mathbb{P}(\sup_{t\leq T}|\langle M^{\tau_k}\rangle_t - \langle M\rangle_t| > \frac{\epsilon}{3}) \\ &\leq 2\mathbb{P}(\tau_k \leq T) + \mathbb{P}(\sup_{t\geq 0}|V_{n,k}^{(2)}(t) - \langle M^{\tau_k}\rangle_t| > \frac{\epsilon}{3})\end{aligned}$$

Due to $(\star)$ there exists some $n_0 \in \mathbb{N}$ with

$$\mathbb{P}(\sup_{t\geq 0}|V_{n,k_0}^{(2)}(t) - \langle M^{\tau_{k_0}}\rangle_t| > \frac{\epsilon}{3}) < \eta \qquad \text{for all } n \geq n_0$$

Hence for all $n \geq n_0$

$$\begin{aligned}\mathbb{P}(\sup_{t\leq T}|V_n^{(2)}(t) - \langle M\rangle_t| > \epsilon) &\leq 2\mathbb{P}(\tau_{k_0} \leq T) + \mathbb{P}(\sup_{t\geq 0}|V_{n,k_0}^{(2)}(t) - \langle M^{\tau_{k_0}}\rangle_t| > \frac{\epsilon}{3}) \\ &\leq 3\eta \qquad \text{for all } n \geq n_0\end{aligned}$$

This implies the assertion, due to $\eta > 0$ arbitrary. $\square$

3.4 Localisation

The main disadvantage so far is that it is quite tedious to examine whether

$$\int H dM$$

exists, since the integral

$$\int H^2 d\mu_M = \mathbb{E} \int_0^T H_t^2 d\langle M \rangle_t$$

has to be calculated resp. estimated. Two main steps can be done to extend the definition of a stochastic integral process.

- localizing the integrand H by cutting at suitable stopping times,
- localizing the integrator M by considering the stopped process M^τ .

Localising means that one finds a stopping time τ such that

- M^τ is a continuous L_2-martingale,
- $H 1_{(0,\tau]} \in L_2(\mu_{M^\tau})$.

Then the integral

$$\int_{(0,\tau]} H dM^\tau \tag{3.2}$$

is well defined and the integral-process $H \cdot M$ can be put by 3.2 on the stochastic interval $(0,\tau]$. Repeating this procedure with a sequence of stopping times

$$\tau_1 \leq \tau_2 \leq ... \quad \sup_{n \in \mathbb{N}} \tau_n = +\infty$$

results in a definition of an integral-process $H \cdot M$ on $[0,\infty) \times \Omega$. This is the main idea which will be presented in the following section.

3.4.1 Local Spaces

First, we explain in general what localization means.

Definition 3.4.1. *Let $\mathcal{G}$ be a set of processes, which all start from zero and have continuous paths. Then X is called local $\mathcal{G}$-process, if there exists a sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that*

(i) $\tau_1 \leq \tau_2 \leq \tau_3 \leq ...$

(ii) $\sup_{n \in \mathbb{N}} \tau_n = +\infty$

(iii) $X^{\tau_n} \in \mathcal{G}$

With $\mathcal{G}_{loc}$ we define the set of local $\mathcal{G}-$ processes.

Main examples:

- $\mathfrak{M}^0 = \{M \in \mathfrak{M} : M_0 = 0\}$

    $\hookrightarrow \mathfrak{M}^0_{\text{loc}}$ as local variant.

- $\mathfrak{M}^0_c = \{M \in \mathfrak{M}^0 : M$ has continuous paths$\}$

    $\hookrightarrow \mathfrak{M}^0_{c,\text{loc}} := (\mathfrak{M}^0_c)_{\text{loc}}$

- $\mathcal{H}^0_2 := \{M \in \mathfrak{M}^0 : \sup\limits_{t\geq 0} \mathbb{E}M_t^2 < \infty\}$

    $\hookrightarrow \mathcal{H}^0_{2,\text{loc}} := (\mathcal{H}^0_2)_{\text{loc}}$

- $\mathcal{H}^0_{2,c} = \{M \in \mathcal{H}^0_2 : M$ has continuous paths$\}$

    $\hookrightarrow \mathcal{H}^0_{2,c,\text{loc}} := (\mathcal{H}^0_{2,c})_{\text{loc}}$

- $b\mathfrak{M}^0_c = \{M \in \mathfrak{M}^0_c : \exists C > 0 : \sup\limits_{t\geq 0} |M_t| < C\}$

    $\hookrightarrow b\mathfrak{M}^0_{c,\text{loc}} := (b\mathfrak{M}^0_c)_{\text{loc}}$

We clarify, when $\mathcal{G}_{\text{loc}}$ is a vector-space.

Proposition 3.4.2. *(i) If $\mathcal{G}$ is closed w.r.t. stopping, then $\mathcal{G}_{loc}$ either.*

(ii) If $\mathcal{G}$ is a vector-space and closed w.r.t. stopping, then $\mathcal{G}_{loc}$ is a vector-space either.

The proof is rather elementary and omitted here.
Note, that a set $\mathcal{A}$ of processes is called closed w.r.t. stopping, if for each $X \in A$ and each stopping time τ the process X^τ belongs to $\mathcal{A}$.
This leads to the following

Corollary 3.4.3. *The sets*

$$\mathfrak{M}^0_{loc}, \quad \mathfrak{M}^0_{c,loc}, \quad \mathcal{H}^0_{2,c,loc}, \quad b\mathfrak{M}^0_{c,loc}$$

are vector-spaces.

From main importance is the fact that the continuous local martingales can be localised into the space of bounded continuous martingales.

Theorem 3.4.4. *The following spaces coincide.*

$$b\mathfrak{M}^0_{c,loc} = \mathcal{H}^0_{2,c,loc} = \mathfrak{M}^0_{c,loc}$$

Proof. Due to

$$b\mathfrak{M}_c^0 \subsetneqq \mathcal{H}_{2,c}^0 \subsetneqq \mathfrak{M}_c^0$$

it follows

$$b\mathfrak{M}_{c,\text{loc}}^0 \subset \mathcal{H}_{2,c,\text{loc}}^0 \subset \mathfrak{M}_{c,\text{loc}}^0.$$

it remains to prove:

$$b\mathfrak{M}_{c,\text{loc}}^0 \supset \mathfrak{M}_{c,\text{loc}}^0$$

Let $M \in \mathfrak{M}_{c,\text{loc}}^0$. Then $M_0 = 0$ and M has continuous paths.
Furthermore there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\tau_1 \leq \tau_2 \leq \ldots \quad \sup_{n\in\mathbb{N}} \tau_n = +\infty$$

and

$$M^{\tau_n} \in \mathfrak{M}_c^0$$

Put

$$\sigma_n := \inf\{t \geq 0 : |M_t| \geq n\}$$

Then

$$\sigma_1 \leq \sigma_2 \leq \ldots \quad \sup_{n\in\mathbb{N}} \sigma_n = +\infty.$$

Consider$(\tau_n \wedge \sigma_n)_{n\in\mathbb{N}}$. This sequence localises M into $b\mathfrak{M}_c^0$, since $M^{\tau_n} \in \mathfrak{M}_c^0$ and therefore $(M^{\tau_n})^{\sigma_n} \in b\mathfrak{M}_c^0$ for all $n \in \mathbb{N}$. □

3.4.2 Quadratic Variation for Local Martingales

We may apply 3.4.4 to extend the definition of the quadratic variation process to local continuous martingales. According to $M \in \mathfrak{M}_{c,\text{loc}}^0$ we define

$$\tau_n := \inf\{t \geq 0 : |M_t| \geq n\}.$$

Then $M^{\tau_n} \in b\mathfrak{M}_c^0$ and therefore the quadratic variation process of M^{τ_n} well defined.

Definition 3.4.5. *Let $M \in \mathfrak{M}_{c,loc}^0$. Then the quadratic variation process of M is uniquely defined by*

$$\langle M \rangle := \sum_{n=1}^{\infty} \langle M^{\tau_n} \rangle \mathbb{1}_{(\tau_{n-1}, \tau_n]}$$

with

$$\tau_n := \inf\{t \geq 0 : |M_t| \geq n\}$$

Note that this definition is reasonable due to 3.3.14 and the quadratic variation of the local martingale M coincide on $(0, \tau_n]$ with the quadratic variation of the bounded martingale M^{τ_n}.
The properties carry over from the L_2-martingales to local martingales.

Theorem 3.4.6. *According to $M \in \mathfrak{M}^0_{c,loc}$ there exists a unique stochastic process A with the following properties:*

(i) $A_0 = 0$

(ii) $(A_t)_{t\geq 0}$ *is adapted with increasing and continuous paths.*

(iii) $(M_t^2 - A_t)_{t\geq 0} \in \mathfrak{M}^0_{c,loc}$

Proof. $A := \langle M \rangle$ satisfies the conditions $(i) - (iii)$ and is unique due to 3.3.17. □

Remark 3.4.7. The following Doob-Meyer decomposition holds:

$$M_t^2 = \underbrace{M_t^2 - \langle M \rangle_t}_{\text{local martingale part}} + \underbrace{\langle M \rangle_t}_{\substack{\text{increasing,}\\ \text{previsible}\\ \text{part}}}$$

So far we have only considered processes that start from the origin. Ba adding an $\mathcal{F}_0$-measurable starting variable we can give the definition for general local martingales.

Definition 3.4.8. *A stochastic process M is called continuous local martingale, if M_0 is $\mathcal{F}_0-$measurable and $M - M_0 \in \mathfrak{M}^0_{c,loc}$ holds, i.e.*

$$M = \underbrace{M_0}_{start} + \underbrace{M - M_0}_{\in \mathfrak{M}^0_{c,loc}}.$$

According to continuous local martingales the quadratic variation process is defined by

$$\langle M \rangle := \langle M - M_0 \rangle$$

As before we can carry over the quadratic variation process to the quadratic covariation process by polarisation.
We denote by FV_c^0 the set of all adapted processes with continuous paths that are of locally bounded variation and start from the origin.
Further we denote by $\mathfrak{M}_{c,\text{loc}}$ the space of all local martingales.
Then

$$\begin{aligned} \langle \cdot \rangle : \quad \mathfrak{M}_{c,\text{loc}} &\longrightarrow FV_c^0 \\ M &\mapsto \langle M \rangle \end{aligned}$$

is a quadratic mapping, i.e.

(i) $\langle cM \rangle = c^2 \langle M \rangle \quad$ for all $c \in \mathbb{R}$

(ii) $\langle M + N \rangle + \langle M - N \rangle = 2(\langle M \rangle + \langle N \rangle) \quad$ for all $M, N \in \mathfrak{M}_{c,\text{loc}}$

Definition 3.4.9. *We define according to $M, N \in \mathfrak{M}_{c,loc}$ the quadratic covariation process by*

$$\langle M, N\rangle := \frac{1}{4}(\langle M+N\rangle - \langle M-N\rangle)$$

Theorem 3.4.10. *The mapping*

$$\langle\cdot,\cdot\rangle : \mathfrak{M}_{c,loc} \times \mathfrak{M}_{c,loc} \longrightarrow FV_c^0$$

is bilinear and symmetric.
According to $M, N \in \mathfrak{M}_{c,loc}$ the quadratic covariation process $\langle M, N\rangle$ is the unique process in FV_c^0 such that

$$MN - \langle M, N\rangle \in \mathfrak{M}_{c,loc}$$

This theorem gives another opportunity to compute the quadratic covariation process by verifying the above characterisation.
The quadratic variation can be used to deduce the martingale property of a local martingale. This is important in many applications.
Clearly true is:

(i) If M is a continuous L_2-martingale with $M_0 = 0$, then $M^2 - \langle M\rangle$ is a martingale.

(ii) If $M \in \mathcal{H}_{2,c}^0$, then $M^2 - \langle M\rangle \in \mathfrak{M}_c^0$

This can be applied to examine the martingale property of a local martingale.

Theorem 3.4.11. *For $M \in \mathfrak{M}_{c,loc}^0$ it holds true:*

(i) If $\mathbb{E}\langle M\rangle_\infty < \infty$, then $M \in \mathcal{H}_{2,c}^0$.

(ii) If $\mathbb{E}\langle M\rangle_t < \infty$ for all $t \geq 0$, then M is a continuous L_2-martingale.

Proof. ad (i): First we show the martingale property of M by verifying

$$\mathbb{E}M_\tau (= \mathbb{E}M_0) = 0$$

for all bounded stopping times τ.
Therefore consider a localising sequence of stopping times

$$\tau_1 \leq \tau_2 \leq ... \quad \sup_{n\in\mathbb{N}} \tau_n = +\infty$$

with $M^{\tau_n} \in b\mathfrak{M}_c^0$.
Then $\mathbb{E}M_\tau^{\tau_n} = 0$, due to $M^{\tau_n} \in b\mathfrak{M}_c^0 \subset \mathcal{H}_{2,c}^0$ and

$$
\begin{aligned}
\mathbb{E}M^2_{\tau_n\wedge\tau} &= \mathbb{E}(M^{\tau_n})^2_\tau \\
&= \mathbb{E}\langle M^{\tau_n}\rangle_\tau \\
&= \mathbb{E}\langle M\rangle^{\tau_n}_\tau \\
&= \mathbb{E}\langle M\rangle_{\tau_n\wedge\tau} \uparrow \mathcal{E}\langle M\rangle_\tau \le \mathbb{E}\langle M\rangle_\infty
\end{aligned}
$$

Hence $(M_{\tau_n\wedge\tau})_{n\in\mathbb{N}}$ is uniformly integrable.
Together with

$$M_{\tau_n\wedge\tau} \overset{n\to\infty}{\longrightarrow} M_\tau \quad \mathbb{P} \text{ a.s.}$$

it follows

$$0 = \mathbb{E}M_{\tau_n\wedge\tau} \overset{n\to\infty}{\longrightarrow} \mathbb{E}M_\tau$$

hence

$$\mathbb{E}M_\tau = 0$$

M is therefore a martingale and due to

$$\sup_{t\ge 0} \mathbb{E}M_t^2 = \sup_{t\ge 0}\sup_{n\in\mathbb{N}} \mathbb{E}M^2_{\tau_n\wedge\tau} \le \mathbb{E}\langle M\rangle_\infty$$

it follows

$$M \in \mathcal{H}^0_{2,c}$$

ad (ii): For each $T > 0$ the stopped process M^T belongs to$\mathcal{H}^0_{2,c}$, since

$$\mathbb{E}\langle M^T\rangle_\infty = \mathbb{E}\langle M\rangle_T < \infty$$

Hence for all $s \le t \le T$:

$$
\begin{aligned}
\mathbb{E}(M_t|\mathcal{F}_s) &= \mathbb{E}(M^T_t|\mathcal{F}_s) \\
&= M^T_s \\
&= M_s
\end{aligned}
$$

Since T is arbitrary, the martingale property follows.
M is an L_2-Martingal, since for $t \le T$:

$$\mathbb{E}M_t^2 = \mathbb{E}(M^T)^2_t = \mathbb{E}\langle M^T\rangle_t = \mathbb{E}\langle M\rangle_t < \infty$$

□

The assertion of the preceding theorem can be improved.

Theorem 3.4.12. *For $M \in \mathfrak{M}^0_{c,loc}$ it holds true:*

(i) If $\mathbb{E}\sqrt{\langle M\rangle_\infty} < \infty$, then M is a uniformly martingale

(ii) If $\mathbb{E}\sqrt{\langle M\rangle_t} < \infty$ for all $t \ge 0$, then M is a continuous martingale.

Proof. This follows from the following two facts

1. for a continuous local martingale M with $M_0 = 0$ the following inequality is true

$$\mathbb{E} \sup_{t \geq 0} |M(t)| \leq c\mathbb{E}\sqrt{\langle M \rangle_\infty}$$

with a constant c that doesn't depend on M.

2. If

$$\mathbb{E} \sup_{t \geq 0} |M(t)| < \infty$$

then M is a uniformly integrable martingale.

□

3.4.3 Stochastic Integral for Local Martingales

The objective is to define the stochastic integral process $H \cdot M$ for local martingales and suitable localised H.

Definition 3.4.13. *According to $M \in \mathfrak{M}^0_{c,loc}$ we define the space $L^2_{loc}(M)$ by the set of all those processes H, that fulfill the following conditions:*

(i) H is previsible and

(ii) $\int\limits_0^t H_s^2 d\langle M \rangle_s < \infty \quad \mathbb{P} \quad a.s. \qquad \text{for all } t \geq 0$.

With $lb\mathcal{P}$ we denote the set of all locally bounded previsible processes, i.e.:
$H \in lb\mathcal{P}$ if and only if

(i) H is previsible

(ii) There exists a sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that

$$\tau_1 \leq \tau_2 \leq \ldots \quad \sup_{n \in \mathbb{N}} \tau_n = +\infty$$

and $H\mathbb{1}_{(0,\tau_n]} \in b\mathcal{P}$ for all $n \in \mathbb{N}$.

Note, that the preceding definition allows two opportunities to localise a previsible process. The space of locally bounded previsible processes is in applications mostly sufficient. But the definition does not depend on a chosen local martingale and is therefore not so comprehensive as the space $L^2_{\text{loc}}(M)$ which allows the extension of the stochastic integral as far as possible.

Theorem 3.4.14. *The following holds true*

(i) $L^2_{loc}(M)$ *is a vector-space for each* $M \in \mathfrak{M}^0_{c,loc}$.

(ii) $L_2(\mu_M) \subset L^2_{loc}(M)$ *for each continuous* L_2-*martingale.*

(iii) $lb\mathcal{P} \subset L^2_{loc}(M)$ *for all* $M \in \mathfrak{M}^0_{c,loc}$.

(iv) If H *is adapted with left continuous paths and existing limits from the right, then* $H \in lb\mathcal{P}$.

Proof. (i) is obvious, since

$$\int_0^t (H_s + K_s)^2 d\langle M\rangle_s \leq 2\left(\int_0^t H_s^2 d\langle M\rangle_s + \int_0^t K_s^2 d\langle M\rangle_s\right) < \infty$$

(ii) For $H \in L_2(\mu_M)$

$$\mathbb{E}\int_0^\infty H_s^2 d\langle M\rangle_s = \int H^2 d\mu_M < \infty,$$

hence

$$\int_0^\infty H_s^2 d\langle M\rangle_s < \infty$$

$\mathbb{P}$-almost sure and this implies $H \in L^2_{\text{loc}}(M)$.
(iii) Let $(\tau_n)_{n\in\mathbb{N}}$ localise H in $b\mathcal{P}$, i.e. $H\mathbb{1}_{(0,\tau_n]} \in b\mathcal{P}$.
Due to $\tau_n \uparrow \infty$

$$\int_0^t H_s^2 d\langle M\rangle_s = \lim_{n\to\infty} \int_0^{\tau_n\wedge t} H_s^2 d\langle M\rangle_s$$

Since $\tau_n \uparrow \infty$, there exists some $n \in \mathbb{N}$ such that $\tau_n(\omega) > t$. Hence

$$\begin{aligned}\int_0^t H_s^2 d\langle M\rangle_s &= \int_0^t \underbrace{H_s^2\mathbb{1}_{(0,\tau_n]}}_{\text{bounded}} d\langle M\rangle_s \\ &\leq C_n\langle M\rangle_t(\omega) < \infty\end{aligned}$$

(iv) Define for $n \in \mathbb{N}$:

$$\tau_n = \inf\{t \geq 0 : |H_t| \geq n\}$$

Then $(\tau_n)_{n\in\mathbb{N}}$ ia a localising sequence, since

$$H(0+) := \lim_{t\downarrow 0} H_t$$

is a finite random variable. □

We are now able to define the stochastic integral process $H \cdot M$ for $M \in \mathfrak{M}^0_{c,\text{loc}}$ and $H \in L^2_{\text{loc}}(M)$ by localisation.
Therefore define stopping times $(\tau_n)_{n\in\mathbb{N}}$ by

$$\tau_n := \inf\{t \geq 0 : \langle M\rangle_t \geq n \text{ or } \int\limits_0^t H_s^2 d\langle M\rangle_s \geq n\}$$

Then due to $H \in L^2_{\text{loc}}(M)$

$$\tau_1 \leq \tau_2 \leq \ldots \quad \sup_{n\in\mathbb{N}} \tau_n = +\infty$$

According to the stopped process M^{τ_n} we have

$$\mathbb{E}\langle M^{\tau_n}\rangle_\infty = \mathbb{E}\langle M\rangle_\infty^{\tau_n} = \mathbb{E}\langle M\rangle_{\tau_n} \leq n < \infty.$$

Hence, $M^{\tau_n} \in \mathcal{H}^0_{2,c}$.
Furthermore $H\mathbb{1}_{(0,\tau_n]} \in L_2(\mu_{M^{\tau_n}})$, due to

$$\begin{aligned}\int H^2\mathbb{1}_{(0,\tau_n]} d\mu_{M^{\tau_n}} &= \mathbb{E}\int\limits_0^\infty H_s^2\mathbb{1}_{(0,\tau_n]}(s) d\langle M^{\tau_n}\rangle_s \\ &= \mathbb{E}\int\limits_0^\infty H_s^2\mathbb{1}_{(0,\tau_n]}(s) d\langle M\rangle_s^{\tau_n} \\ &= \mathbb{E}\int\limits_0^{\tau_n} H_s^2 d\langle M\rangle_s \\ &\leq n < \infty\end{aligned}$$

Definition 3.4.15. *According to $M \in \mathfrak{M}^0_{c,loc}$ and $H \in L^2_{loc}(M)$ we define the stochastic integral process $H \cdot M$ by*

$$H \cdot M := \sum_{n=1}^\infty (H\mathbb{1}_{(0,\tau_n]} \cdot M^{\tau_n})\mathbb{1}_{(\tau_{n-1},\tau_n]}$$

Remark 3.4.16.

- The stochastic intervals $((\tau_{n-1},\tau_n])_{n\in\mathbb{N}}$ build a disjoint decomposition of $(0,\infty) \times \Omega)$.
- The integral process starts from the origin.
- On $(0,\tau_n]$ the integral process $H \cdot M$ coincides with $H\mathbb{1}_{(0,\tau_n]} \cdot M^{\tau_n}$, since for $m < n$ due to compatibility w.r.t stopping, see 3.3.14

$$\begin{aligned}\left(H\mathbb{1}_{(0,\tau_n]} \cdot M^{\tau_n}\right)^{\tau_m} &= H\mathbb{1}_{(0,\tau_n]}\mathbb{1}_{(0,\tau_m]} \cdot (M^{\tau_n})^{\tau_m} \\ &= H\mathbb{1}_{(0,\tau_m]} \cdot M^{\tau_m}\end{aligned}$$

- $(H \cdot M)^{\tau_n} = H\mathbb{1}_{(0,\tau_n]} \cdot M^{\tau_n}$ for all $n \in \mathbb{N}$.

To compute the integral process in practise one has to find a localising sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$. Each sequence that fulfills

- $H1_{(0,\tau_n]} \in L_2(\mu_{M^{\tau_n}})$,
- M^{τ_n} is an L_2-martingale,
- $\sup \tau_n = \infty$,

can be chosen.
To verify properties of a stochastic properties, in particular for a stochastic integral process, a localisation technique can be useful.

Proposition 3.4.17. *Let M, N be two stochastic processes and $(\tau_n)_{n\in\mathbb{N}}$ a sequence of increasing stopping times with $\sup_{n\in\mathbb{N}} \tau_n = \infty$. If for each $n \in \mathbb{N}$ the stopped process M^{τ_n} is indistinguishable from N^{τ_n}, then M is indistinguishable from N.*

This means that one has only to ensure that M and N coincide on each stochastic interval $(0, \tau_n]$.
An application of this techniques leads to the following properties of the generally defined stochastic integral process.

Theorem 3.4.18. *Let $M \in \mathfrak{M}^0_{c,loc}$, $H \in L^2_{loc}(M)$. Then*

(i) $H \cdot M \in \mathfrak{M}^0_{c,loc}$

(ii) $\langle H \cdot M\rangle_t = \int\limits_0^t H_s^2 d\langle M\rangle_s$ *for all* $t \geq 0$.

(iii) $(H \cdot M)^\tau = H\mathbb{1}_{(0,\tau]} \cdot M^\tau = H\mathbb{1}_{(0,\tau]} \cdot M = H \cdot M^\tau$

(iv) If K previsible and $K \in L^2_{loc}(H \cdot M)$, then
- $KH \in L^2_{loc}(M)$
- $K \cdot (H \cdot M) = KH \cdot M$

(v) $(H + K) \cdot M = H \cdot M + K \cdot M$ *for all* $H, K \in L^2_{loc}(M)$

(vi) For $N \in \mathfrak{M}^0_{c,loc}$ and $H \in L^2_{loc}(M) \cap L^2_{loc}(N)$ it holds true
- $H \in L^2_{loc}(M + N)$
- $H \cdot (M + N) = H \cdot M + H \cdot N$

(vii) For all $M, N \in \mathfrak{M}_{c,loc}$ and $H \in L^2_{loc}(M)$

$$\langle H \cdot M, N\rangle = \int_0^{\cdot} H_s d\langle M, N\rangle_s.$$

In particular for $K \in L^2_{loc}(N)$

$$\langle H \cdot M, K \cdot N \rangle = \int_0^\cdot H_s K_s d\langle M, N \rangle_s.$$

Proof. All these properties were verified for stochastic integral process that could be defined by the Doléans-measure. By localisation these properties carry over to the generally defined integral-process. □

Finally we consider the behaviour of the integral process when the time goes to infinity. For integral processes in $\mathcal{H}_{2,c}$ this is clear, since a convergence takes place in L_2 as well as point-wise a.s.. If one integrates w.r.t. a local martingale, the situation is more delicate. The behaviour of the quadratic variation process helps to ensure an almost sure convergence.

Theorem 3.4.19. *Let M be a continuous local martingale. Then M converges on $\{\langle M \rangle_\infty < \infty\}$ point-wise $\mathbb{P}$-a.s..*

Proof. We may assume $M_0 = 0$. It holds true

$$\{\langle M \rangle_\infty < \infty\} = \bigcup_{C>0} \{\langle M \rangle_\infty \leq C\}.$$

Define according to $C > 0$ the stopping time σ_C by

$$\sigma_C = \inf\{t \geq 0 : \langle M \rangle_t > C\}.$$

Then

$$\{\sigma_C = +\infty\} = \{\langle M \rangle_\infty \leq C\}$$

and for the stopped process M^{σ_C} we have

$$\mathbb{E}\langle M^{\sigma_C} \rangle_\infty = \mathbb{E}\langle M \rangle_\infty^{\sigma_C} = \mathbb{E}\langle M \rangle_{\sigma_C} \leq C.$$

It follows, that M^{σ_C} is a $\mathcal{H}_{2,c}$ martingale, that $\mathbb{P}$-almost sure converges. On the event $\{\sigma_C = +\infty\} = \{\langle M \rangle_\infty \leq C\}$ the processes M and M^{σ_C} coincide Hence M converges on each $\{\langle M \rangle_\infty \leq C\}$ and therefore also on $\{\langle M \rangle_\infty < \infty\}$. □

One may apply this to integral-processes and obtain

Theorem 3.4.20. *Let $M \in \mathfrak{M}_{c,loc}$ and $H \in L^2_{loc}(M)$. On the event*

$$\left\{ \int_0^\infty H_s^2 d\langle M \rangle_s < \infty \right\}$$

the stochastic integral process $H \cdot M$ converges point-wise $\mathbb{P}$-almost sure.

3.5 Ito Calculus

In calculus the chain rule, product rule and factor rule gives applicants tools at hand in order to be able to calculate derivatives of more complex functions. Even without any knowledge how a derivative is defined people can use calculus efficiently. Integration can be seen as inverse mapping of taking derivatives. And therefore the rules for taking derivatives find its counterpart in rules of integration, like substitution rule, integration by parts etc. Stochastic calculus, also called Ito-calculus, can be seen as calculus for stochastic processes.

3.5.1 Ito-Formula

The Ito-formula is a generalisation of the chain-rule.
Let $x : [0, \infty) \longrightarrow \mathbb{R}$ be continuously differentiable and $f : \mathbb{R} \longrightarrow \mathbb{R}$ a C^1-function.
Chain Rule: Then

$$(f \circ x)'(t) = f'(x(t))x'(t) \qquad \text{for all } t \geq 0$$

Alternative in integral-form:

$$f(x(t)) - f(x(0)) = \int_0^t (f \circ x)'(s)ds = \int_0^t f'(x(s)) \underbrace{x'(s)ds}_{\substack{\text{Radon-Niko-}\\ \text{dym deri-}\\ \text{vative}}} = \int_0^t f'(x(s))dx(s)$$

This means, the chain rule can be expressed alternatively by

$$f(x(t)) - f(x(0)) = \int_0^t f'(x(s))dx(s)$$

or shortly

$$df(x(t)) = f'(x(t))dx(t).$$

This differential notation is justified resp. motivated by the observation

$$f(x(t)) - f(x(0)) = \int_0^t 1 \, df(x(s)) = \int_0^t f'(x(s))dx(s)$$

The first generalisation of the chain rule is therefore
Let $x : [0, \infty) \longrightarrow \mathbb{R}$ be continuous and locally of bounded variation and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1-function, then

$$f(x(t)) - f(x(0)) = \int_0^t f'(x(s))dx(s) \qquad \text{for all } t \geq 0.$$

But: Continuous martingales have no paths of locally bounded variations. This means that the above formula must be modified. This leads to the second generalisation, the Ito-formula.
Let X be a continuous semi-martingale and $f : \mathbb{R} \longrightarrow \mathbb{R}$ some C^2-function. Then:

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s$$

In differential notation:

$$df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t) d\langle X \rangle_t$$

Objective: Rigorous derivation of the Ito-formula.
First we have to define the term semi-martingale.
We consider in this section a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$, that satisfies the usual conditions

Definition 3.5.1. *A stochastic-process X is called continuous semi-martingale, if there exists a decomposition of the form*

$$X = X_0 + M + A$$

such that

$$M \in \mathfrak{M}^0_{c,loc}, \ A \in FV^0_c \text{ and } X_0 \text{ is } \mathcal{F}_0 - \text{measurable.}$$

This decomposition is unique, since continuous local martingales with paths that are locally of bounded variation are constant, 3.3.17.
We call

(i) X_0 *the starting variable,*

(ii) M *the local martingale part,*

(iii) A *the locally bounded variation part*

of X.

The integration w.r.t. a semi-martingale can be explained by separate integration according to the martingale and bounded variation part.
We define the set $L_{\text{loc}}(A)$ by

$$L_{\text{loc}}(A) := \{H : H \text{ ist progressively measureable and } \int_0^t |H_s| d||A||_s < \infty \text{ for all } t \geq 0\}.$$

Then for

$$H \in L_{\text{loc}}(A) \cap L^2_{\text{loc}}(M) =: L_{\text{loc}}(X)$$

the stochastic integral process

$$H \cdot X$$

can be defined by

$$H \cdot X := H \cdot M + H \cdot A$$

whereat

$$(H \cdot X)_t = \int\limits_0^t H_s dA_s$$

is path-wise defined.
Notation:

$$(H \cdot X)_t = \int\limits_0^t H_s dX_s = \int\limits_0^t H_s dM_s + \int\limits_0^t H_s dA_s$$

The stochastic integral process $H \cdot X$ is again a semi-martingale with $H \cdot M$ as local martingale part and $H \cdot A$ as bounded variation part.
Note:

$$lb\mathbb{P} \subset L_{\text{loc}}(X)$$

This means that processes from a large vector-space can be integrated. In particular each cáglád process, left continuous with right-hand limits, can be integrated. Hence no integrability condition has to be verified.

Definition 3.5.2. *Let X be a continuous semi-martingale of the form*

$$X = X_0 + M + A.$$

The quadratic variation process of X is defined by

$$\langle X \rangle := \langle M \rangle.$$

Polarisation leads to the quadratic covariation by

$$\langle X, Y \rangle = \frac{1}{4}(\langle X + Y \rangle - \langle X - Y \rangle)$$

for all semi-martingales X, Y.

Remark 3.5.3. Let $X = X_0 + M + A, Y = Y_0 + N + B$ be two semi-martingales. Then

(i) $\langle X, Y \rangle = \langle M, N \rangle$

(ii) $\langle H \cdot X \rangle = \langle H \cdot M \rangle = \int\limits_0^\cdot H_s^2 d\langle M \rangle_s = \int\limits_0^\cdot H_s^2 d\langle X \rangle_s$ for all $H \in L_{\text{loc}}(X)$

(iii) $\langle H \cdot X, K \cdot Y \rangle = \langle H \cdot M, K \cdot N \rangle = \int\limits_0^\cdot H_s K_s d\langle M, N \rangle_s = \int\limits_0^\cdot H_s K_s d\langle X, Y \rangle_s$ for all $H \in L_{\text{loc}}(X), K \in L_{\text{loc}}(Y)$

Remark 3.5.4. The quadratic covariation process of two semi-martingales can also be seen as the covariation of their paths.
More precise: Let X, Y be semi-martingales. Let π_n be a lattice

$$\pi_n : 0 = t_0^{(n)} < t_1^{(n)} < ... \qquad \sup_i t_i^{(n)} = +\infty$$

Then for all $T \geq 0$

$$\sup_{0 \leq t \leq T} |KV_n(t) - \langle X, Y \rangle_t| \overset{n\to\infty}{\longrightarrow} 0$$

converges in probability.
Here

$$KV_n(t) = \sum_i (X_{t_i^{(n)} \wedge t} - X_{t_{i-1}^{(n)} \wedge t})(Y_{t_i^{(n)} \wedge t} - Y_{t_{i-1}^{(n)} \wedge t})$$

denotes the quadratic covariation of X and Y along the lattice π.

The way to the Ito-formula can be gone by first establishing the general integration by parts formula and then approximating C^2-functions by polynomials in a suitable way. First we consider the case of real function that are locally of bounded variation.

Theorem 3.5.5. *Let $f, g : [0, \infty) \longrightarrow \mathbb{R}$ be right continuous functions, that are locally of bounded variation.*
Then

$$f(t)g(t) - f(0)g(0) = \int_0^t f(s-)dg(s) + \int_0^t g(s-)df(s) + \sum_{0<s\leq t} \Delta f(s)\Delta g(s) \qquad \textit{for all } t \geq 0$$

whereat

$$f(s-) := \lim_{u \uparrow s} f(u)$$

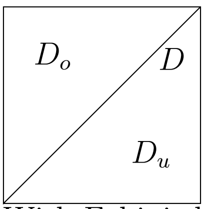
and

$$\Delta f(s) := f(s) - f(s-)$$

$\Delta f(s)$ *measures the jump-size at* s*. This is relevant at points where* f *is discontinuous.*

Proof. The proof mainly follows from an application of Fubini's theorem.
The functions f and g can be seen as distribution functions of signed measures μ_f, μ_g. The function fg corresponds to the product measure $\mu_f \otimes \mu_g$, since

$$\mu_f \otimes \mu_g((0, t] \times (0, t]) = \mu_f((0, t])\mu_g((0, t]) = f(t)g(t).$$



$$D_o := \{(s, r) : 0 < s \leq t, s < r \leq t\}$$

$$D_u := \{(s, r) : 0 < s \leq t, r < s\}$$

$$D := \{(s, s) : 0 < s \leq t\}$$

With Fubini the measure of the rectangle can be calculated which leads to the claimed formula. $\square$

Remark 3.5.6. If f, g are continuous, then the sum cancels out and it remains.

$$f(t)g(t) - f(0)g(0) = \int_0^t f(s)dg(s) + \int_0^t g(s)df(s).$$

If f, g are absolutely-continuos w.r.t. Lebesgue-measure then

$$df(t) = f'(t)dt$$

and

$$dg(t) = g'(t)dt.$$

Therefore

$$f(t)g(t) - f(0)g(0) = \int_0^t f(s)g'(s)ds + \int_0^t g(s)f'(s)ds.$$

This coincides with the usual integration by parts formula

Since by definition for a local martingale M

$$M_t^2 = M_0^2 + 2\int_0^t M_s dM_s + \langle M \rangle_s$$

again a polarisation argument leads to the integration by parts formula for local martingales.

Theorem 3.5.7 (Integration by parts for local martingales)**.** *Let $M, N \in \mathfrak{M}_{c,loc}$. Then*

$$M_t N_t - M_0 N_0 = \int_0^t M_s dN_s + \int_0^t N_s dM_s + \langle M, N \rangle_t \quad \textit{for all } t \geq 0 \ \mathbb{P} \quad \textit{a.s.}$$

A bit more difficult to prove is the mixed integration by parts formula

Theorem 3.5.8 (mixed integration by parts)**.** *Let $M \in \mathfrak{M}_{c,loc}, A \in FV_c$. Then*

$$M_t A_t - M_0 A_0 = \underbrace{\int_0^t M_s dA_s}_{\substack{\textit{Lebesgue-} \\ \textit{Stieltjes-} \\ \textit{Integral}}} + \underbrace{\int_0^t A_s dM_s}_{\substack{\textit{stochastic} \\ \textit{integral}}} \ \underbrace{(+\langle A, M \rangle_t)}_{\substack{=0, \textit{ due to } A \\ \textit{bounded variation}}} \qquad \textit{for all } t \geq 0 \ \mathbb{P} \quad \textit{a.s.}$$

All together we obtain the integration by parts formula for semi-martingales.

Theorem 3.5.9. *Let X, Y be continuous semi-martingales. Then*

$$X_tY_t - X_0Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t \quad \textit{for all } t \geq 0 \ \mathbb{P} \quad \textit{a.s.}$$

Theorem 3.5.10. *Let X be a continuous semi-martingale. Then*

$$X_t^2 - X_0^2 = 2\int_0^t X_s dX_s + \langle X\rangle_t \quad \textit{for all } t \geq 0 \ \mathbb{P} \quad \textit{a.s.}$$

This integration by parts formula can be exploited to derive the Ito-formula.

Theorem 3.5.11 (Itō-Formel). *Let X be a continuous semi-martingale and $f : \mathbb{R} \longrightarrow \mathbb{R}$ some C^2-function. Then*

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s) d\langle X\rangle_s \quad \textit{for all } t \geq 0 \ \mathbb{P} \quad \textit{a.s.}$$

If X has the representation

$$X = X_0 + M + A$$

then $f \circ X$ is a semi-martingale with decomposition

$$f \circ X = f(X_0) + \underbrace{f'(X) \cdot M}_{\substack{\textit{local martingale-}\\ \textit{part}}} + \underbrace{f'(X) \cdot A + \frac{1}{2} f''(X) \cdot \langle X\rangle}_{\substack{\textit{bounded}\\ \textit{variation-}\\ \textit{part}}}$$

In differential notation the Ito-formula is

$$\begin{aligned} df(X_t) &= f'(X_t)dX_t + \frac{1}{2} f''(X_t) d\langle X\rangle_t \\ &= f'(X_t)dM_t + f'(X)_t dA_t + \frac{1}{2} f''(X_t) d\langle X\rangle_t \end{aligned}$$

Proof. Let $X = X_0 + M + A$ be a semi-martingale. The integration by parts formula is used to prove that the set

$$\mathcal{A} := \left\{ f \in C^2(\mathbb{R}) : \text{Ito-formula is valid for } \left(f(X_t)\right)_{t \geq 0} \right\}$$

is not only a vector-space but also aan algebra, hence closed under products. From this it follows that all polynomials belong to $\mathcal{A}$. Finally by localising and uniformly approximating C^2-functions by polynomials on compact sets the validity of Ito's formula carries over to C^2-functions.

$\square$

A multidimensional counterpart is the following

Theorem 3.5.12 (Multidimensional Ito-formula)**.** *Let X be some d-dimensional process.*
$X = (X^{(1)}, ..., X^{(d)})$ is called continuous semi-martingale, if each component is a continuous semi-martingale.
Let f be a $C^2(\mathbb{R}^d)$-function, then

$$f(X_t) - f(X_0) = \int_0^t \bigtriangledown f(X_s) dX_s + \frac{1}{2} \int_0^t H_f(X_s) d\langle X \rangle_s.$$

Thereby $\bigtriangledown f(x) = \begin{pmatrix} \partial_1 f(x) \\ \vdots \\ \partial_d f(x) \end{pmatrix}$ *denotes the gradient of f in $x \in \mathbb{R}^d$*
and $H_f(x) = \begin{pmatrix} \partial_{11} f(x) & \cdots & \partial_{1d} f(x) \\ \vdots & \ddots & \dots \\ \partial_{d1} f(x) & \cdots & \partial_{dd} f(x) \end{pmatrix}$ *the Hessian-matrix of f in $x \in \mathbb{R}^d$.*
We define further

$$\int_0^t \bigtriangledown f(X_s) dX_s := \sum_{i=1}^d \int_0^t \partial_i f(X_s) dX_s^{(i)}$$

and

$$\int_0^t H_f(X_s) d\langle X \rangle_s := \sum_{i,j=1}^d \int_0^t \partial_{i,j} f(X_s) d\langle X^{(i)}, X^{(j)} \rangle_s$$

The smootheness assumption can be weakened when the local martingale part vanishes.

Theorem 3.5.13. *Let X be a continuous real-valued semi-martingale and B some FV_c-process. Let f be a $C^{1,2}-$function. Then*

$$f(B_t, X_t) = f(B_0, X_0) + \int_0^t \partial_1 f(B_s, X_s) dB_s + \int_0^t \partial_2 f(B_s, X_s) dX_s + \frac{1}{2} \int_0^t \partial_{22} f(X_s) d\langle X \rangle_s$$

Often in applications a local version of Ito's formula is needed.

Theorem 3.5.14. *Let D be an open subset of $\mathbb{R}^d$ and X a continuous $d-$dimensional semi-martingale with paths in D. Then for a C^2-function $f : D \longrightarrow \mathbb{R}$*

$$f(X_t) - f(X_0) = \int_0^t \nabla f(X_s) dX_s + \frac{1}{2} \int_0^t H_f(X_s) d\langle X \rangle_s$$

3.5.2 First Applications

We will give some applications of Ito's formula.

Example 3.5.15. Let W be a Wiener-process. Then

$$W_t^3 = tW_t + 3 \int_0^t W_s^2 dW_s - \int_0^t s dW_s + 2 \int_0^t W_s ds$$

Proof. We apply the Ito-formula to $f(x) = x^3$. Then

$$\begin{aligned} dW_t^3 &= 3W_t^2 dW_t + \frac{1}{2} 6W_t dt \\ &= 3W_t^2 dW_t + 3W_t dt \\ &= 3W_t^2 dW_t + 2W_t dt + W_t dt \end{aligned}$$

We obtain by using integration by parts

$$dtW_t = tdW_t + W_t dt \Leftrightarrow W_t dt = dtW_t - tdW_t$$

Hence after plugging in

$$dW_t^3 = 3W_t^2 dW_t + 2W_t dt + W_t dt = d\,tW_t + 3W_t^2 dW_t - tdW_s + 2W_t dt$$

□

Example 3.5.16. Let W be a Wiener-process. We are searching for a semi-martingale representation of $(W_t^{2n})_{t \geq 0}$ in order to compute $\mu_{2n}(t) := \mathbb{E} W_t^{2n}$.
We apply the Ito-formula to $f(x) := x^{2n}$. Then

$$\begin{aligned} dW_t^{2n} &= 2nW_t^{2n-1} dW_t + \frac{1}{2} 2n(2n-1) W_t^{2n-2} dt \\ &= \underbrace{2nW_t^{2n-1} dW_t}_{=:M_t} + \underbrace{n(2n-1) W_t^{2n-2} dt}_{\text{locally of bounded variation}} \end{aligned}$$

The process M_t is as stochastic integral process a local martingale. We show that M is indeed a martingale by using 3.4.11.

$$\langle M\rangle_t = \int_0^t (2nW_s^{2n-1})^2 d\langle W\rangle_s = \int_0^t 4n^2 W_s^{4n-2} ds$$

and

$$\mathbb{E}\langle M\rangle_t = \int_0^t 4n^2 \mathbb{E}W_s^{\overbrace{4n-2}^{\text{even power}}} ds = \int_0^t 4n^2 \mathbb{E}(\frac{W_s}{\sqrt{s}})^{4n-2} s^{n-1} ds < \infty$$

Hence 3.4.11 implies that M is a martingale. Thus

$$\mathbb{E}M_t = \mathbb{E}M_0 = 0.$$

Therefore

$$\begin{aligned}\mathbb{E}W_t^{2n} &= \mathbb{E}\int_0^t 2nW_s^{2n-1} dW_s + n(2n-1)\mathbb{E}\int_0^t W_s^{2(n-1)} ds \\ &= n(2n-1)\int_0^t \mathbb{E}W_s^{2(n-1)} ds\end{aligned}$$

By induction we obtain

$$\mathbb{E}W_t^{2n} = (2n-1)(2n-3)\cdot \ldots \cdot 1 \cdot t^n$$

Example 3.5.17 (Brownian bridge). Let W be some Wiener-process. According to a terminal time-point $T > 0$ and a terminal point in space $b \in \mathbb{R}$ a stochastic-process $(X_t)_{0\leq t<T}$ has to be constructed that behaves on $[0, T)$ as a Wiener-process - conditioned on $W_T = b$. This process is called Brownian bridge with terminal point b at T.
We define therefore

$$X_t = b\frac{t}{T} + (T-t)\int_0^t \frac{1}{T-s} dW_t \quad \text{for all } 0 \leq t < T$$

Then $M_t := \int_0^t \frac{1}{T-s} dW_s$, $0 \leq t < T$ is an L_2-martingale but no $\mathcal{H}_2$-martingale and

$$\mathbb{E}M_t^2 = \int_0^t \left(\frac{1}{T-s}\right)^2 ds = \frac{1}{T-t} - \frac{1}{t} < \infty \quad \text{for all } 0 \leq t < T.$$

But

$$\sup_{t<T} \mathbb{E}M_t^2 = \sup_{t<T} \frac{1}{T-t} - \frac{1}{t} = +\infty$$

With integration by parts we obtain the semi-martingale representation of X by

$$\begin{aligned}(T-t)M_t &= \int_0^t T - s dM_s + \int_0^t M_s d(T-s) \\ &= \int_0^t \frac{T-s}{T-s} dW_s - \int_0^t M_s ds \\ &= (W_t - W_0) - \int_0^t M_s ds \\ &= W_t - \int_0^t M_s ds\end{aligned}$$

Hence X can be written as

$$X_t = \underbrace{b\frac{t}{T} - \int_0^t M_s ds}_{\text{bounded var.}} + \underbrace{W_t}_{\text{martingale}} \qquad \text{for all } 0 \le t < T$$

It is valid

$$\mathbb{E}X_t = b\frac{t}{T}$$

and

$$\mathbb{V}\text{ar}X_t = (T-t)^2 \int_0^T \left(\frac{1}{T-s}\right)^2 \mathbb{1}_{(0,t]} ds = (T-t)^2 \left(\frac{1}{T-t} - \frac{1}{t}\right)$$

and for $s < t$:

$$\begin{aligned}\text{Cov}(X_t, X_s) &= \mathbb{E}((X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)) \\ &= \mathbb{E}((T-t)M_t(T-s)M_s) \\ &= (T-t)(T-s)\mathbb{E}M_t M_s \\ &= (T-t)(T-s)\mathbb{E}M_s^2 + (T-t)(T-s)\underbrace{\mathbb{E}(M_t - M_s)M_s}_{=0} \\ &= (T-t)(T-s)\frac{s}{T(T-s)} \\ &= s - s\frac{t}{T}\end{aligned}$$

The process M has independent and normally distributed increments, i.e. $M_t - M_s$ is independent of $\mathcal{F}_s$ and normally distributed for all $0 \le s \le t < T$.

$$M_t - M_s = \int_s^t \frac{1}{T-u} dW_u$$

Since $f(u) := \frac{1}{T-u} \in L_2([s,t))$ there exists a sequence

$$f^{(n)} = \sum_{i=1}^{l(n)} y_t^{(n)} \mathbb{1}_{(t_{i-1}^{(n)}, t_t^{(n)}]}$$

with

$$||f^{(n)} - f||_{L^2([s,t))} \longrightarrow 0$$

whereat

$$t_0^{(n)}, ..., t_{l(n)}^{(n)}$$

is a decomposition of the interval $[s,t)$. Hence

$$\int_s^t f^{(n)}(u) dW_u = \sum_{i=1}^{l(n)} y_i^{(n)} (W_{t_i^{(n)}} - W_{t_{i-1}^{(n)}}) \stackrel{L_2(\mathbb{P})}{\longrightarrow} \int_s^t f(u) dW_u$$

with $W_{t_i^{(n)}} - W_{t_{i-1}^{(n)}}$ independent of $\mathcal{F}_s$ and normally distributed. Since the independence and distribution remains unchanged in the L_2-limit, $W_t - W_s$ is independent of $\mathcal{F}_s$ and normally distributed
According to k time-points

$$0 < t_1 < t_s < ... < t_k < T$$

the distribution of $(X_{t_1}, ..., X_{t_k})$ is a $k-$dimensional normal distribution.
The random variable X_t is normally distributed as proven above with the parameters

$$\mathbb{E}X_t = b\frac{t}{T} \text{ and } \mathbb{V}\text{ar}X_t = (T-t)\frac{t}{T}$$

For $t_1 < t_2 < ... < t_k$ the random vector $(M_{t_1}, ..., M_{t_k})$ has a $k-$dimensional normal distribution , since

$$M_{t_1}, M_{t_2} - M_{t_1}, ..., M_{t_k} - M_{t_{k-1}}$$

are stochastic independent and normally distributed. Then

$$X_{t_i} = b\frac{t_i}{T} + (T - t_i) M_{t_i}$$

is a linear transformation of M, which leads to a $k-$dimensional normal distribution for $(X_{t_1}, ..., X_{t_n})$.
With help of the Ito-formula we show, that X is a solution of the following stochastic differential equation.

$$dX_t = \frac{b - X_t}{T-t} dt + dW_t$$

with initial condition $X_0 = 0$.
Exploit the semi-martingale representation of X:

$$\begin{aligned} dX_t &= dW_t + b\frac{1}{T}dt - M_t dt \\ &= dW_t + \left(\frac{b}{T} - M_t\right) dt \\ &= dW_t + \left(\frac{b}{T} - \underbrace{\frac{X_t - b\frac{t}{T}}{T-t}}_{X_t \text{ expressed by} M_t}\right) dt \end{aligned}$$

3.5.3 Doléans Exponential

In this part we introduce a class of positive martingales that can be used to change measures. This is of importance in finance to compute an equivalent martingale measure. Let's start with the easiest ordinary differential equation for real functions

$$z'(t) = z(t)$$

with some initial condition z_0. This differential equation can be written as integral equation in the form

$$z(t) = z_0 + \int_0^t z(s)ds$$

resp. short

$$dz(t) = z(t)dt.$$

Surely the unique solution is given by

$$z(t) = z_0 e^t \quad \text{for all } t \geq 0.$$

In stochastic analysis the easiest stochastic differential equation is of the form

$$dZ(t) = Z(t)dX(t) \quad \text{with initial condition} \;\; Z_0 \tag{3.3}$$

with a given semi-martingale X that starts from the origin. We say that a stochastic process Z is a solution to the above stochastic differential equation, if the following integral equation is valid

$$Z(t) = Z_0 + \int_0^t Z(s)dX(s) \quad \text{for all } t \geq 0. \tag{3.4}$$

This means that we give a stochastic differential equation sense by considering its correspondent integral equation. With Ito's formula we can solve the equation (3.3) by considering the approach

$$Z(t) = f(X(t), \langle X\rangle_t)$$

for a suitable function f.
Ito's formula implies

$$dZ_t = \partial_1 f(X_t, \langle X \rangle_t) dX_t + \partial_2 f(X_t, \langle X \rangle_t) d\langle X \rangle_t + \frac{1}{2} \partial_{11} f(X_t, \langle X \rangle_t) d\langle X \rangle_t$$

We have to find a function f such that

$$df(X_t, \langle X \rangle_t) = f(X_t, \langle X \rangle_t) dX_t$$

Hence we guess:

I: $\partial_1 f = f$

II: $\partial_2 f = -\frac{1}{2}\partial_{11} f$

The first equation provides

$$f(x, y) = e^x h(y)$$

the second equation

$$\begin{aligned} h'(y)e^x &= -\frac{1}{2} e^x h(y) \\ \Rightarrow \qquad h'(y) &= -\frac{1}{2} h(y) \\ \Rightarrow \qquad h(y) &= e^{-\frac{1}{2}y} \end{aligned}$$

Thus

$$f(x, y) = e^x e^{-\frac{1}{2}y} = \exp(x - \frac{1}{2}y)$$

satisfies the equations I and II.
This is the reason for the following definition

Definition 3.5.18.

$$\mathcal{E}(X)_t := \exp(X_t - \frac{1}{2}\langle X \rangle_t) \quad \textit{for all } t \geq 0$$

is called exponential semi-martingale of X.

The Ito-formula implies that $\mathcal{E}(X)$ solves the integral equation

$$\mathcal{E}(X)_t = 1 + \int_0^t \mathcal{E}(X)_s dX_s$$

resp.

$$d\mathcal{E}(X)_t = \mathcal{E}(X)_t \, dX_t$$

with initial condition $\mathcal{E}(X)_0 = 1$. By multiplying with the initial random variable Z_0 we obtain a solution to (3.3)

Theorem 3.5.19. *Let X be a continuous semi-martingale with $X(0) = 0$. The process*

$$Z_t = Z_0 \mathcal{E}(X)_t = Z_0 \exp(X_t - \frac{1}{2}\langle X \rangle_t)$$

solves uniquely the integral equation

$$Z_t = Z_0 + \int_0^t Z_s dX_s \quad \textit{for all } t \geq 0$$

resp. the stochastic differential equation

$$dZ_t = Z_t dX_t \quad \textit{for all } t \geq 0$$

with initial random variable Z_0.

Proof. For all $t \geq 0$

$$\begin{aligned}
Z_t &= Z_0 \mathcal{E}(X)_t \\
&= Z_0 (1 + \int_0^t \mathcal{E}(X)_s dX_s) \\
&= Z_0 + Z_0 \int_0^t \mathcal{E}(X)_s dX_s \\
&\overset{Z_0 \ \mathcal{F}_0-\text{mb}}{=} Z_0 + \int_0^t Z_0 \mathcal{E}(X)_s dX_s \\
&= Z_0 + \int_0^t Z_s dX_s
\end{aligned}$$

The uniqueness can be shown in the following way.
For a solution Y we consider

$$N_t := (\mathcal{E}(X)_t)^{-1} = \exp(-X_t + \frac{1}{2}\langle X \rangle_t)$$

Then

$$\begin{aligned}
dN_t &= N_t d(-X_t + \frac{1}{2}\langle X \rangle_t) + \frac{1}{2} N_t d \underbrace{\langle -X_t + \frac{1}{2} \rangle \langle X \rangle_t)}_{=\langle X \rangle_t} \\
&= -N_t dX_t + N_t d\langle X \rangle_t
\end{aligned}$$

Integration by parts implies

$$\begin{aligned} dY_tN_t &= Y_t dN_t + N_t dY_t + d\langle Y, N\rangle_t \\ &= -Y_tN_t dX_t + Y_tN_t d\langle X\rangle_t + N_t \underbrace{Y_t dX_t}_{\substack{=dY_t, \text{ since } Y_t \\ \text{solution}}} - \underbrace{N_tY_t d\langle X\rangle_t}_{(\star)} \\ &= 0 \end{aligned}$$

Hence

$$Y_tN_t - Y_0N_0 = \int_0^t d(Y_sN_s) = 0$$

which implies

$$Y_t = Z_0 N_t^{-1} = Z_0 \mathcal{E}(X)_t$$

To prove $(\star)$: we use the theorem

$$\langle \int_0^{\cdot} H_s dX_s, \int_0^{\cdot} K_s dX_s \rangle = \int_0^{\cdot} H_s K_s d\langle X\rangle_s$$

resp. in differential notation:

$$d\langle \int_0^{\cdot} H_s dX_s, \int_0^{\cdot} K_s dX_s \rangle = H.K.d\langle X\rangle_..$$

Due to

$$dY_t = Y_t dX_t$$

and

$$dN_t = -N_t dX_t + N_t d\langle X\rangle_t$$

it follows

$$d\langle Y, N\rangle_t = -N_tY_t d\langle X\rangle_t$$

□

3.5.4 Linear Stochastic Differential Equations

The stochastic differential equation

$$dZ_t = Z_t dX_s$$

with initial value Z_0 has applications in finance, since it provides a suitable model for the price evolution of a stock.
Let $(S_t)_{t\geq 0}$ denote the price process of a stock. The evolution depends mainly on two ingredients

- μ_t denoting the random rate of return

- σ_t denoting the volatility.

For small h we obtain

$$S_{t+h} - S_t \approx S_t\mu_t h + S_t\sigma_t(W_{t+h} - W_t)$$

with Wiener-process W.
This means

$$\Delta S_t \approx S_t\mu_t\Delta t + S_t\sigma_t\Delta W_t$$

This leads in the limit to the stochastic differential equation

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t)$$

with initial value S_0.
The solution of this equation is a reasonable approach for modelling the stock price behaviour. More formally we assume

- μ is some progressively measurable process with

$$\int_0^t |\mu_s| ds < \infty \qquad \text{for all } t \geq 0$$

  and

- σ is somee previsible process with

$$\int_0^t \sigma_s^2 ds < \infty \qquad \text{for all } t \geq 0.$$

Then $\sigma \in L^2_{\text{loc}}(W)$ and

$$X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \qquad \text{for all } t \geq 0$$

is a continuous semi-martingale that can be written as

$$dX_t = \mu_t dt + \sigma_t dW_t$$

in differential form.
Hence using 3.5.19

$$S_t = S_0\mathcal{E}(X)_t = S_0 \exp(\int_0^t \mu_s ds) \exp(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds)$$

We obtain the classical Black-Scholes model when we assume that the coefficients μ and σ are non-random constants don't depending on time.

Orenstein-Uhlenbeck Process

A further process of interest in finance is the Orenstein-Uhlenbeck process. It is defined as a solution to the following linear stochastic differential equation.

$$dX_t = -\alpha X_t dt + \sigma dW_t \tag{3.5}$$

with initial value $X_0 = \zeta$, an $\mathcal{F}_0-$measurable random variable. The coefficients of this equation are constants $\alpha, \sigma > 0$.
A solution of the above equation can be computed by a variation of constants technique which is well known for ordinary differential equations. The above Orenstein-Uhlenbeck equation decomposes into

- a homogeneous equation $dY_t = -\alpha Y_t dt$ and
- a random inhomogeneity $\sigma dW(t)$

The homogeneous equation

$$dY_t = -\alpha Y_t dt$$

is an ordinary differential equation and its solution is given by

$$Y_t = e^{-\alpha t}$$

Then $Y_0 = 1$ and

$$d\frac{1}{Y_t} = \alpha \frac{1}{Y_t} dt$$

If X is a solution of 3.5 then

$$\begin{aligned}
d\frac{X_t}{Y_t} &= X_t d\frac{1}{Y_t} + \frac{1}{Y_t} dX_t + \underbrace{d\langle X, \frac{1}{Y}\rangle_t}_{\substack{=0,\text{ since } \frac{1}{Y_t} \text{ deter-}\\ \text{ministic, hence}\\ \text{of bounded var.}}} \\
&= \alpha \frac{X_t}{Y_t} dt + \frac{1}{Y_t}(\underbrace{-\alpha X_t dt + \sigma dW_t}_{\substack{=dX_t,\text{ due to } X_t \text{ solution}\\ \text{of 3.5}}}) \\
&= \frac{1}{Y_t} \sigma dW_t \\
&= e^{\alpha t} \sigma dW_t
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{X_t}{Y_t} &= \frac{X_0}{Y_0} + \int_0^t d\frac{X_s}{Y_s} \\
&= \frac{X_0}{Y_0} + \int_0^t e^{\alpha s} \sigma dW_s \\
&= \zeta + \int_0^t e^{\alpha s} \sigma dW_s
\end{aligned}$$

Thus we obtain

$$\begin{aligned}X_t &= \zeta Y_t + Y_t \int_0^t e^{\alpha s} \sigma dW_s \\ &= \zeta e^{-\alpha t} + e^{-\alpha t} \int_0^t e^{\alpha s} \sigma dW_s \\ &= \zeta e^{-\alpha t} + \int_0^t e^{\alpha(s-t)} \sigma dW_s\end{aligned}$$

From this explicit representation we can deduce some properties of the Orenstein-Uhlenbeck process.

- Expectation
$$\mathbb{E} X_t = e^{-\alpha t} \mathbb{E} \zeta =: m(t)$$

- Variance
$$\mathbb{V}\mathrm{ar} X_t = e^{-2\alpha t} \mathbb{V}\mathrm{ar} \zeta + \int_0^t e^{2\alpha(s-t)} \sigma ds = e^{-2\alpha t} \mathbb{V}\mathrm{ar} \zeta + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) =: v(t)$$

- Marginal distributions:

  Note, that ζ and $\int_0^t e^{\alpha(s-t)} \sigma dW_s$ are independent. If ζ is normally distributed or constant then
$$X_t \sim \mathcal{N}(m(t), v(t))$$

- Limiting distribution:

  Due to $\lim_{t \to \infty} m(t) = 0$ and $\lim_{t \to \infty} v(t) = \frac{\sigma^2}{2\alpha}$ the process X_t converges in distribution to a $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})$-distributed random variable.

- Mean reversion:

  The Ornstein-Uhlenbeck-process is a mean-reverting diffusion with return-level 0 and return-rate α. This means that the process X has the tendency to return to 0 wherever it is located. This tendency is perturbed by the noise $\sigma dW(t)$.

- Stationary distribution:

  If ζ is a $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})-$distributed random variable, then the distribution of X_t doesn't change. This means that the Orenstein-Uhlenbeck process is a Markov-process with $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})$ as stationary distribution.

Vasicek-Prozess

A slight modification of the Orenstein-Uhlenbeck equation leads to the Vasicek-process which is of interest in modelling bond markets.

Definition 3.5.20. *The solution of the stochastic differential equation*

$$dX_t = \vartheta(\mu - X_t)dt + \sigma dW_t \tag{3.6}$$

with initial value ζ, return-level $\mu \in \mathbb{R}$ and return-rate $\vartheta > 0$ is called Vasicek-process.

A solution can be calculated with the help of the Orenstein-Uhlenbeck equation.
Let X be a solution of 3.6. Then

$$Z_t = X_t - \mu$$

solves the equation

$$dZ_t = d(X_t - \mu) = dX_t = \vartheta(\mu - X_t)dt + \sigma dW_t = -\vartheta Z_t dt + \sigma dW_t$$

and is therefore an Orenstein-Uhlenbeck process. Hence

$$X_t - \mu = e^{-\vartheta t}(\zeta - \mu) + \int_0^t e^{\vartheta(s-t)}\sigma dW_s$$

$$\Leftrightarrow \qquad X_t = e^{-\vartheta t}\zeta + \mu(1 - e^{-\vartheta t}) + \int_0^t e^{\vartheta(s-t)}\sigma dW_s$$

For the expectation and variance we obtain

$$\mathbb{E}X_t = e^{-\vartheta t}\mathbb{E}\zeta + \mu(1 - e^{-\vartheta t}) =: m(t) \longrightarrow \mu$$

and

$$\mathbb{V}\mathrm{ar}X_t = e^{2\vartheta t}\mathbb{V}\mathrm{ar}\zeta + \frac{\sigma^2}{2\vartheta}(1 - e^{-2\vartheta t}) =: v(t) \longrightarrow \frac{\sigma^2}{2\vartheta}$$

If ζ is normally distributed, then $X_t \sim \mathcal{N}(m(t), v(t))$ and X_t converges in distribution to a $\mathcal{N}(\mu, \frac{\sigma^2}{2\vartheta})-$ distributed random variable.

General One-dimensional Linear Stochastic Differential Equation

Definition 3.5.21. *The general linear stochastic differential equation reads as follows*

$$dX_t = (X_t\mu_t + a_t)dt + (X_t\sigma_t + \eta_t)dW_t \tag{3.7}$$

with initial value ζ, some $\mathcal{F}_0$-measurable random variable.
Requirements:

- *μ is progressively measurable with $\int_0^t |\mu_s|ds < \infty$ for all $t \geq 0$,*

- *a is progressively measurable with* $\int_0^t |a_s| ds < \infty$ *for all* $t \geq 0$,

- $\sigma \in L^2_{loc}(W)$ *and* $\eta \in L^2_{loc}(W)$.

The solution can be calculated with a variation of constants technique.

$$dX_t = \underbrace{X_t(\mu_t dt + \sigma_t dW_t)}_{\text{Black-Scholes Equation}} + \underbrace{a_t dt + \eta_t dW_t}_{\text{Inhomogenity}} \tag{3.8}$$

We first consider the homogeneous equation

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t)$$

with $S_0 = 1$ which is solved by

$$S_t = \exp\left(\int_0^t \mu_s ds\right) \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right).$$

Then, according to a solution X of (3.7) the stochastic differential of $\left(\frac{X_t}{S_t}\right)$ has to be determined. Ito's formula provide

$$\begin{aligned}
d\frac{1}{S_t} &= -\frac{1}{S_t^2} dS_t + \frac{1}{2} 2 \frac{1}{S_t^3} d\langle S\rangle_t \\
&= -\frac{1}{S_t^2} S_t(\mu_t dt + \sigma_t dW_t) + \frac{1}{S_t^3} S_t^2 \sigma_t^2 dt \\
&= -\frac{1}{S_t}(\mu_t dt + \sigma_t dW_t) + \frac{1}{S_t}\sigma_t^2 dt
\end{aligned}$$

Then, we continue with integration by parts

$$\begin{aligned}
d\frac{X_t}{S_t} &= X_t d\frac{1}{S_t} + \frac{1}{S_t} dX_t + d\langle X, \frac{1}{S}\rangle_t \\
&= -X_t \frac{1}{S_t}(\mu_t dt + \sigma_t dW_t) + \frac{X_t}{S_t}\sigma_t^2 dt \\
&\quad + \frac{1}{S_t}\big((X_t\mu_t + a_t)dt + (X_t\sigma_t + \eta_t)dW_t\big) \\
&\quad - \frac{X_t}{S_t}\sigma_t^2 dt - \frac{1}{S_t}\sigma_t\eta_t dt \\
&= \frac{1}{S_t}(a_t - \sigma_t\eta_t)dt + \frac{1}{S_t}\eta_t dW_t
\end{aligned}$$

The right-hand side does not depend on X. Hence we can determine the left-hand side by integration.

$$\frac{X_t}{S_t} = \frac{X_0}{S_0} + \int_0^t d\left(\frac{X}{S}\right)_s = \zeta + \int_0^t \frac{1}{S_u}(a_u - \sigma_u\eta_u)du + \int_0^t \frac{1}{S_u}\eta_u dW_u$$

and therefore a solution is given by

$$X_t = \zeta S_t + S_t \int_0^t \frac{1}{S_u}(a_u - \sigma_u \eta_u) du + S_t \int_0^t \frac{1}{S_u} \eta_u dW_u$$

Why is the integral

$$\int_0^t \frac{\eta_u}{S_u} dW_u$$

well-defined? Due to our assumptions $\eta_u \in L^2_{\text{loc}}(W)$. We have to examine if $\frac{\eta_u}{S_u} \in L^2_{\text{loc}}(W)$:

$$\int_0^t \left(\frac{\eta_u}{S_u}\right)^2 d\langle W\rangle_u = \int_0^t \left(\frac{\eta_u}{S_u}\right)^2 du \underset{\text{stetig}}{\overset{S_u}{\leq}} \left(\sup_{u\in[0,t]} \frac{1}{S_u}\right)^2 \int_0^t \eta_u^2 du < \infty$$

3.6 Three Main Theorems

In the following we will introduce three main theorems that are of high importance in applications of stochastic analysis, in particular in finance.

3.6.1 Theorem of Lévy

The first gives a characterisation of the Wiener-process by its martingale properties. It is the so called Lévy's theorem.

Theorem 3.6.1. *Let $W = (W^{(1)}, ..., W^{(d)})$ be some $d-$dimensional continuous local martingale with*
$W_0^{(i)} = 0$ *for all* $i = 1, ..., d$ *and* $\langle W^{(i)}, W^{(j)}\rangle_t = \delta_{ij} t$ *with* $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$
Then, W is a $d-$dimensional Wiener-process. This means, that the coordinates of W are one-dimensional independent Wiener-processes.

Proof. The main ideas of the proof will be given without going into detail.
We have to show that W is a stochastic process with independent and stationary increments, i.e. for all $s < t$

- $W_t - W_s$ is independent of $\mathcal{F}_s$ and
- $W_t - W_s \sim \mathcal{N}(0, \underbrace{(t-s)I_d}_{\text{covariation-matrix}})$

For each $\vartheta \in \mathbb{R}^d$

$$X_\vartheta(t) := (\vartheta, W_t) = \sum_{k=1}^{d} \vartheta_k W_t^{(k)}$$

is a local martingale with

$$\langle X_\vartheta \rangle_t = \sum_{k=1}^{d} \vartheta_k^2 \langle W^{(k)} \rangle_t + \sum_{\substack{k,l=1 \\ k \neq l}}^{d} \vartheta_k \vartheta_l \underbrace{\langle W^{(k)}, W^{(l)} \rangle_t}_{=0} = |\vartheta|^2 t$$

Hence

$$M_\vartheta(t) := \exp(iX_\vartheta(t) + \frac{1}{2}|\vartheta|^2 t) \quad t \geq 0$$

is as solution to

$$dZ(t) = iZ(t)dX_\vartheta(t)$$

a complex-valued, exponential local martingale. Indeed, it is a true martingale since

$$\begin{aligned}
\mathbb{E}\left(\langle Re(M_\vartheta) \rangle_t + \langle Im(M_\vartheta) \rangle_t\right) &= \mathbb{E} \int_0^t \left((Im(M_\vartheta(s))^2 + Re(M_\vartheta(s))^2\right) \underbrace{|\vartheta|^2 ds}_{=d\langle X_\vartheta \rangle_s} \\
&= \mathbb{E} \int_0^t |M_\vartheta(s)|^2 |\vartheta|^2 ds \\
&= \mathbb{E} \int_0^t e^{|\vartheta|^2 s} |\vartheta|^2 ds \\
&= \int_0^t e^{|\vartheta|^2 s} |\vartheta|^2 ds < \infty
\end{aligned}$$

Due to the martingale property of $(M_\vartheta(t))_{t \geq 0}$ we obtain for $s \leq t$

$$1 = \mathbb{E}\left(\frac{M_\vartheta(t)}{M_\vartheta(s)} \middle| \mathcal{F}_s\right) = \mathbb{E}\exp(i(\vartheta, W_t - W_s)|\mathcal{F}_s)\exp(\frac{1}{2}|\vartheta|^2(t-s))$$

Hence

$$\mathbb{E}\exp(i(\vartheta, W_t - W_s)|\mathcal{F}_s) = \exp(-\frac{1}{2}|\vartheta|^2(t-s)))$$

for all $\vartheta \in \mathbb{R}^d$. But this means that the Fourier-transform of the conditional distribution of the increment $W(t) - W(s)$ given $\mathcal{F}_s$ coincides with the Fourier-transform of a $\mathbb{N}(0,(t-s)I_d)$ distribution. This implies the independence and distribution properties (i) and (ii). $\square$

Application of Lévy's Theorem

We give an application that can be used in finance.
Let W be some 1-dimensional Wiener-process and $\sigma \in L^2_{\mathrm{loc}}(W)$. Thus we may define

$$M_t := \int_0^t \sigma_s dW_s \quad t \geq 0$$

and

$$N_t := \int_0^t |\sigma_s| dW_s \quad t \geq 0$$

We will apply Lévy's theorem in order to show that the processes M and N have the same distribution.
Note, that

$$sgn(\sigma_t)\sigma_t = |\sigma_t|.$$

Hence

$$N_t = \int_0^t sgn(\sigma_s)\sigma_s dW_s.$$

Define

$$A := \{(t, \omega) : \sigma_t(\omega) = 0\}.$$

Then A is a previsible set and

$$\begin{aligned} N_t &= \underbrace{\int_0^t \mathbb{1}_A(s)|\sigma_s| dW_s}_{=0} + \int_0^t \mathbb{1}_{A^c}(s) sgn(\sigma_s)\sigma_s dW_s \\ &= \int_0^t \mathbb{1}_{A^c}(s) sgn(\sigma_s)\sigma_s dW_s \end{aligned}$$

We put

$$B_t := \int_0^t \mathbb{1}_A(s) dW_s + \int_0^t \mathbb{1}_{A^c}(s) sgn(\sigma_s) dW_s$$

Then $(B_t)_{t\geq 0}$ is a local martingale with

$$\begin{aligned}
\langle B\rangle_t &= \langle \int_0^{\cdot} \mathbb{1}_A(s) + \mathbb{1}_{A^c}(s) sgn(\sigma_s) dW_s\rangle_t \\
&= \int_0^t (\mathbb{1}_A(s) + \mathbb{1}_{A^c}(s) \underbrace{sgn(\sigma_s)}_{sgn(\sigma_s)^2=1})^2 d\langle W\rangle_s \\
&= \int_0^t \mathbb{1}_A(s) + \mathbb{1}_{A^c}(s) ds \\
&= \int_0^t 1 ds \\
&= t
\end{aligned}$$

Lévy's theorem implies, that B is a Wiener-process. Hence

$$\begin{aligned}
N_t &= \int_0^t \mathbb{1}_A(s) \underbrace{|\sigma_s|}_{\substack{=\sigma_s \\ \text{on } A}} dW_s + \int_0^t \mathbb{1}_{A^c}(s) sgn(\sigma_s)\sigma_s dW_s \\
&= \int_0^t \sigma_s \underbrace{(\mathbb{1}_A(s) + \mathbb{1}_{A^c}(s) sgn(\sigma_s)) dW_s}_{=dB_s} \\
&= \int_0^t \sigma_s dB_s
\end{aligned}$$

Hence, M and N are integral-processes of σ according to two different Wiener-processes. This implies that their distribution coincide. The impact in finance relies in the fact that volatility can be assumed to be positive. The distribution of a stock-price process remains unchanged.

3.6.2 Martingale Representation Theorem

The next step is the so called martingale representation theorem. This will be used in finance to compute replicating trading strategies. The main statement is that each local martingale on a Wiener-filtration can be represented as stochastic integral-process.
First we introduce the Wiener-filtration.

Definition 3.6.2. *Let $(W_t)_{t\geq 0}$ be some Wiener-Prozess on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$. We define a filtration $(\mathcal{G}_t)_{t\geq 0}$, that satisfies the usual conditions and* *is generated by W. Denote by*

$$N := \{A \subset \Omega : \exists B \in \mathcal{F}_\infty : A \subset B \text{ und } \mathbb{P}(B) = 0\}$$

the set of negligible sets and define for $t \geq 0$:

- *(generated by W)* $\mathcal{F}_t^{(0)} := \sigma(W_s : s \leq t)$ *with* $\mathcal{F}_\infty^{(0)} := \sigma(W_s : s \geq 0)$
- *(all null-sets in $\mathcal{F}_0^{(1)}$)* $\mathcal{F}_t^{(1)} := \sigma(\mathcal{F}_t^{(0)} \cup N)$
- *(right continuous)* $\mathcal{G}_t := \mathcal{F}_{t+}^{(1)} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^{(1)}$ *with* $\mathcal{G}_\infty := \sigma(\mathcal{G}_t : t \geq 0)$

Then, $(\mathcal{G}_t)_{t\geq 0}$ is called Wiener-filtration and W is a Wiener-process according to $\mathcal{G}$.

The filtration $(\mathcal{G}_t)_{t\geq 0}$ can be seen as the smallest filtration with the following properties:

- the filtration satisfies the usual conditions and
- the process W is a Wiener-process according to this filtration

We start by giving an L_2-version of the martingale-representation theorem.

Theorem 3.6.3. *Let W be some d-dimensional Wiener-process with Wiener-filtration $(\mathcal{G}_t)_{t\geq 0}$. Then there exists according to $Y \in L_2(\mathcal{G}_\infty)$ some $(\mathcal{G}_t)_{t\geq 0}$-previsible process $H = (H^{(1)}, ..., H^{(d)})$ such that*

$$\sum_{k=1}^{d} \mathbb{E} \int_0^\infty (H_s^{(k)})^2 ds < \infty$$

and

$$\begin{aligned} Y &= \mathbb{E}Y + \int_0^\infty H_s dW_s \\ &:= \mathbb{E}Y + \sum_{k=1}^{d} \int_0^\infty H_s^{(k)} dW_s^{(k)} \end{aligned}$$

This means, that the corresponding $\mathcal{H}_2$-martingale has an integral-process representa-

tion of the form

$$\begin{aligned}\mathbb{E}(Y|\mathcal{G}_t) &= \mathbb{E}Y + (H \cdot W)_t \\ &:= \mathbb{E}Y + \sum_{k=1}^{d} (H^{(k)} \cdot W^{(k)})_t \\ &= \mathbb{E}Y + \sum_{k=1}^{d} \int_0^t H_s^{(k)} dW_s^{(k)}\end{aligned}$$

Proof. Again we give the main idea of the proof. We restrict ourselves to the case $d = 1$. First we note, that Blumenthal's $0-1$ law states, that $\mathcal{G}_0$ is a trivial σ-field, i.e. $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{G}_0$.
Then Y is independent of $\mathcal{G}_0$ and

$$\mathbb{E}(Y|\mathcal{G}_0) = \mathbb{E}Y \quad \mathbb{P} \text{ a.s.} \quad .$$

Without loss of generality we put $\mathbb{E}Y = 0$ and consider the integral operator

$$I : L_2(\mu_W) \longrightarrow L_{2,0}(\mathcal{G}_\infty); H \to \int H dW$$

with

$$L_{2,0}(\mathcal{G}_\infty) = \{X \in L_2(\Omega, \mathcal{G}_\infty, \mathbb{P}) : \mathbb{E}X = 0\}.$$

It remains to show that I is a surjective mapping. The space $V = I(L_2(\mu_W))$ is a closed subspace of $L_{2,0}(\mathcal{G}_\infty)$ and we can consider the orthogonal complement

$$V^\perp := \{Z \in L_{2,0}(\mathcal{G}_\infty) : \mathbb{E}ZX = 0 \ \forall X \in V\}.$$

This means that Z is perpendicular to all $X \in V$.
Note, that each $M_\infty \in L_{2,0}(\mathcal{G}_\infty)$ corresponds to some unique $\mathcal{H}_2$-martingale defined by

$$M_t = \mathbb{E}(M_\infty|\mathcal{G}_t) \quad t \geq 0$$

and

$$M_0 = \mathbb{E}(M_\infty|\mathcal{G}_0) = \mathbb{E}(M_\infty) = 0.$$

Furthermore for each $M_\infty \in V$ and $Z_\infty \in V^\perp$ the process $(M_t Z_t)_{t \geq 0}$ is a uniformly integrable martingale, since for each stopping time τ

$$\begin{aligned}\mathbb{E}(M_\tau Z_\tau) &= \mathbb{E}(M_\tau \mathbb{E}(Z_\infty|\mathcal{G}_\tau)) \\ &= \mathbb{E}(\mathbb{E}(M_\tau Z_\infty|\mathcal{G}_\tau)) \\ &= \mathbb{E}M_\tau Z_\infty \\ &= \mathbb{E}\underbrace{M_\infty^\tau}_{\in V} Z_\infty \\ &= 0\end{aligned}$$

We have to prove
$$V^{\perp} = \{0\}$$
and consider $Z_\infty \in V^{\perp}$. We show $Z_\infty = 0$ $\mathbb{P}$ a.s. .
As in the proof of Lévy's theorem we may consider the complex-valued martingale
$$M_\vartheta(t) = \exp(i(\vartheta, W_t) + \frac{1}{2}|\vartheta|^2 t) \quad t \geq 0.$$
Stopping at T provides that $M_\vartheta^T \in \mathcal{H}_2$ and therefore as an integral-process in V. Thus
$$\left(Z_t M_\vartheta^T(t)\right)_{t\geq 0}$$
is a uniformly integrable martingale. Hence
$$\mathbb{E}(Z_t \exp(i(\vartheta, W_t - W_s))|\mathcal{G}_s) = Z_s \exp(-\frac{1}{2}|\vartheta|^2(t-s)) \quad \text{for all } 0 \leq s < t \leq T, \vartheta \in \mathbb{R}^d$$
By iterated conditioning w.r.t. $j = 1, ..., n$ and $0 \leq t_1 < ... < t_n \leq T$ we obtain
$$\begin{aligned}\mathbb{E}Z_T \exp(i(\sum_{j=1}^n (\vartheta_j, W_{t_j} - W_{t_{j-1}})) &= \mathbb{E}Z_0 \exp(-\frac{1}{2}\sum_{k=1}^n (t_j - t_{j-1})|\vartheta|^2) \\ &= 0 \quad \text{for all } \vartheta_1, ..., \vartheta_n \in \mathbb{R}^d\end{aligned}$$
A further approximation argument shows that
$$\mathbb{E}Z_T f(W_{t_n} - W_{t_{n-1}}, ..., W_{t_1}) = 0$$
for all $0 \leq t_1 < ... < t_n \leq T$ and all $f : (\mathbb{R}^d)^n \longrightarrow \mathbb{C}$ bounded and continuous.
This implies $Z_T = 0$, due to
$$\begin{aligned}&\mathbb{E}Z_T f(W_{t_n} - W_{t_{n-1}}, ..., W_{t_1}) = 0 \quad \text{for all } f \in C_b((\mathbb{R}^d)^n, \mathbb{C}) \\ \Rightarrow\ &\mathbb{E}Z_T \mathbb{1}_A = 0 \quad \text{for all } A \in \sigma(W_{t_1}, ..., W_{t_n}) \\ \Rightarrow\ &\mathbb{E}Z_T \mathbb{1}_A = 0 \quad \text{for all } A \in \mathcal{G}_T \\ \Rightarrow\ &Z_T = 0\end{aligned}$$
Since $T > 0$ is arbitrary, $Z_\infty := \lim_{T\to\infty} Z_T = 0$ □

Remark 3.6.4. The integrand in the integral representation is unique:
For $H, K \in L_{2,d}(\mu_W)$ with
$$I(H) = I(K)$$
we have
$$0 = I(H) - I(K) = I(H - K)$$
and
$$||H - K||_{L_{2,d}(\mu_W)} = ||I(H - K)||_{L_{2,0}(\mathcal{G}_\infty)} = ||0||_{L_{2,0}(\mathcal{G}_\infty)} = 0$$
Hence $H = K$

By localising we obtain a local version of the preceding martingale representation theorem.

Theorem 3.6.5. *Let $(M_t)_{t\geq 0}$ be a local martingale according to a Wiener-process filtration $(\mathcal{G}_t)_{t\geq 0}$ generated by some Wiener-process W. Then, M has continuous paths and there exists a previsible d-dimensional process $H = (H^{(1)}, ..., H^{(d)})$ such that*

$$\sum_{k=1}^{d} \int_0^t (H_s^{(k)})^2 ds < \infty \qquad \text{for all } t \geq 0 \ \mathbb{P} \quad a.s.$$

and

$$\begin{aligned} M_t &= M_0 + \int_0^t H_s dW_s \qquad \text{for all } t \geq 0 \\ &:= M_0 + \sum_{k=1}^{d} \int_0^t H_s^{(k)} dW_s^{(k)}. \end{aligned}$$

Remark 3.6.6. We don't assume continuity of paths for M. This shows that this theorem is more than only a local version of 3.6.3.

Proof. a) Continuity of paths M:

We may assume that $M_0 = 0$ and can localise M into a uniformly integrable martingale which has limiting variable $M_\infty \in L_1(\mathcal{G}\infty)$. The point is that M needs not to be L_2-integrable such that 3.6.3 is not applicable. But M_∞ can be approximated in L_1 by a sequence of bounded random-variables $M_\infty^{(n)}$ that have all integral-representations with continuous paths. By taking a suitable subsequence and using Borel-Cantelli one can show that the paths of M are approximated uniformly by the paths of the continuous martingale that belong to $M_\infty^{(n)}$. As uniform limit the paths of M are therefore continuous themselves.

b) How to show the integral representation:

Due to a) M has continuous paths and can therefore be localised in $b\mathfrak{M}_c$ by

$$\tau_n = \inf\{t \geq 0 : |M_t| \geq n\}.$$

Each M^{τ_n} has an integral representation due to 3.6.3

$$M^{\tau_n} = H^{(n)} \cdot W$$

with a previsible process $H^{(n)} \in L_{2,d}(\mu_W)$.

Note that $M_0 = 0$ is assumed.

The sequence of previsible processes $(H^{(n)})_{n\in\mathbb{N}}$ is consistent in some way, i.e.

$$H^{(n)} \mathbb{1}_{(0,\tau_k]} = H^{(k)} \mathbb{1}_{(0,\tau_k]} \qquad \text{for all } 1 \leq k \leq n$$

The sought-after process H can be built by the sequence $H^{(n)}$ by

$$H = \sum_{n=1}^{\infty} H^{(n)} \mathbb{1}_{(\tau_{n-1},\tau_n]}$$

Then $H \in L^2_{\text{loc}}(W)$ due to

$$\begin{aligned}\mathbb{E}\int_0^\infty |H_s|^2 \mathbb{1}_{(0,\tau_n]} ds &= \mathbb{E}\int_0^\infty |H^{(n)}|^2 \mathbb{1}_{(0,\tau_n]} ds \\ &= \mathbb{E}\int_0^\infty |H_s^{(n)}|^2 ds < \infty\end{aligned}$$

which implies

$$\int_0^t |H_s|^2 ds < \infty \qquad \text{for all } t \geq 0 \ \mathbb{P} \ \text{ a.s.} \quad .$$

Hence we can integrate H w.r.t. W and it follows

$$\begin{aligned}M^{\tau_n} &= H^{(n)} \mathbb{1}_{(0,\tau_n]} \cdot W \\ &= H \mathbb{1}_{(0,\tau_n]} \cdot W \\ &= (H \cdot W)^{\tau_n} \qquad \text{for all } n \in \mathbb{N}\end{aligned}$$

This implies that M is indistinguishable from $H \cdot M$.

□

3.6.3 Theorem of Girsanov

As last main theorem we would give a very general version of Girsanov's theorem. As seen in the Black-Scholes model this is of great importance in finance since it allows to compute equivalent martingale measures with the help of suitable exponential martingales. The first version clarifies the structure of density processes of equivalent measures.

Theorem 3.6.7. *Let $\mathbb{P}, Q$ be equivalent probability measures on $(\Omega, \mathcal{F}_\infty)$ with density process*

$$\left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_t} =: L_t \quad t \geq 0$$

that has continuous paths $\mathbb{P}-$almost sure.

(i) if we define the local martingale X by

$$X_t = \int_0^t \frac{1}{L_s} dL_s$$

for all $t \geq 0$, then

$$L_t = L_0 \exp(X_t - \frac{1}{2}\langle X \rangle_t) \quad \textit{for all } t \geq 0$$

and L is a solution to

$$dL(t) = L(t)dX(t).$$

(ii) If M is a continuous local martingale w.r.t. $\mathbb{P}$, then

$$N_t = M_t - \langle M, X \rangle_t \quad \textit{for all } t \geq 0$$

defines a local $Q-$martingale, whose quadratic variation w.r.t. Q coincide with that w.r.t. M.

Note:

$$N_t = M_t - \langle M, X \rangle_t = M_t - \int_0^t \frac{1}{L_s} d\langle M, L \rangle_s$$

Proof. (i): Due to Q equivalent to $\mathbb{P}$ the density-process is strictly positive $\mathbb{P}-$ almost sure. $\left(\frac{1}{L_t}\right)_{t\geq 0}$ is assumed to have continuous paths and can therefore be integrated according to L_t Hence, the local martingale

$$X_t = \int_0^t \frac{1}{L_s} dL_s \quad \text{for all } t \geq 0$$

is well defined and we can apply Ito's formula to

$$Y_t = \ln L_t.$$

It follows

$$\begin{aligned} dY_t &= \frac{1}{L_t} dL_t - \frac{1}{2}\frac{1}{L_t^2} d\langle L_t \rangle \\ &= dX_t - \frac{1}{2} d\langle X \rangle_t \end{aligned}$$

Thus

$$\ln \frac{L_t}{L_0} = Y_t - Y_0 = X_t - \underbrace{X_0}_{=0} - \frac{1}{2}\langle X \rangle_t$$

and therefore

$$L_t = L_0 \exp(X_t - \frac{1}{2}\langle X \rangle_t).$$

(ii): we have to show: $(N_t L_t)_{t\geq 0}$ is a local $\mathbb{P}-$martingale.

By integration by parts we get

$$dM_tL_t = M_tdL_t + L_tdM_t + d\langle L, M\rangle_t$$

and

$$d\langle M, X\rangle_tL_t = \langle M, X\rangle_tdL_t + L_td\langle M, X\rangle_t = \langle M, X\rangle_tdL_t + \frac{L_t}{L_t}d\langle M, L\rangle_t$$

Hence the stochastic differential of N_tL_t is given by

$$dN_tL_t = dM_tL_t - d\langle M, X\rangle_tL_t = N_tdL_t + L_tdM_t.$$

This implies that NL is a local $\mathbb{P}-$martingale and therefore N a local $Q-$martingale. $M = N - \langle M, X\rangle$ is a semi-martingale w.r.t. Q with local martingale part N. Hence

$$\underbrace{\langle M\rangle^{\mathbb{P}}}_{\text{w.r.t. } Q} = \underbrace{\langle N\rangle^{Q}}_{\text{w.r.t. } \mathbb{P}}$$

This means, that by the transition from $\mathbb{P}$ to Q only a term of locally bounded variation is added and this doesn't change the quadratic variation.

□

The question that occurs is the following:
Which properties must a positive martingale L fulfil such that an equivalent martingale measure can be defined with L as density process.

Theorem 3.6.8. *There exists according to $\mathbb{P}$ an equivalent probability measure Q on $(\Omega, \mathcal{F}_\infty)$ with density process*

$$\left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t \quad \textit{for all } t \geq 0$$

if and only if $(L_t)_{t\geq 0}$ is a uniformly integrable martingale with

$$L_\infty = \lim_{t\to\infty} L_t > 0 \quad \mathbb{P} \quad a.s.$$

and

$$\mathbb{E}L_\infty = 1$$

Proof. "'$\Rightarrow$"' Due to $\mathbb{P}$ equivalent to Q on $\mathcal{F}_\infty$ there exists some $\mathcal{F}_\infty-$measurable random variable $D > 0$ and

$$\left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_\infty} = D.$$

The density L_t on $(\Omega, \mathcal{F}_t)$ fulfills

$$L_t = \left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathbb{E}(D|\mathcal{F}_t) \quad \text{for all } t \geq 0$$

Hence, $(L_t)_{t\geq 0}$ is uniformly integrable with

$$L_\infty = \lim_{t\to\infty} L_t = D > 0 \quad \mathbb{P} \text{ a.s.}$$

and

$$\mathbb{E}L_\infty = \mathbb{E}D = 1$$

"'$\Leftarrow$"' Define the probability measure Q by

$$\left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_\infty} = L_\infty$$

Due to $L_\infty > 0$ $\mathbb{P}$ a.s. the measures Q and $\mathbb{P}$ are equivalent.
For each $t > 0$ we obtain

$$L_t = \mathbb{E}(L_\infty|\mathcal{F}_t) = \left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_t}.$$

Hence $(L_t)_{t\geq 0}$ defines a density process of Q according to $\mathbb{P}$. □

In the case of a Wiener-filtration the structure of the density-process can be given more precisely.

Theorem 3.6.9. *Let $(W_t)_{t\geq 0}$ be some $d-$dimensional Wiener-process and $(\mathcal{G}_t)_{t\geq 0}$ the corresponding Wiener-filtration. Then*
(i) If Q is according to $\mathbb{P}$ an equivalent probability measure on $(\Omega, \mathcal{G}_\infty)$, then there exists a previsible process $(H_t)_{t\geq 0}$ with

$$\mathbb{P}\left(\int_0^t |H_s|^2 ds < \infty \text{ for all } t \geq 0\right) = 1$$

and

$$L_t = \left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \exp\left(\int_0^t H_s dW_s - \frac{1}{2}\int_0^t |H_s|^2 ds\right)$$

for all $t \geq 0$. Furthermore according to Q

$$\overline{W_t} = W_t - \int_0^t H_s ds \quad t \geq 0$$

defines a $d-$dimensional Wiener-process.
(ii) If according to some previsible $\mathbb{R}^d-$valued process $(H_t)_{t\geq 0}$

$$L_t = \exp\left(\int_0^t H_s dW_s - \frac{1}{2}\int_0^t |H_s|^2 ds\right) \quad t \geq 0$$

defines a uniformly integrable $\mathbb{P}-$martingale with

$$L_\infty = \lim_{t\to\infty} L_t > 0 \quad \mathbb{P} \quad a.s.$$

then there exists a probability measure Q equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_\infty)$, such that

$$\left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{G}_t} = L_t \quad t \geq 0.$$

According to Q a $d-$dimensional Wiener-process is defined by

$$\overline{W} = W_t - \int_0^t H_s ds.$$

Proof. (i) The assertion follows from Girsanov part I, the martingale representation theorem part II and Lévy's theorem.
$(L_t)_{t\geq 0}$ is a strictly positive martingale w.r.t. a Wiener-filtration and has therefore continuous paths.
For

$$X_t = \int_0^t \frac{1}{L_s} dL_s$$

we obtain as in 3.6.7

$$L_t = \left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{G}_t} = L_0 \exp(X_t - \frac{1}{2}\langle X\rangle_t) \quad \text{for all } t \geq 0$$

Since $\mathcal{G}_0$ is trivial $L_0 \equiv 1$.
Due to 3.6.5 the local martingale $(X_t)_{t\geq 0}$ has a representation

$$X_t = \int_0^t H_s dW_s$$

with

$$\mathbb{P}\left(\int_0^t |H_s|^2 ds < \infty \text{ for all } t \geq 0\right) = 1$$

(Note: $X_0 = 0$). Due to

$$X_t = \sum_{k=1}^d \int_0^t H_s^{(k)} dW_s^{(k)}$$

we obtain

$$
\begin{aligned}
\langle X\rangle_t &= \sum_{k=1}^{d}\langle\int_0^{\cdot} H_s^{(k)}dW_s^{(k)}\rangle_t + \sum_{\substack{j,l=1\\ j\neq l}}^{d}\underbrace{\langle\int_0^{\cdot} H_s^{(j)}dW_s^{(k)}, \int_0^{\cdot} H_s^{(l)}dW_s^{(l)}\rangle_t}_{=0} \\
&= \sum_{i=1}^{d}\int_0^t (H_s^{(i)})^2 ds \\
&= \int_0^t |H_s|^2 ds
\end{aligned}
$$

It remains to show: $\overline{W_t} = W_t - \int_0^t H_s dW_s$ is a $d-$dimensional Wiener-process w.r.t. Q. With Girsanov part I this follows for the $i-$th coordinate, since

$$
\begin{aligned}
W_t^{(i)} - \langle W^{(i)}, X\rangle_t &= W_t^{(i)} - \langle W^{(i)}, \sum_{j=1}^{d}\int_0^{\cdot} H_s^{(i)}dW_s^{(i)}\rangle_t \\
&= W_t^{(i)} - \sum_{j=1}^{d}\int_0^t H_s^{(i)} d\underbrace{\langle W^{(i)}, W_s^{(j)}\rangle_s}_{=0 \text{ for } i\neq j} \\
&= W_t^{(i)} - \int_0^t H_s^{(i)} ds
\end{aligned}
$$

is a martingale w.r.t. Q for all $1 \leq i \leq d$. Furthermore

$$\langle \overline{W}^{(i)}\rangle = \langle W^{(i)}\rangle_t = t$$

and

$$\langle \overline{W}^{(i)}, \overline{W}^{(j)}\rangle_t = \langle W^{(i)}, W^{(j)}\rangle_t = 0 \quad \text{for all } i \neq j.$$

Therefore the assumption of Lévy's theorem are satisfied and $\overline{W}$ defines a Wiener-process w.r.t. Q.
(ii) follows from Girsanov part II. Note

$$\mathbb{E}L_t = 1 \quad \text{for all } t \geq 0$$

and

$$\mathbb{E}L_\infty \overset{\text{u.i.m.}}{=} \mathbb{E}\lim_{t\to\infty} L_t = 1 > 0$$

□

We give two applications in finance. Let $(W_t)_{t\geq 0}$ be some Wiener-Process according to $(\mathcal{F}_t)_{t\geq 0}$. The process

$$L_t = \exp(\vartheta W_t - \frac{1}{2}\vartheta^2 t) \quad \text{with } \vartheta \neq 0$$

is a strictly positive martingale with $\mathbb{E}L_t = 1$, but $L_t \overset{t\to\infty}{\longrightarrow} 0$ $\mathbb{P}$ a.s. , which shows that $(L_t)_{t\geq 0}$ is not uniformly integrable. Therefore there exists no probability measure Q on $(\Omega, \mathcal{F}_\infty)$ equivalent to $\mathbb{P}$ with density-process

$$\left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t \quad \text{for all } t \geq 0.$$

According to a finite time-interval $[0, T]$ this can be done, since by stopping in T the process L is transformed into a uniformly integrable martingale.
More precise:
$L_t^T := L_{t\wedge T}$ defines a uniformly integrable martingale with

$$L_\infty^T = \lim_{t\to\infty} L_{t\wedge T} = L_T > 0.$$

Girsanov implies that there exists a probability measure Q_T on $(\Omega, \mathcal{F}_\infty)$ equivalent to $\mathbb{P}$ with density process

$$\left.\frac{dQ_T}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^T = \begin{cases} L_t & t < T \\ L_T & t \geq T \end{cases}$$

A slight generalisation is the following
Let $\vartheta : [0, \infty) \longrightarrow \mathbb{R}$ be measurable with

$$\int_0^\infty \vartheta(s)^2 < \infty.$$

We define

$$L_t = \exp(\int_0^t \vartheta_s dW_s - \frac{1}{2}\int_0^t \vartheta_s^2 ds) \quad t \geq 0.$$

Then $\int_0^\infty \vartheta_s dW_s$ is a $\mathcal{N}(0, \int_0^\infty \vartheta_s^2 ds)-$ distributed random variable and

$$L_\infty = \lim_{t\to\infty} L_t = \exp(\int_0^\infty \vartheta_s dW_s - \frac{1}{2}\int_0^\infty \vartheta_s^2 ds) > 0$$

Furthermore for $p > 1$:

$$\begin{aligned}
\mathbb{E}L_t^p &= \mathbb{E}\big(\exp(\int\limits_0^t \vartheta_s dW_s - \frac{1}{2}\int\limits_0^t \vartheta_s^2 ds)\big)^p \\
&= \mathbb{E}\exp(p\int\limits_0^t \vartheta_s dW_s - \frac{1}{2}p\int\limits_0^t \vartheta_s^2 ds) \\
&= e^{-\frac{1}{2}p\int_0^t \vartheta_s^2 ds}\mathbb{E}\exp(p\int\limits_0^t \vartheta_s dW_s) \\
\text{Ist } Y \sim \mathcal{N}(0,\sigma^2) \Rightarrow \mathbb{E}\exp(\lambda Y) = \exp(\tfrac{1}{2}\lambda^2\sigma^2) &= e^{-\frac{1}{2}p\int_0^t \vartheta_s^2 ds}\exp(\frac{1}{2}p^2\int\limits_0^t \vartheta_s^2 ds) \\
&= \exp(\underbrace{\frac{1}{2}p(p-1)}_{>0}\int\limits_0^t \vartheta_s^2 ds) \\
&\le \exp(\frac{1}{2}p(p-1)\int\limits_0^\infty \vartheta_s^2 ds)
\end{aligned}$$

Hence $\sup_{t\ge 0}\mathbb{E}L_t^p < \infty$, which shows the uniform integrability of $(L_t)_{t\ge 0}$.
Girsanov implies that according to $\mathbb{P}$ there exists an equivalent probability measure Q on $(\Omega, \mathcal{F}_\infty)$ with density process

$$\frac{dQ}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = L_t \quad \text{for all } t \ge 0.$$

4 Ito-Process Models of Finance

The aim of this chapter is

- general modelling of financial markets without jumps,
- Characterisation of arbitrage-free markets,
- arbitrage-free pricing of financial derivatives.

4.1 Basic Concepts

4.1.1 Motivation

We have introduced the Black-Scholes model in chapter 2 and seen that even with such a simple model a market-price valuation of financial derivatives is possible. But one cannot ignore the fact that from a pricing perspective only one parameter, the stock-volatility σ determines the prices of derivatives. From that point of view the model is too simple and should be extended to cover real financial markets better. With the framework of stochastic analysis the mathematicians have the tools at hand to fulfill this demand.
The pitfalls of the Black-Scholes model are observable by the so called smile-effect. To explain this we consider a stock with an initial price $x > 0$ and fix a maturity T for those calls with different strikes

$$K_1 < K_2 < K_3 < \cdots < K_n$$

that are quoted in the market. Thus we observe the call's market prices

$$c(K_1) > c(K_2) > \cdots > c(K_n).$$

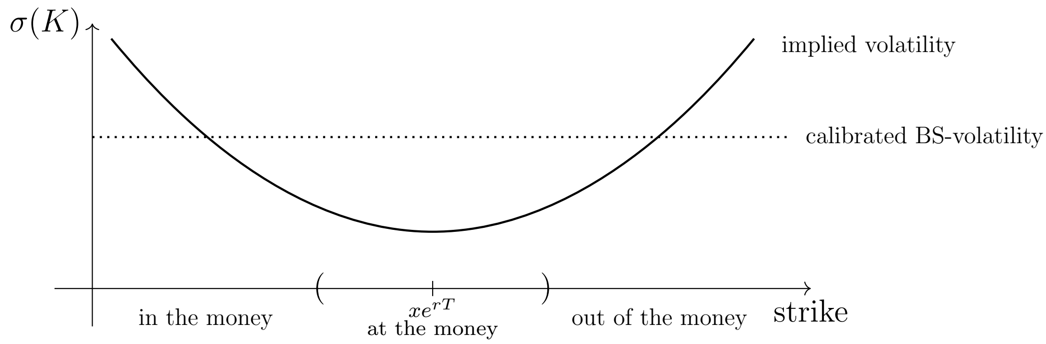
For each call we look at that volatility that explains the market price best. This means that we have to solve the equation

$$c(x, T, K, \sigma) = c_{market}(K)$$

in σ. This solution

$$\sigma_{impl}(K)$$

denotes the so called implied volatility of the call option with strike K. If the BS-modell is correct all implied volatilities must coincide. But in real markets we observe the following dependence on the strike



A Black-Scholes model with a calibrated BS-volatility would under estimate the implied volatility for options that are deep out of the money and deep in the money and would over estimate the implied volatility for options that are at the money.
From this smile-effect we get a first idea how the model can be improved to fully explain observable market prices of calls by their prices evaluated in the model. The idea is that the volatility of the stock should depend on stock-price and time to cover also different maturities of calls. As we will see later this idea leads to a complete diffusion model that is able to fully explain market-prices of calls and puts of different maturities and strikes. The formula that stands behind this is the so called formula of Dupire.

4.1.2 Technical Remarks

In a continuous time market-model we consider the price-process of risky assets over a finite-time interval $[0, T)$ and assume that these are semi-martingales. To be more precise we assume that we have fixed a filtered probability space

$$(\Omega, (\mathcal{F}_t)_{0 \leq t < T}, \mathcal{F}_T, \mathbb{P})$$

that satisfies the usual conditions. The sigma field $\mathcal{F}_T$ can be seen as collection of all information up to time T,

$$\mathcal{F}_T = \sigma(\{\mathcal{F}_t : t < T\}).$$

The term semi-martingale can be defined in this setting as in stochastic analysis if we replace ∞ by T.

Definition 4.1.1. *A stochastic-process $M = (M_t)_{0 \leq t < T}$ with $M_0 = 0$ is called local-martingale with time-horizon T if there exists an increasing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that*

(i) $\sup_{n \in N} \tau_n = T$,

(ii) M^{τ_n} *is a martingale for each* $n \in \mathbb{N}$.

We say $M \in \mathfrak{M}^0_{c,loc}$.
A stochastic-process M with $\mathcal{F}_0$-measurable starting variable M_0 is called a local-martingale with time horizon T if $M - M_0 \in \mathfrak{M}^0_{c,loc}$ and we denote by $\mathfrak{M}_{c,loc}$ the space of all local-martingales with time-horizon T.

Since the time-horizon is fixed in our financial modelling it will be omitted in the notation. In finance it is reasonable to know if local-martingales can be extended at T.

Definition 4.1.2. *A local-martingale $M \in \mathfrak{M}_{c,loc}$ can be extended at T if*

$$\lim_{t \uparrow T} M_t = M_T$$

exists a.s..

By the quadratic variation process we can decide if a local-martingale can be extended.

Proposition 4.1.3. *Let $M \in \mathfrak{M}_{c,loc}$ with*

$$\lim_{t \uparrow T} \langle M \rangle_t = \langle M \rangle_T < \infty \quad \mathbb{P} \quad a.s. \quad .$$

Then M can be extended at T.

A semi-martingale with time horizon T can be defined in the following way.

Definition 4.1.4. *An adapted stochastic process $(A_t)_{0 \leq t < T}$ is called continuous FV-process with horizon T, if A has continuous paths, that have finite variation on each interval $[0, t], t < T$ almost surely. With FV_c resp. FV_c^0 we denote the space of all continuous FV-processes resp. the subspace of continuous $FV-$processes that are starting from the origin.*
A process X is called continuous semi-martingale with time-horizon T, if there exists a decomposition

$$X = X_0 + M + A$$

with $M \in \mathfrak{M}^0_{c,loc}, A \in FV_c^0$ and $\mathcal{F}_0$ measurable X_0.

By localising in T instead of ∞ the stochastic integral-process can be defined.

Definition 4.1.5. *For $M \in \mathfrak{M}_{c,loc}$ we define the space $L^2_{loc}(M)$ by all those previsible $(H(t))_{0 \leq t < T}$ such that $\int_0^t H(s)^2 d\langle M \rangle_s < \infty$ $\mathbb{P}-$ almost sure for all $0 \leq t < T$.*
For each $H \in L^2_{loc}(M)$ we can define the integral-process

$$\left(\int_0^t H_s dM_s \right)_{0 \leq t < T}$$

which is a continuous local martingale. If this local-martingale can be extended at T, i.e.

$$\lim_{t \nearrow T} \int_0^t H_s dM_s \, \mathbb{P} \quad a.s.$$

then we may define

$$\int_0^T H_s dM_s = \lim_{t \nearrow T} \int_0^t H_s dM_s.$$

For $A \in FV_c$ *the space* $L_{loc}(A)$ *can be defined by all those progressively measurable process* K *such that*

$$\int_0^t |K_s| dFV_{[0,s]}(A) < \infty \quad \mathbb{P}- \quad a.s. \quad \text{for all } 0 \leq t < T.$$

Then path-wise $\left(\int_0^t K_s dA_s \right)_{0 \leq t < T}$ *can be defined.*

4.1.3 Model Specification

In the following we establish a model of a financial market that consists of the following ingredients.

- trading period $[0, T)$
- d risky assets (stocks)
- numeraire asset (often a money market account) serves as clearing asset. This means that prices will be quoted not in money but in shares of the numeraire asset.

We stipulate the following assumptions on the market
1. Assumption
The source of randomness is given by an n-dimensional Wiener-process

$$W = (W_1, ..., W_n).$$

There exists some filtered probability space $(\Omega, (\mathcal{F}_t)_{0 \leq t < T}, \mathbb{P})$ and a n-dimensional Wiener-process W such that $(\mathcal{F}_t)_{0 \leq t < T}$ coincides with that Wiener-filtration that is generated from W. In particular the filtration satisfies the usual conditions.
Shortly: The Wiener-process W determines the randomness in the market.
2. Assumption For each $1 \leq i \leq d$ let $(S_i(t))_{0 \leq t < T}$ denote the price-process of the $i-$th risky asset. Then $(S_i(t))_{t \geq 0}$ is assumed to be a strictly positive continuous semi-martingale. In particular

$$\mathbb{P}(S_i(t) > 0 \quad \text{for all } 0 \leq t < T) = 1 \quad i = 1, ..., d.$$

3. Assumption
The price process $(N(t))_{0 \leq t < T}$ of the numeraire asset is a strictly positive continuous semi-martingale.

Remark 4.1.6. - The first assumption is the actual restrictive requirement on the market. It determines which latent randomness drives the prices of the risky assets.

- The second assumption is necessary and therefore natural in order to obtain markets that are free of arbitrage.

  Shortly: All reasonable markets must fulfill the second assumption.

- The third assumption can be seen similar to the second.

- The existence of a numeraire asset is important in order to ensure a sufficient flexibility of trading. That often enables an opportunity to replicate the derivative's payoff by a trading strategy which leads to a unique price settlement. In most of the models the numeraire asset is given by a money market account and this will be denoted by $(\beta(t))_{0\le t<T}$.

Definition 4.1.7. *A numeraire asset is called money-market account, if* $(N(t))_{0\le t<T}$ *is a strictly positive* FV_c- *process.*

Remark 4.1.8. Often the money-market account is seen as risk-free asset. This is so far justified as its fluctuations are considerably less than those of the risky assets.

<u>Conclusions:</u>
1. Price process of the i-th risky asset:
Ito's formula will be applied to

$$X_i(t) = \ln S_i(t) \quad \text{ for all } 0 \le t < T.$$

This provides

$$dX_i(t) = \frac{1}{S_i(t)} dS_i(t) - \frac{1}{2}\frac{1}{S_i(t)^2} d\langle S_i\rangle_t.$$

The semi-martingale

$$Y_i(t) = \int_0^t \frac{1}{S_i(u)} dS_i(u), \quad \text{ for all } 0 \le t < T$$

has a quadratic variation process

$$\langle Y_i\rangle_t = \int_0^t \frac{1}{S_i(u)^2} d\langle S_i\rangle_u.$$

Hence

$$dX_i(t) = dY_i(t) - \frac{1}{2} d\langle Y_i\rangle_t$$

and therefore

$$X_i(t) = X_i(0) + Y_i(t) - \frac{1}{2}\langle Y_i\rangle_t.$$

Thus we obtain

$$\begin{aligned}
S_i(t) &= S_i(0)\exp\left(\ln S_i(t) - \ln S_i(0)\right) \\
&= S_i(0)\exp\left(Y_i(t) - \frac{1}{2}\langle Y_i\rangle_t\right).
\end{aligned}$$

This means that S_i satisfies the stochastic differential equation

$$dS_i(t) = S_i(t)dY_i(t), \qquad \text{for all } 0 \le t < T$$

with initial value $S_i(0) \in (0,\infty)$.
The semi-martingale Y_i haa a decomposition of the form

$$Y_i(t) = M_i(t) + C_i(t), \qquad \text{for all } 0 \le t < T$$

with $M \in \mathfrak{M}^0_{c,\mathrm{loc}}, C \in FV^0_c$.
Due to $\langle Y_i\rangle = \langle M_i\rangle$ we obtain

$$S_i(t) = S_i(0)\exp\left(M_i(t) - \frac{1}{2}\langle M_i\rangle_t\right)\exp(C_i(t)), \qquad \text{for all } 0 \le t < T.$$

Hence S_i fulfills the SDE

$$dS_i(t) = S_i(t)(dM_i(t) + dC_i(t)).$$

4. Assumption
For each $1 \le i \le d$ the process C_i has $\mathbb{P}-$almost sure absolutely continuous paths w.r.t. Lebesgue-measure, i.e. there exist progressively measurable processes $(\mu_i(t))_{0\le t<T}$ with

$$\int_0^t |\mu_i(s)|ds < \infty \qquad \text{for all } 0 \le t < T$$

such that

$$C_i(t) = \int_0^t \mu_i(s)ds.$$

Hence

$$S_i(t) = S_i(0)\exp\left(M_i(t) - \frac{1}{2}\langle M_i\rangle_t\right)\exp\left(\int_0^t \mu_i(s)ds\right)$$

resp.

$$dS_i(t) = S_i(t)(dM_i(t) + \mu_i(t)dt)$$

with initial value $S_i(0)$.

Due to the fact that the filtration is a Wiener-filtration the local martingale M_i can br represented by

$$\begin{aligned} M_i(t) &= \int_0^t \sigma_i(s)dW(s) \\ &= \sum_{j=1}^n \int_0^t \sigma_{ij}(s)dW_j(s) \end{aligned}$$

with previsible processes $\sigma_{i1}, ..., \sigma_{in}$, such that

$$\int_0^t |\sigma_i(s)|^2 ds < \infty \qquad \text{for all } 0 \leq t < T.$$

The price-process of the $i-$th risky assets therefore satisfies

$$S_i(t) = S_i(0) \exp\left(\int_0^t \sigma_i(s)dW(s) - \frac{1}{2}\int_0^t |\sigma_i(s)|^2 ds\right) \exp\left(\int_0^t \mu_i(s)ds\right)$$

for all $0 \leq t < T$ and thus

$$\begin{aligned} dS_i(t) &= S_i(t)(\mu_i(t)dt + \sigma_i(t)dW(t)) \\ &= S_i(t)(\mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)dW_j(t)). \end{aligned}$$

2. Numeraire Asset:
An analogous argumentation yields the existence of an
$\mathbb{R}^d-$valued previsible process $(\sigma_N(t))_{0\leq t<T}$ and some progressively measurable process $(\mu_N(t))_{0\leq t<T}$ such that

$$\int_0^t |\sigma_N(s)|^2 ds < \infty \qquad \text{for all } 0 \leq t < T$$

and

$$\int_0^t |\mu_N(s)| ds < \infty \qquad \text{for all } 0 \leq t < T.$$

Hence

$$N(t) = N(0) \exp\left(\int_0^t \sigma_N(s)dW(s) - \frac{1}{2}\int_0^t |\sigma_N(s)|^2 ds\right) \exp\left(\int_0^t \mu_N(s)ds\right)$$

for all $0 \le t < T$.
Therefore

$$dN(t) = N(t)(\mu_N(t)dt + \sigma_N(t)dW(t))$$
$$= N(t)(\mu_N(t)dt + \sum_{j=1}^{n} \sigma_{Nj}(t)dW_j(t).$$

If $\sigma_N(t) \equiv 0$, then the numeraire asset is a money-market account with random interest rate $r(t) = \mu_N(t)$. Hence

$$\frac{N(t)}{N(0)} = \exp\left(\int_0^t r(s)ds\right) = \beta(t) \quad \text{for all } 0 \le t < T.$$

and

$$d\beta(t) = \beta(t)r(t)dt, \quad \beta(0) = 1.$$

Some examples are given in the following.

Example 4.1.9. Examples are

a) The classic one-dimensional Black-Scholes model
- constant volatility $\sigma > 0$
- constant rate of return $\mu \in \mathbb{R}$
- constant interest rate $r \in \mathbb{R}$
- one-dimensional Wiener-process, that drives the stock prices.

This means:

$$dS(t) = S(t)(\mu dt + \sigma dW(t))$$

with initial value $S_0 \in (0, \infty)$.

$$S(t) = S(0)e^{\mu t} \exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)$$

and

$$dN(t) = N(t)rdt, \quad N(0) = 1$$

hence

$$N(t) = e^{rt} = \beta(t), \quad \text{for all } 0 \le t < T.$$

b) The classic multi-dimensional Black-Scholes Modell
- d stocks
- n independently driving Wiener-processes
- constant volatility matrix $\sigma \in \mathbb{R}^{d \times n}$
- d konstant rate of returns $\mu_1, ..., \mu_d$.

This means

$$dS_i(t) = S_i(t)(\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t))$$

with $S_i(0) \in (0, \infty)$ for all $1 \le i \le d$ respectively

$$S_i(t) = S_i(0)e^{\mu_i t} \exp\left(\sum_{j=1}^{n} \sigma_{ij} W_j(t) - \frac{1}{2} \sum_{j=1}^{n} \sigma_{ij}^2 t \right) \quad \text{for all } 1 \le i \le d.$$

The money-market account behaves as in a).

c) The multi-dimensional Black-Scholes model with deterministic coefficients as in b), but replace $\mu_1, ..., \mu_d$ and σ by functions $\mu_1, ..., \mu_d : [0, T) \longrightarrow \mathbb{R}$ and $\sigma : [0, T) \longrightarrow \mathbb{R}^{d \times n}$ such that

$$\int_0^t |\mu(s)| ds < \infty \quad 0 \le t < T$$

and

$$\int_0^t \|\sigma(s)\|^2 ds < \infty \quad 0 \le t < T.$$

Then

$$dS_i(t) = S_i(t)(\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t) dW_j(t)), \quad \text{for all } 1 \le i \le d.$$

d) The multi-dimensional diffusion model

- volatility matrix $\sigma : [0, T) \times (0, \infty)^d \longrightarrow \mathbb{R}^{d \times n}$
- rate of returns function $\mu : [0, T) \times (0, \infty)^d \longrightarrow \mathbb{R}^d$

Then, the $d-$dimensional price-process of the risky assets are a strong solution of

$$dS_i(t) = S_i(t)(\mu_i(t, S(t))dt + \sum_{j=1}^{n} \sigma_{ij}(t, S(t)) dW_j(t)), \quad \text{for all } 1 \le i \le d.$$

The money-market account satisfies

$$d\beta(t) = \beta(t) r(t, S(t)) dt$$

i.e.

$$\beta(t) = \exp\left(\int_0^t r(u, S(u)) du \right).$$

Important: in a diffusion-model S is a $d-$dimensional Markov-process as strong solution of some SDE.

4.1.4 Trading

In the following trading will be mathematically specified. We consider a financial market with d risky assets and a numeraire asset as stated in the preceding section.

- maximal trading period $[0, T)$
- trading is modelled by a previsible process (K, H) such that $K = (K(t))_{0\leq t<T}$ is real-valued and integrable w.r.t. $(N(t))_{0\leq t<T}$.
  The process $H = (H(t))_{0\leq t<T}$ is $d-$dimensional and integrable according to S. This means
  $$\int_0^t H_i(u)dS_i(u)$$
  is well defined for each $0 \leq t < T$ and $i = 1, \cdots, d$.
- $K(t)$ corresponds to the number of shares of the numeraire asset at t.
- $H_i(t)$ corresponds to the number of shares of the i-th risky asset at t.
- A trading strategy (K, H) leads to an evolution of wealth by
  $$\begin{aligned} V(t) &= K(t)N(t) + \sum_{i=1}^{d} H_i(t)S_i(t) \\ &= K(t)N(t) + H(t) \cdot S(t) \quad 0 \leq t < T \end{aligned}$$
- The gains-process is defined by
  $$\begin{aligned} G(t) &= \int\limits_{(0,t]} K(u)dN(u) + \sum_{i=1}^{d} \int\limits_{(0,t]} H_i(u)dS_i(u) \\ &= \int\limits_{(0.t]} K(u)dN(u) + \int\limits_{(0,t]} H(u)dS(u) \quad \text{for all } 0 \leq t < T \end{aligned}$$

Definition 4.1.10. *A trading-strategy (K, H) is called self-financing, if the increment of wealth is only caused by the gain of trading, i.e.*

$$V(t) - V(0) = \int\limits_{(0,t]} K(u)dN(u) + \int\limits_{(0,t]} H(u)dS(u) \quad \textit{for all } 0 \leq t < T$$

or in differential notation:

$$\begin{aligned} dV(t) &= K(t)dN(t) + H(t)dS(t) \\ &= K(t)dN(t) + \sum_{i=1}^{d} H_i(t)dS_i(t). \end{aligned}$$

The wealth process can also be quoted in shares of the numeraire asset.
Let

$$S_i^\star(t) := \frac{S_i(t)}{N(t)} \quad \text{for all } 1 \leq i \leq d, 0 \leq t < T$$

and

$$V^\star(t) := \frac{V(t)}{N(t)} \quad \text{for all } 0 \leq t < T.$$

An important fact is that self-financing strategies are uniquely determined by the initial capital $V(0)$ and the trading H in the risky assets.

Proposition 4.1.11. *A trading strategy (K, H) is self-financing if and only if*

$$V^\star(t) = \frac{V(0)}{N(0)} + \int_0^t H(u)dS^\star(u) \quad 0 \leq t < T$$

holds.

Proof. This follows with integration by parts for semi-martingales.
We consider only the case $d = 1 = n$.
'$\underline{\Leftarrow}$' It holds

$$V^\star(t) = V^\star(0) + \int_0^t H(u)dS^\star(u) \quad \text{for all } 0 \leq t < T.$$

and we have to show

$$V(t) = V(0) + \int_0^t K(u)dN(u) + \int_0^t H(u)dS(u) \quad \text{for all } 0 \leq t < T.$$

With integration by parts we obtain

$$dV(t) = d(V^\star(t)N(t)) = V^\star(t)dN(t) + N(t)dV^\star(t) + d\langle V^\star, N\rangle_t.$$

Due to our prerequisite we have

$$dV^\star(t) = H(t)dS^\star(t)$$

Therefore we have to compute $dS^\star(t)$. It holds:

$$dS^\star(t) = d\frac{S(t)}{N(t)} = S(t)d\frac{1}{N(t)} + \frac{1}{N(t)}dS(t) + d\langle S, \frac{1}{N}\rangle_t$$

and

$$\begin{aligned} d\frac{1}{N(t)} &= -\frac{1}{N(t)^2}dN(t) + \frac{1}{N(t)^3}d\langle N\rangle_t \\ &= -\frac{1}{N(t)^2}dN(t) + \frac{1}{N(t)}\sigma_N^2(t)dt \end{aligned}$$

since $dN(t) = N(t)(\mu_N(t)dt + \sigma_N^2(t)dW(t))$.
Hence

$$dS^\star(t) = \frac{1}{N(t)}dS(t) - \frac{S(t)}{N(t)^2}dN(t) + \frac{S(t)}{N(t)}\sigma_N^2(t)dt - \frac{S(t)}{N(t)}\sigma(t)\sigma_N(t)dt.$$

One may notice

$$\begin{aligned} d\langle V^\star, N\rangle_t &= H(t)d\langle S^\star, N\rangle_t \\ &= H(t)\frac{S(t)}{N(t)}\sigma(t)N(t)\sigma_N(t)dt - H(t)\frac{S(t)}{N(t)^2}N(t)\sigma_N(t)N(t)\sigma_N(t)dt \\ &= H(t)S(t)(\sigma(t)\sigma_N(t) - \sigma_N(t)^2)dt. \end{aligned}$$

All together we obtain

$$\begin{aligned} dV(t) &= V^\star(t)dN(t) + N(t)H(t)dS^\star(t) + H(t)S(t)(\sigma(t)\sigma_N(t) - \sigma_N(t)^2)dt \\ &= V^\star(t)dN(t) + H(t)dS(t) - H(t)S^\star(t)dN(t) \\ &\quad + H(t)S(t)\sigma_N^2(t)dt - H(t)S(t)\sigma_N(t)\sigma(t)dt \\ &\quad + H(t)S(t)(\sigma(t)\sigma_N(t) - \sigma_N(t)^2)dt \\ &= H(t)dS(t) + (V^\star(t) - H(t)S^\star(t))dN(t) \\ &= H(t)dS(t) + K(t)dN(t) \end{aligned}$$

due to $V^\star(t) = \frac{V(t)}{N(t)} = \frac{K(t)N(t)+H(t)S(t)}{N(t)} = K(t) + H(t)S^\star(t)$.
'$\Rightarrow$'
This follows more or less with the same lines. □

Remark 4.1.12. The wealth process of a self-financing trading strategy (K, H) quoted in shares of N is uniquely determined by its initial quote $V^\star(0) = \frac{V(0)}{N(0)}$ and the trading in the risky assets by H, since

$$V^\star(t) = V^\star(0) + \int_0^t H(u)dS^\star(u) \quad \text{for all } 0 \le t < T.$$

Contrary, according to an initial quote $V^\star(0)$ and a trading H in the risky assets there exists a unique previsible process $(K(t))_{0\le t<T}$ such that (K,H) is self-financing with wealth-process

$$V^\star(t) = V^\star(0) + \int_0^t H(u)dS^\star(u) \quad \text{for all } 0 \le t < T.$$

Computation of $(K(t))_{0\le t<T}$:
On one hand we have

$$V^\star(t) = K(t) + H(t)S^\star(t)$$

and on the other hand

$$V^\star(t) = V^\star(0) + \int_0^t H(u)dS^\star(u).$$

Hence

$$K(t) = V^\star(0) + \int_0^t H(u)dS^\star(u) - H(t)S^\star(t).$$

4.2 The Fundamental Theorem of Asset Pricing

4.2.1 No Admissible Arbitrage Opportunities

We consider a financial market with

1. d risky assets $S_1, \cdots, S_d$
2. a numeraire asset N

and assume that all price-processes are driven by some n-dimensional Wiener-process W. In the following we would like to give a precise analogon of the fundamental theorem in continuous time. As it turns out this question is by far more difficult than in discrete time and the right mathematical formulation of an arbitrage opportunity not that obvious in order to give a meaningful probabilistic characterisation of arbitrage-free markets. The main difficulty comes from the fact that due to continuous trading several artificial strategies which have no use in reality have to be excluded. We start with a first try, a one to one correspondent formulation of arbitrage opportunities.

Definition 4.2.1. *A self-financing trading strategy (ϕ, H) is called an arbitrage opportunity iff its value process V fulfills*

1. $V(0) \le 0$
2. $V(T) \ge 0$
3. $\mathbb{P}(V(T) - V(0) > 0) > 0$

Note that the existence of a limit $\lim\limits_{t \nearrow T} V(t)$ is assumed.
An equivalent formulation by only considering trading in the risky assets can be done in the following way.

Proposition 4.2.2. *Assume* $N(T) := \lim\limits_{t \nearrow T} N(t) > 0\ \mathbb{P}\quad a.s.\quad .$
Then there exists an arbitrage opportunity if and only if there exists some previsible process $(H(t))_{0 \leq t < T}$ *such that*

$$\int_0^T H(u) dS^\star(u) \geq 0 \text{ and } \mathbb{P}\left(\int_0^T H(u) dS^\star(u) > 0\right) > 0.$$

Proof. The assumption ensures, that $\lim\limits_{t \nearrow T} V(t)$ exists if and only if $\lim\limits_{t \nearrow T} V^\star(t)$ exists.
Due to

$$V^\star(T) = \underbrace{V^\star(0)}_{=0} + \int_0^T H(u) dS^\star(u)$$

the claim follows. $\square$

This notion of a risk-free profit opportunity is too simple. Even in the Black-Scholes model arbitrage opportunities exist and therefore further restrictions have to be introduced that exclude strategies which have no practical use.

Proposition 4.2.3. *In the Black-Scholes model there are arbitrage opportunities.*

Proof. We consider a Black-Scholes model with $\mu = r$. Then

- $\beta(t) = e^{rt}$
- $dS(t) = S(t)(rdt + \sigma dW(t)), \quad \sigma > 0$
- $S^\star(t) = e^{-rt} S(t)$ is a martingale,
- $dS^\star(t) = S^\star(t) \sigma dW(t)$

We are looking for a previsible process $(H(t))_{0 \leq t < T}$ with

$$\lim_{t \nearrow T} \int_0^t H(u) dS^\star(u) = 1.$$

Then, with the help of H an arbitrage-opportunity can be constructed.
We consider the following approach

$$V^*(t) = \int_0^t H(u) dS^\star(u) = \int_0^t \underbrace{H(u) \sigma S^\star(u)}_{f(u)} dW(u).$$

and choose $H(u)$ such that

$$f(u) = H(u)\sigma S^\star(u), \quad 0 \le u < T$$

is some deterministic function with

$$\int_0^t f(u)^2 du < \infty \quad \text{for all } 0 \le t < T$$

but

$$\int_0^T f(u)^2 du = \infty.$$

One may take

$$f(t) = \frac{1}{\sqrt{T-t}}.$$

Then

$$M(t) := \int_0^t f(u) dW(u)$$

is an L_2 martingale with

$$\langle M \rangle_t = \int_0^t f(u)^2 du \quad \text{for all } 0 \le t < T$$

and

$$\langle M \rangle_T = \lim_{t \nearrow T} \langle M \rangle_t = \infty.$$

We consider the stopping time τ by

$$\tau := \inf\{0 \le t < T : M(t) = 1\}.$$

Then

$$\mathbb{P}(\tau < T) = 1.$$

Define H by

$$H(u) = \begin{cases} \frac{f(u)}{\sigma S^\star(u)} & \text{if } u \le \tau \\ 0 & \text{if } u > \tau \end{cases}.$$

Then

$$\begin{aligned} \int_0^t H(u) dS^\star(u) &= \int_0^t f(u) \mathbb{1}_{(0,\tau]}(u) dW(u) \\ &= \int_0^{t\wedge\tau} f(u) dW(u) \\ &= M(t \wedge \tau) \end{aligned}$$

hence

$$\lim_{t \nearrow T} \int_0^t H(u) dS^\star(u) = M(\tau) = 1.$$

$\square$

Conclusion: The set of possible trading strategies is too rich and must be reasonably reduced.
Request: While trading you must not get into debt up to an arbitrary amount.

Definition 4.2.4. *A self-financing trading strategy (K, H) is called admissible, if there exists some $c > 0$ such that*

$$V^\star(t) \geq -c \quad \textit{for all } 0 \leq t < T.$$

By applying (K, H) a debt of the trader quoted w.r.t. to the numeraire does not exceed c.

Definition 4.2.5. *A financial market is called free of arbitrage if there exists no admissible arbitrage-opportunities. We say that the market fulfills the (NA)-property.*

Excluding non-admissible trading strategies leads to the property that the Black-Scholes model is free of arbitrage, since an equivalent martingale measure exists. In general we would like to define the term equivalent local martingale measure.

Definition 4.2.6. *We consider a financial market as defined in 4.1.3. A probability measure $\mathbb{P}^\star$ on $(\Omega, \mathcal{F}_T)$ is called equivalent local martingale measure, if:*

(i) $\mathbb{P}^\star \sim \mathbb{P}$ *on* $(\Omega, \mathcal{F}_T)$,

(ii) $(S_i^\star(t))_{0 \leq t < T}$ *is a local martingale according to* $\mathbb{P}^\star$ *for all* $1 \leq i \leq d$.

Note that local martingales that are bounded below by a constant form a super-martingale. This can be exploited to show that financial markets with an equivalent local martingale measure are free of arbitrage.

Theorem 4.2.7. *If there exists an equivalent local martingale measure, then there are no admissible arbitrage opportunities.*

Proof. Let (K, H) be some self-financing admissible trading strategy. Then

$$V^\star(t) = V^\star(0) + \int_0^t H(u) dS^\star(u) \quad \text{for all } 0 \leq t < T.$$

Hence $V^\star$ is a local martingale w.r.t. $\mathbb{P}^\star$ with

$$V^\star(t) \geq -c \quad \text{for all } 0 \leq t < T.$$

Therefore $V^\star$ is a super-martingale, that converges due to the martingale-convergence theorem $\mathbb{P}^\star$ a.s. for $t \nearrow T$. Fatou's lemma implies for all s:

$$\begin{aligned}\mathbb{E}^\star(V^\star(T)|\mathcal{F}_s) &= \mathbb{E}^\star(\liminf_{t\nearrow T} V^\star(t)|\mathcal{F}_s)\\ &\overset{\text{Fatou}}{\leq} \liminf_{t\nearrow T} \mathbb{E}^\star(V^\star(t)|\mathcal{F}_s)\\ &\leq V^\star(s).\end{aligned}$$

In particular

$$\mathbb{E}^\star V^\star(T) \leq V^\star(0).$$

□

In the case of financial markets that are driven by a Wiener-process the existence of an equivalent local martingale measure can be deduced from the parameters

- the rate of returns $(\mu(t))_{0\leq t<T}$,
- the volatility matrix $(\sigma(t))_{0\leq t<T}$,
- the interest-rate $(r(t))_{0\leq t<T}$.

by an application of Girsanov's theorem.
This implies that one can examine from the parameters of the model if the market fulfills the (NA)-condition.

Theorem 4.2.8. *We consider a financial market as specified in 4.1.3 and assume that there exists* $N(T) := \lim_{t\nearrow T} N(t)$ *with* $N(T) > 0$ $\mathbb{P}$ *a.s.* .
Then there exists an equivalent local martingale measure $\mathbb{P}^\star$ *if and only if there exists an* $n-$*dimensional previsible process* $(\vartheta(t))_{0\leq t<T}$ *such that*

(i) $\int_0^T |\vartheta(s)|^2 ds < \infty \quad \mathbb{P}$ *a.s.* ,

(ii) $\mu(t)+\sigma(t)(\vartheta(t)-\sigma_N(t)) = (\mu_N(t)-|\sigma_N(t)|^2+\sigma_N(t)\vartheta(t))\mathbb{1}$ *for all* $0 \leq t < T$,

(iii) $\mathbb{E}\exp\left(\int_0^T \vartheta(s)dW(s) - \frac{1}{2}\int_0^T |\vartheta(s)|^2 ds\right) = 1.$

Note, $\mathbb{1} = (\underbrace{1,...,1}_{d})^T$

Proof. '$\Rightarrow$' Let $\mathbb{P}^\star$ be some equivalent local martingale measure. Then the density process

$$L_t = \left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} \qquad \text{for all } 0 \le t < T$$

is a uniformly integrable martingale with

$$L_T = \lim_{t \nearrow T} L_t > 0 \quad \mathbb{P} \ \text{ a.s.} \quad .$$

Furthermore L has a representation as exponential martingale of the from

$$L_t = \exp\left(M(t) - \frac{1}{2}\langle M\rangle_t\right) \qquad \text{for all } 0 \le t < T$$

with local martingale M.
Define

$$\langle M\rangle(T) := \lim_{t \nearrow T} \langle M\rangle_t.$$

Since

$$\frac{M(t)}{\langle M\rangle_t} \longrightarrow 0 \qquad \text{für } t \nearrow T$$

on $\{\langle M(T)\rangle = \infty\}$ we obtain

$$\ln L_t = M(t) - \frac{1}{2}\langle M\rangle_t = \langle M\rangle_t \left(\frac{M(t)}{\langle M\rangle_t} - \frac{1}{2}\right) \longrightarrow -\infty$$

on $\{\langle M\rangle_T = \infty\}$.
But $L_T > 0 \quad \mathbb{P}$ a.s. and therefore

$$\mathbb{P}(\{\langle M\rangle_T = \infty\}) = 0.$$

This implies

$$\langle M\rangle_T < \infty \quad \mathbb{P} \ \text{ a.s.} \quad .$$

Due to the martingale representation theorem there exists some previsible process $(\vartheta(t))_{0 \le t < T}$ such that

$$\int_0^t |\vartheta(s)|^2 ds < \infty \qquad \text{for all } 0 \le t < T$$

and

$$M(t) = \int_0^t \vartheta(s) dW(s) \qquad \text{for all } 0 \le t < T.$$

Due to $\langle M\rangle_T < \infty \quad \mathbb{P}$ a.s.

$$\int_0^T |\vartheta(s)|^2 ds < \infty \quad \mathbb{P} \ \text{ a.s.}$$

and

$$M(T) = \lim_{t \nearrow T} M(t) = \int_0^T \nu(s) dW(s).$$

This implies

$$L_T = \exp\left(\int_0^T \nu(s) dW(s) - \frac{1}{2} \int_0^T |\nu(s)|^2 ds \right)$$

and therefore

$$1 = \mathbb{E} L_T = \mathbb{E} \exp\left(\int_0^T \nu(s) dW(s) - \frac{1}{2} \int_0^T |\nu(s)|^2 ds \right).$$

Therefore (i) and (iii) hold.
ad (ii): For $1 \leq i \leq d$:

$$dS_i(t) = S_i(t)(\mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t) dW_j(t)$$

$$dN(t) = N(t)(\mu_N(t)dt + \sum_{j=1}^n \sigma_{Nj}(t) dW_j(t)$$

Due to Ito's formula

$$\begin{aligned} d\frac{1}{N(t)} &= -\frac{1}{N(t)^2} dN(t) + \frac{1}{N(t)^3} d\langle N \rangle_t \\ &= -\frac{1}{N(t)}(\mu_N(t)dt + \sigma_N(t)dW(t)) + \frac{1}{N(t)}|\sigma_N(t)|^2 dt \\ &= \frac{1}{N(t)}((|\sigma_N(t)|^2 - \mu_N(t))dt - \sigma_N(t)dW(t)). \end{aligned}$$

Integration by parts implies

$$\begin{aligned} d\frac{S_i(t)}{N(t)} &= S_i(t) d\frac{1}{N(t)} + \frac{1}{N(t)} dS_i(t) + d\langle S_i, \frac{1}{N} \rangle_t \\ &= \frac{S_i(t)}{N(t)}((|\sigma_N(t)|^2 - \mu_N(t))dt - \sigma_N(t)dW(t)) \\ &\quad + \frac{S_i(t)}{N(t)}(\mu_i(t)dt + \sigma_i(t)dW(t)) \\ &\quad - \frac{S_i(t)}{N(t)} \sigma_N(t)\sigma_i(t)dt. \end{aligned}$$

Hence, we obtain

$$dS_i^\star(t) = S_i^\star(t)((|\sigma_N(t)|^2 + \mu_i(t) + \sigma_N(t)\sigma_i(t) - \mu_N(t))dt + (\sigma_i(t) - \sigma_N(t))dW(t)). \quad (4.1)$$

Due to Girsanov's theorem

$$W_j^\star(t) = W_j(t) - \int_0^t \vartheta(s)ds \quad 1 \le j \le n, 0 \le t < T$$

are n independent Wiener-processes w.r.t. $\mathbb{P}^\star$.
Plugged into 4.1 we get

$$\begin{aligned} dS_i^\star(t) &= S_i^\star(t)(|\sigma_N(t)|^2 + \mu_i(t) - \sigma_N(t)\sigma_i(t) - \mu_N(t) + (\sigma_i(t) - \sigma_N(t))\vartheta(t))dt \\ &\quad + S_i^\star(t)(\sigma_i(t) - \sigma_N(t))dW^\star(t). \end{aligned}$$

Hence $S_i^\star$ is a local martingale if and only if the dt-term vanishes, i.e.

$$\begin{aligned} &|\sigma_N(t)|^2 + \mu_i(t) + (\sigma_i(t) - \sigma_N(t))\vartheta(t) = \mu_N(t) + \sigma_N(t)\sigma_i(t) \\ \Leftrightarrow &\mu_i(t) + \sigma_i(t)\vartheta(t) - \sigma_i(t)\sigma_N(t) = \mu_N(t) - |\sigma_N(t)|^2 + \sigma_N(t)\vartheta(t). \end{aligned}$$

Hence (ii) follows.
'$\Leftarrow$' Due to (i) and (ii) Girsanov implies that there exists an equivalent probability measure $\mathbb{P}^\star$ with

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \vartheta(s)dW(s) - \frac{1}{2}\int_0^t |\vartheta(s)|^2 ds\right).$$

(ii) implies, that

$$S_i^\star(t) = \frac{S_i(t)}{N(t)}, \quad 0 \le t < T$$

is a local $\mathbb{P}^\star-$ martingale for all $1 \le i \le d$. □

Remark 4.2.9. If N denotes a money market account, then the condition (ii) in 4.2.8 is given by

$$\mu(t) + \sigma(t)\vartheta(t) = r(t)\mathbb{1} \quad \text{for all } 0 \le t < T.$$

Example 4.2.10. a) one-dimensional Black-Scholes model

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sigma dW(t)) \\ d\beta(t) &= \beta(t) r dt \end{aligned}$$

Put $\vartheta = -\frac{\mu - r}{\sigma}$ and

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\vartheta W(t) - \frac{1}{2}\vartheta^2 t\right).$$

Then $\mathbb{P}^\star$ is an equivalent local martingale measure.
According to $\mathbb{P}^\star$ it holds

$$dS(t) = S(t)(rdt + \sigma dW^\star(t)$$

with $W^\star(t) = W(t) - \vartheta t$ or equivalent

$$dS^\star(t) = S^\star(t)\sigma dW^\star(t).$$

b) multi-dimensional Black-Scholes model

$$\begin{aligned} dS_i(t) &= S_i(t)(\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)) \quad \text{for all } 1 \le i \le d, 0 \le t < T \\ d\beta(t) &= \beta(t) r dt \end{aligned}$$

If the equation

$$\mu + \sigma\vartheta = r\mathbb{1}$$

can be solved by $\vartheta \in \mathbb{R}^n$, then there exists an equivalent local martingale measure $\mathbb{P}^\star$ with

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\sum_{j=1}^{n} \vartheta_j W_j(t) - \frac{1}{2}|\vartheta|^2 t\right)$$

and

$$W^\star(t) = W(t) - \vartheta t$$

is a $n-$dimensional Wiener-process w.r.t. $\mathbb{P}^\star$.
In addition

$$dS_i(t) = S_i(t)(r dt + \sum_{j=1}^{n} \sigma_{ij} dW_j^\star(t)).$$

In the case $n = d$ and invertible σ

$$\vartheta = \sigma^{-1}(-(\mu - r\mathbb{1})).$$

c) Black-Scholes with deterministic coefficients

$$\begin{aligned} dS_i(t) &= S_i(t)(\mu_i(t) dt + \sum_{j=1}^{n} \sigma_{ij}(t) dW_j(t)) \\ d\beta(t) &= \beta(t) r dt \end{aligned}$$

If the equation

$$\mu(t) + \sigma(t)\vartheta(t) = r(t)\mathbb{1} \quad \text{for all } 0 \le t < T$$

can be solved by $\vartheta(t) \in \mathbb{R}^n$ and is

$$\int_0^T |\vartheta(s)|^2 ds < \infty$$

then an equivalent local martingale measure can be defined by

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \vartheta(s)dW(s) - \frac{1}{2}\int_0^t |\vartheta(s)|^2 ds\right) \quad 0 \le t < T$$

and

$$W^\star(t) = W(t) - \int_0^t \vartheta(s)dt$$

defines a Wiener-process w.r.t $\mathbb{P}^\star$ such that

$$dS_i(t) = S_i(t)(r(t)dt + \sum_{j=1}^n \sigma_{ij}(t)dW_j^\star(t))$$

according to $\mathbb{P}^\star$ for all $1 \le i \le d$.

4.2.2 No Free Lunch with Vanishing Risk

Note, that the statements in theorem 4.2.7 are not equivalent. The prerequisite of no admissible arbitrage opportunities is not sufficient to deduce the existence of an equivalent local martingale measure. A slight stronger condition has to be assumed. Delbaen and Schachermeyer introduced the so called No Free Lunch with Vanishing Risk condition and proved the First Fundamental Theorem. In the book of Eberlein and Kallsen a very readable definition of NFLVR is given, see Definition 11.48 there.

Definition 4.2.11. *A non-negative random $\mathcal{F}_T$-measurable variable C with $\mathbb{P}(C > 0) > 0$ is called a free lunch with vanishing risk if, for any $\epsilon > 0$ there exists some admissible strategy (ϕ, H) such that its wealth-process V fulfills*

(i) $V(0) < \epsilon$,

(ii) $V(T) \ge C$.

A financial market fulfills the NFLVR condition if no free lunch with vanishing risk exists.

Delbaen and Schachermayer proved in Theorem 1.1 the famous most general First Fundamental Theorem of Asset Pricing.

Theorem 4.2.12. *In a financial market with locally bounded price processes the following statements are equivalent.*

(i) The NFLVR condition holds

(ii) There exists an equivalent local martingale measure $\mathbb{P}^*$, i.e. some probability measure $\mathbb{P}^* \approx \mathbb{P}$ such that the discounted price process S^* is a local-martingale relative to $\mathbb{P}^*$.

Proof. If there exists an equivalent local martingale measure $\mathbb{P}^*$, then as in 4.2.7 one can show that there is no free lunch with vanishing risk. That from the NFLVR condition an equivalent local martingale measure can be constructed is difficult to show and follows with a careful application of tools from functional analysis. We refer to the original paper of Delbaen and Schachermayer. □

Note that price-processes with continuous paths in particular those driven by some Wiener-process are locally bounded and therefore NFLVR implies the existence of an equivalent local martingale measure.
The question arises when an equivalent martingale measure exists. To reply this one should be aware that the NFLVR condition strongly depends on the choice of the numeraire asset since in the definition of admissible trading strategies it is important how the discounted value process is achieved. One outstanding numeraire asset is the so called market portfolio

$$M = N + S_1 + \cdots + S_d.$$

Definition 4.2.13. *A self-financing strategy is called allowable or market-admissible if it is admissible w.r.t. the market portfolio M. Hence there exists some $c > 0$ such that*

$$\hat{V}(t) = \frac{V(t)}{M(t)} \geq -c$$

for each $0 \leq t \leq T$. We say that a financial market fulfills the NFLVR(M) condition if no free lunch with vanishing risk can be financed by market-admissible trading strategies.

Note that each admissible trading strategy is also an allowable trading strategy but not vice versa. Thus the NFLVR(M)-condition is stronger than the NFLVR-condition and it can be shown that from the stronger requirement the existence of an equivalent martingale measure can be deduced.

Theorem 4.2.14. *In a financial market the following statements are equivalent.*

(i) The NFLVR(M)-condition holds.

(ii) There exists an equivalent martingale measure $\mathbb{P}^*$, i.e. $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ and S^* is a martingale w.r.t. $\mathbb{P}^*$.

Proof. $\Longrightarrow$: First one has to verify that there exists an equivalent martingale measure $\mathbb{P}^*$ w.r.t. the numeraire N if and only if there exists an equivalent martingale measure $\hat{\mathbb{P}}$ w.r.t. the numeraire M. If $\mathbb{P}^*$ is such an equivalent martingale measure, then

$$\frac{M}{N} = 1 + S_1^* \cdots + S_d^*$$

is a positive martingale w.r.t. $\mathbb{P}^*$ After normalisation an equivalent probability measure $\hat{\mathbb{P}}$ can be defined by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}|_{\mathcal{F}_t} = L(t) = \frac{M(t)}{N(t)}\frac{N(0)}{M(0)}$$

for all $0 \le t \le T$. Due to Bayes-formula the martingale property of S^* w.r.t. $\mathbb{P}^*$ implies the martingale property of

$$\frac{S^*}{L} = \frac{S}{M}\frac{M(0)}{N(0)}$$

w.r.t. $\hat{\mathbb{P}}$. Hence $\hat{S} = \frac{S}{M}$ is a martingale w.r.t. $\hat{\mathbb{P}}$.

To show the other direction we assume that $\hat{\mathbb{P}}$ is a martingale measure w.r.t. M as numeraire. But then $\frac{N}{M}$ defines a positive martingale w.r.t. $\hat{\mathbb{P}}$ and an equivalent probability measure $\mathbb{P}^*$ can be defined by

$$\frac{d\mathbb{P}^*}{d\hat{\mathbb{P}}}|_{\mathcal{F}_t} = \frac{1}{L}(t) = \frac{N(t)}{M(t)}\frac{M(0)}{N(0)}$$

for all $0 \le t \le T$ and

$$S^* = \frac{\hat{S}}{L}$$

defines a martingale w.r.t. $\mathbb{P}^*$.

In the next step from the NFLVR(M)-condition the existence of an equivalent local martingale measure $\hat{\mathbb{P}}$ can be deduced. But $\hat{S} = \frac{S}{M}$ is bounded and a bounded local martingale is a martingale. Thus $\hat{\mathbb{P}}$ is a martingale measure w.r.t. the numeraire M. But this implies that there exists also a martingale measure $\mathbb{P}^*$ w.r.t. N as numeraire.

$\Longleftarrow$: This is obviously true.

$\square$

One main question is the following. in which markets and for which strategies hold the law of one price? An answer can be given by so called maximal allowable strategies.

Definition 4.2.15. *A self-financing trading strategy with wealth process V is called maximal allowable, maximal market-admissible or regular if its terminal value $V(T)$ can not be dominated by another terminal value of an allowable self-financing trading strategy with the same initial value, i.e. for each allowable self-financing trading strategy with wealth-process U and $U(0) = V(0)$ holds $U(T) \le V(T)$.*

For maximal allowable strategies and markets that fulfill the NFLVR(M)-condition a version of the law of one price holds.

Theorem 4.2.16. *Suppose that the market satisfies NFLVR(M). If H, K are maximal allowable strategies with the same final value $V_H(T) = V_K(T)$, then $V_H(t) = V_K(t)$ for all $t \le T$. Accordingly we have $V_H \le V_K$ if we only assume $V_H(T) \le V_K(T)$ for the final values.*

Proof. We consider the market-portfolio M as numeraire asset. Then the discounted price processes $\hat{S} = \frac{S}{M}$ is a bounded process. Due to the NFLVR(M)-condition the set of equivalent martingale measures is non-empty. Since H is maximal allowable Delbaen and Schachermayer showed that there exists an equivalent martingale measure $\hat{\mathbb{P}}$ such that $\hat{V}_H$ and $\hat{V}_K$ are martingales w.r.t. $\hat{\mathbb{P}}$. This shows the assertion. $\square$

4.3 Pricing of European Derivatives

European derivatives are contracts that ensure its holder a pay-off C at a pre-described fixed time point T. Such a payoff C is usually called a claim with maturity T. In this section we would like to show how such a claim can be priced in an arbitrage-free market. In principle one has to distinguish between two types of markets, the so called complete and incomplete markets.

4.3.1 Pricing in a Complete Market

We consider a financial market as in 4.1.3 specified. This means that there are d risky assets and one numeraire asset in the market with price-processes of the form

$$\begin{aligned} dS_i(t) &= S_i(t)(\mu_i(t)dt + \sum_{j=1}^{n} \overline{\sigma}_{ij}(t)dW_j(t)) \quad 1 \le i \le d \\ dN(t) &= N(t)(\mu_N(t)dt + \sum_{j=1}^{n} \sigma_{Nj}(t)dW_j(t)). \end{aligned}$$

We assume that there exists some equivalent local martingale measure $\mathbb{P}^\star$. Then the market has no admissible arbitrage opportunities.
With Girsanov there exists some Wiener-process $W^\star$ according to $\mathbb{P}^\star$ such that

$$\begin{aligned} dS_i^\star(t) &= S_i^\star(t)\sigma_i(t)dW^\star(t) \\ &= S_i^\star(t)\sum_{j=1}^{n} \sigma_{ij}(t)dW_j^\star(t) \qquad \text{for all } 1 \le i \le d, 0 \le t < T. \end{aligned}$$

Hereby

$$\sigma_{ij}(t) := \overline{\sigma}_{ij}(t) - \sigma_{Nj}(t)$$

and

$$S_i^\star(t) = \frac{S_i(t)}{N(t)}.$$

Shortly:

$$dS^\star(t) = S^\star(t)\sigma(t)dW^\star(t)$$

A derivative can be seen as a contract that ensures at a fixed date T a random payment C. This will be now formalised by the term contingent claim.

Definition 4.3.1. *A contingent T-claim C is some $\mathcal{F}_T$-measurable random variable. C is called attainable w.r.t. the initial capital $x \in \mathbb{R}$, if there exists some previsible process H such that*

(i) $\frac{x}{N(0)} + \int_0^T H(u)dS^\star(u) = \frac{C}{N(T)} =: C^\star$

(ii) $\left(\int_0^t H(u)dS^\star(u)\right)_{0\le t<T}$ *is a uniformly integrable* $\mathbb{P}^\star$*-martingale.*

Then H *is called hedge for* C *according to the initial capital* $x \in \mathbb{R}$.
x *is that amount of money in Euro, that a seller needs to completely eliminate the risk from his short position.*

Remark 4.3.2. Let H be some hedge according to an initial capital x for a T-Claim C, then a self-financing trading strategy $(K(t), H(t))_{0\le t<T}$ can be uniquely defined by

$$K(t) = \frac{x}{N(0)} + \int_0^t H(u)dS^\star(u) - H(t)S^\star(t) \quad \text{for all } 0 \le t < T.$$

Its wealth-process $(V(t))_{0\le t<T}$ fulfills

$$\begin{aligned} V^\star(t) &= \frac{x}{N(0)} + \int_0^t H(u)dS^\star(u) \quad \text{for all } 0 \le t < T \\ &= \mathbb{E}^\star(C^\star|\mathcal{F}_t) \end{aligned}$$

respectively

$$V(t) = N(t)V^\star(t) = N(t)\mathbb{E}^\star(C^\star|\mathcal{F}_t) \quad 0 \le t < T.$$

In particular $C^\star$ is integrable w.r.t. $\mathbb{P}^\star$ and

$$\frac{x}{N(0)} = \mathbb{E}^\star C^\star = \mathbb{E}^\star \frac{C}{N(T)}$$

and

$$V(T) = C.$$

If $C^\star \ge -a$ for some $a \in \mathbb{R}_{>0}$, then (K, H) is admissible.

Proof. $\left(\int_0^t H(u)dS^\star(u)\right)_{0\le t<T}$ is a uniformly integrable $\mathbb{P}^\star-$martingale with

$$C^\star = \frac{x}{N(0)} + \int_0^T H(u)dS^\star(u).$$

Hence

$$\begin{aligned} \mathbb{E}^\star(C^\star|\mathcal{F}_t) &= \mathbb{E}^\star(\frac{x}{N(0)} + \int_0^T H(u)dS^\star(u)|\mathcal{F}_t) \\ &= \frac{x}{N(0)} + \int_0^t H(u)dS^\star(u) \\ &= V^\star(t) \quad 0 \le t < T. \end{aligned}$$

In particular

$$\mathbb{E}^\star C^\star = V^\star(0) = \frac{x}{N(0)}.$$

$\square$

Corollary 4.3.3. *Let H_1 and H_2 be hedges for C according to the initial capital x_1 respectively x_2. Then*

$$x_1 = x_2 \quad \text{and} \quad \int_0^t H_1(u)dS^\star(u) = \int_0^t H_2(u)dS^\star(u)$$

for all $0 \leq t < T$.

Proof. Due to Theorem 4.3.2

$$\frac{x_1}{N(0)} = \mathbb{E}^\star C^\star = \frac{x_2}{N(0)} \Rightarrow x_1 = x_2$$

and

$$\begin{aligned}\frac{x_1}{N(0)} + \int_0^t H_1(u)dS^\star(u) &= \mathbb{E}^\star(C^\star|\mathcal{F}_t) \\ &= \frac{x_2}{N(0)} + \int_0^t H_2(u)dS^\star(u).\end{aligned}$$

Due to $x_1 = x_2$ it holds $\int_0^t H_1(u)dS^\star(u) = \int_0^t H_2(u)dS^\star(u)$ for all $0 \leq t < T$. $\square$

Note that the value-process of a contingent claim C is uniquely given by

$$\left(\mathbb{E}^\star(C^\star|\mathcal{F}_t)\right)_{0\leq t<T}.$$

Theorem 4.3.4. *Let C be a contingent T-claim according to an initial capital x and hedge $(H(t))_{0\leq t<T}$. Let $C^\star \geq -a$ for some $a > 0$.*
Then for each equivalent local martingale measure $\mathbb{P}_1^\star$

$$\begin{aligned}\mathbb{E}_1^\star(C^\star|\mathcal{F}_t) &= \frac{x}{N(0)} + \int_0^t H(u)dS^\star(u) \\ &= \mathbb{E}^\star(C^\star|\mathcal{F}_t) \quad 0 \leq t < T.\end{aligned}$$

Proof. 1. step: show the assertion for bounded $C^\star$.
Due to Theorem 4.3.2

$$\frac{x}{N(0)} + \int_0^t H(u)dS^\star(u) = \mathbb{E}^\star(C^\star|\mathcal{F}_t) \quad \text{for all } 0 \le t < T.$$

Since $-a \le C^\star \le b$ it holds

$$-a \le \frac{x}{N(0)} + \int_0^t H(u)dS^\star(u) \le b \quad \text{for all } 0 \le t < T.$$

Hence the local $\mathbb{P}_1^\star$-martingale $\left(\int_0^t H(u)dS^\star(u)\right)_{0\le t<T}$ is bounded and therefore uniformly integrable w.r.t. $\mathbb{P}_1^\star$.
Hence

$$\begin{aligned}\frac{x}{N(0)} + \int_0^t H(u)dS^\star(u) &= \mathbb{E}_1^\star(x + \int_0^T H(u)dS^\star(u)|\mathcal{F}_t) \\ &= \mathbb{E}_1^\star(C^\star|\mathcal{F}_t) \quad \text{for all } 0 \le t < T.\end{aligned}$$

In particular

$$\mathbb{E}_1^\star C^\star = \frac{x}{N(0)} = \mathbb{E}^\star C^\star.$$

2. step: $C^\star \ge -a$
This can be shown by reduction to the first step. But the argumentation is a bit complex and omitted here. □

We may conclude
If C is an attainable contingent claim then the value-process of each replicating trading strategy coincide with

$$(\mathbb{E}^*(C^*|\mathcal{F}_t))_{0\le t\le T}$$

and therefore this process can be seen as arbitrage-free value process for the claim C.

Definition 4.3.5. *Let $C \ge 0$ be an attainable contingent Claim. Then the value*

$$\mathbb{E}^\star C^\star$$

is called initial arbitrage-free price for C, quoted in shares of the numeraire asset.
$N(0)\mathbb{E}^\star C^\star$ denotes then the initial arbitrage-free price in Euro.
$(\mathbb{E}^\star(C^\star|\mathcal{F}_t))_{0\le t<T}$ is called arbitrage-free price-process for C, quoted in shares of the numeraire asset and $(N(t)\mathbb{E}^\star(C^\star|\mathcal{F}_t))_{0\le t<T}$ denotes the arbitrage-free price process in Euro.

Remark 4.3.6. This definition is reasonable, since in an extended market where C is traded with price-process $(N(t)\mathbb{E}^\star(C^\star|\mathcal{F}_t))_{0\leq t<T}$ the probability measure $\mathbb{P}^\star$ remains to be an equivalent local martingale measure. Therefore the extended market has no admissible arbitrage-opportunities.

The question arises, how to examine if claims are attainable. A simple answer can be given in so called complete markets.

Definition 4.3.7. *A financial market is called complete if and only if there exists a unique local equivalent martingale measure.*

By the coefficients of the market, the drift processes and volatilities, one can decide if a market is complete.

$$dS_i(t) = S_i(t)(\mu_i(t)dt + \sum_{j=1}^{n} \overline{\sigma}_{ij}(t)dW_j(t)) \quad \text{for all } 1 \leq i \leq d$$
$$dN(t) = N(t)(\mu_N(t)dt + \sum_{j=1}^{n} \sigma_{Nj}(t)dW_j(t)$$

The condition (ii) from 4.2.8 can be transformed to

$$\mu(t) + \sigma(t)\vartheta(t) = \mu_N(t)\mathbb{1} + \sigma(t)\sigma_N(t)$$

whereat

$$\sigma_{ij}(t) := \overline{\sigma}_{ij}(t) - \sigma_{Nj}(t) \quad \text{for all } 1 \leq i \leq d, 1 \leq j \leq n.$$

If the dimension of the source of uncertainty is larger then the numer of risky assets. the market is not complete.

Theorem 4.3.8. *If $n > d$, then the market is not complete.*

Proof. If there exists no local equivalent martingale measure then the market is not complete either due to the definition of completeness.
Hence we can assume that there exists an equivalent local martingale measure $\mathbb{P}^\star$ with previsible $\mathbb{R}^n$-valued density process $(\vartheta(t))_{0\leq t<T}$ such that

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \vartheta(s)dW(s) - \frac{1}{2}\int_0^t |\vartheta(s)|^2 ds\right) \quad \text{for all } 0 \leq t < T$$

and

$$\sigma(t)\vartheta(t) = \mu_N(t)\mathbb{1} - \mu(t) + \sigma(t)\sigma_N(t)$$

for $\lambda \otimes \mathbb{P}-$a.s. (t, ω) is satisfied.

Due to $d < n$ $Kernel(\sigma(t)) \neq \{0\}$ for all $0 \leq t < T$. Choose $\eta(t) \in Kernel(\sigma(t))$ with $|\eta(t)| = 1$ for all $0 \leq t < T$. We use this η to define a further equivalent local martingale measure.
We know:

$$W^\star(t) = W(t) - \int_0^t \vartheta(s)ds \quad \text{for all } 0 \leq t < T$$

is a Wiener-process w.r.t. $\mathbb{P}^\star$ and

$$dS^\star(t) = S^\star(t)\sigma(t)dW^\star(t).$$

Let

$$L(t) := \exp\left(\int_0^t \eta(s)dW^\star(s) - \frac{1}{2}\int_0^t |\eta(s)|^2 ds\right) \quad \text{for all } 0 \leq t < T.$$

Due to the Novikov condition $(L(t))_{0\leq t<T}$ is a uniformly integrable martingale.
(Alternatively we can also apply Lévy's theorem: Put

$$B(t) = \int_0^t \eta(s)dW^\star(s), 0 \leq t < T.$$

Then B is a local martingale with $\langle B\rangle_t = \int_0^t \underbrace{|\eta(s)|^2}_{=1} ds = t$. Hence B is a Wiener-process and

$$L(t) = \exp(B(t) - \frac{1}{2}t)$$

is a $\mathbb{P}^\star$- martingale.)
We define a further probability measure $\mathbb{P}^{\star\star}$ by

$$\left.\frac{d\mathbb{P}^{\star\star}}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t} = L(t) \quad \text{for all } 0 \leq t < T.$$

Then

$$W^{\star\star}(t) = W^\star(t) - \int_0^t \eta(s)ds \quad 0 \leq t < T$$

is a Wiener-process according to $\mathbb{P}^{\star\star}$ and

$$\begin{aligned} dS^\star(t) &= S^\star(t)\sigma(t)dW^\star(t) \\ &= S^\star(t)\sigma(t)(dW^{\star\star}(t) + \eta(t)dt) \\ &= S^\star(t)(\underbrace{\sigma(t)\eta(t)}_{\substack{=0, \text{ due to} \\ \eta\in\text{Kernel } \sigma(t)}} dt + \sigma(t)dW^{\star\star}(t)) \\ &= S^\star(t)\sigma(t)dW^{\star\star}(t). \end{aligned}$$

Hence $\mathbb{P}^{\star\star}$ is a further equivalent local martingale measure. □

If there are more risky assets then dimension of uncertainty the market is complete if there exists a local martingale measure.

Theorem 4.3.9. *Let $n < d$ and $Kernel(\sigma(t)) = \{0\}$ a.s., then the market is complete, if*

$$\mu_N(t)\mathbb{1} - \mu(t) + \sigma(t)\sigma_N(t) = \sigma(t)\vartheta(t)$$

can be solved a.s. by ϑ and

$$\mathbb{E}\exp\left(\int_0^T \vartheta(s)dW(s) - \frac{1}{2}\int_0^T |\vartheta(s)|^2 ds\right) = 1.$$

Proof. Due to $Kernel(\sigma(t) = \{0\})$ the above $\vartheta(t)$ is uniquely determined. Since

$$\mathbb{E}\exp\left(\int_0^T \vartheta(s)dW(s) - \frac{1}{2}\int_0^T |\vartheta(s)|^2 ds\right) = 1$$

there exists an equivalent local martingale measure $\mathbb{P}^\star$ which is unique. Therefore the market is complete. □

If the number of risky assets coincide with the dimension of uncertainty then the market is complete.

Theorem 4.3.10. *We suppose that there exists an equivalent local martingale measure. If $n = d$ and $\sigma(t)$ is invertible for $\lambda \otimes \mathbb{P}$-almost sure (t, ω), then the market is complete.*

Proof.

$$\mu(t) + \sigma(t)\vartheta(t) = \mu_N(t)\mathbb{1} + \sigma(t)\sigma_N(t)$$

can be uniquely solved if and only if $\sigma(t)$ is invertible. □

Theorem 4.3.11. *We suppose that there exists an equivalent local martingale measure. If $n = d$ and*

$$\lambda \otimes \mathbb{P}\left(\{(t, \omega) : \sigma(t) \text{ ist nicht invertierbar}\}\right) > 0,$$

then the market is not complete.

Proof. Due to

$$\lambda \otimes \mathbb{P}\left(\{(t, \omega) : \sigma(t) \text{ is not invertable}\}\right) > 0$$

$Kernel(\sigma(t)) \neq \{0\}$ and a further equivalent local martingale measure can be defined. Therefore the market is not complete. □

In a complete market each integrable claim is attainable.

Theorem 4.3.12. *We consider a complete market with a unique equivalent local martingale measure $\mathbb{P}^\star$. Let C denote a T-claim with $\mathbb{E}^\star|C^\star| < \infty$.*
Then there exists according to the initial capital $x := N(0)\mathbb{E}^\star C^\star$ a replicating trading strategy for C.

Proof. Due to the martingale representation theorem there exists a representation of the uniformly integrable $\mathbb{P}^\star$-martingale

$$\mathbb{E}^\star(C^\star|\mathcal{F}_t) = \mathbb{E}^\star C^\star + \int_0^t \alpha(u)dW^\star(u)$$

as stochastic integral process w.r.t. a Wiener-process according to $\mathbb{P}^\star$.
Due to completeness $n \leq d$ and $\sigma(t)$ is invertible for all t.
We have to determine a d-dimensional previsible process H such that

$$\int_0^t \alpha(u)dW^\star(u) = \int_0^t H(u)dS^\star(u).$$

$$\begin{aligned}
\sum_{j=1}^n \int_0^t \alpha_j(s)dW_j^\star(s) &= \sum_{i=1}^d \int_0^t H_i(u)dS_i^\star(u) \\
&= \sum_{i=1}^d \sum_{j=1}^n \int_0^t H_i(u)S_i^\star(u)\sigma_{ij}(u)dW_j^\star(u) \\
&= \sum_{j=1}^n \int_0^t \sum_{i=1}^d H_i(u)S_i^\star(u)\sigma_{ij}(u)dW_j^\star(u)
\end{aligned}$$

Hence we have to solve

$$\alpha_j(u) = \sum_{i=1}^d H_i(u)S_i^\star(u)\sigma_{ij}(u) \quad \text{for all } 1 \leq j \leq n$$

respectively

$$\alpha(u) = \sigma^T(u)\begin{pmatrix} H_1(u)S_1^\star(u) \\ \vdots \\ H_d(u)S_d^\star(u) \end{pmatrix}.$$

Since $Kernel(\sigma(t)) = \{0\}$, this equation can uniquely be solved by $(H(u))_{0\leq u<T}$. □

The preceding theorem is not of real usage in practise since only the existence of α can be guaranteed. It makes no statement how such an α can be determined. Often a PDE-approach leads to a concrete hedge.

4.3.2 PDE Approach

We consider a complete diffusion-model with $n = d$, i.e.

$$\begin{aligned} dS_i(t) &= S_i(t)\left(r(t,S(t))dt + \sum_{j=1}^{n} \sigma_{ij}(t,S(t))dW_j^\star(t)\right) \qquad \text{for all } 1 \le i \le d \\ d\beta(t) &= \beta(t) r(t,S(t))dt \end{aligned}$$

w.r.t. the unique equivalent local martingale measure $\mathbb{P}^\star$ with $W^\star$ denoting some Wiener-process according to $\mathbb{P}^\star$.
Let $C^\star$ be a T-claim of the form

$$C = h(S(T))$$

with

$$\mathbb{E}^\star \frac{|h(S(T))|}{\beta(T)} < \infty.$$

Then there exists a replicating trading strategy with value-process $(V(t))_{0\le t<T}$, such that

$$\begin{aligned} V(t) &= \beta(t)\mathbb{E}^\star\left(\frac{h(S(T))}{\beta(T)}|\mathcal{F}_t\right) \\ &= \mathbb{E}^\star\left(h(S(T))\exp\left(-\int_t^T r(u,S(u))du\right)|\mathcal{F}_t\right). \end{aligned}$$

In the diffusion-model S is a $d-$dimensional Markov-process. Hence

$$\begin{aligned} V(t) &= \mathbb{E}^\star\left(h(S(T))\exp\left(-\int_t^T r(u,S(u))du\right)|S(t)\right) \\ &= v(t,S(t)) \end{aligned}$$

with

$$\begin{aligned} v(t,x) &= \mathbb{E}^\star\left(h(S(T))\exp\left(-\int_t^T r(u,S(u))du\right)|S(t)=x\right) \qquad \text{for all } 0 \le t < T, x \in (0,\infty)^d \\ &= \int h(y_T)\exp\left(-\int_t^T r(u,y_u)du\right) K_t(x,dy). \end{aligned}$$

and

$$K_t(x,\cdot) = \mathbb{P}^\star(S(u))_{u\ge t} \in \cdot|S_t = x)$$

as well as $(y_u)_{u\ge t}$ the realisation of a path of S after time t.

In many diffusion-models from the integrability condition $\mathbb{E}^\star|C^\star| < \infty$ the smoothness of v can be deduced. Then Ito's formula can be applied and it follows:

$$
\begin{aligned}
dv(t,S(t)) &= \partial_t v(t,S(t))dt + \sum_{i=1}^d \partial_{x_i} v(t,S(t))dS_i(t) + \frac{1}{2}\sum_{i,j=1}^d \partial_{x_i}\partial_{x_j} v(t,S(t))d\langle S_i,S_j\rangle_t \\
&= \partial_t v(t,S(t))dt + \sum_{i=1}^d \partial_{x_i} v(t,S(t))S_i(t)\Big(r(t,S(t))dt + \sum_{j=1}^d \sigma_{ij}(t,S(t))dW_j^\star(t)\Big) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^d \Big(\partial_{x_i}\partial_{x_j} v(t,S(t))S_i(t)S_j(t)(\sigma\sigma^T)_{ij}(t,S(t))\Big)dt
\end{aligned}
$$

due to

$$
\begin{aligned}
d\langle S_i,S_j\rangle_t &= d\langle \sum_{k=1}^d \int_0^\cdot S_i(u)\sigma_{ik}(u,S(u))dW_k^\star(u), \sum_{l=1}^d \int_0^\cdot S_j(u)\sigma_{jk}(u,S(u))dW_l^\star(u)\rangle_t \\
&= \sum_{k=1}^d\sum_{l=1}^d d\langle \int_0^\cdot S_i(u)\sigma_{ik}(u,S(u))dW_k^\star(u), \int_0^\cdot S_j(u)\sigma_{jl}(u,S(u))dW_l^\star(u)\rangle_t \\
&= \sum_{k,l=1}^d S_i(u)\sigma_{ik}(u,S(u))S_j(u)\sigma_{jk}(u,S(u))d\langle W_k^\star, W_l^\star\rangle_u \\
&= S_i(u)S_j(u)(\sigma\sigma^T)_{ij}(u,S(u))du
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
dv(t,S(t)) = \Big(&\partial_t v(t,S(t)) + \sum_{i=1}^d \partial_{x_i} v(t,S(t))S_i(t)r(t,S(t)) \\
&+ \frac{1}{2}\sum_{i,j=1}^d \partial_{x_i}\partial_{x_j} v(t,S(t))S_i(t)S_j(t)(\sigma\sigma^T)_{ij}(t,S(t))\Big)dt \\
&+ \sum_{i=1}^d \partial_{x_i} v(t,S(t))S_i(t)\sum_{j=1}^d \sigma_{ij}(t,S(t))dW_j^\star(t).
\end{aligned}
$$

Since $(\beta^{-1}(t)v(t,S(t)))_{0\le t<T}$ is a $\mathbb{P}^\star$-martingale, we get

$$
\partial_t v(t,x) + \sum_{i=1}^d \partial_{x_i} v(t,x)x_i r(t,x) + \frac{1}{2}\sum_{i,j=1}^d \partial_{x_i}\partial_{x_j} v(t,x)x_i x_j(\sigma\sigma^T)_{ij}(t,x) = r(t,x)v(t,x)
$$

for all $0 \le t < T, x \in (0,\infty)^d$.
Thus the function

$$
v : (0,T)\times(0,\infty)^d \longrightarrow \mathbb{R}
$$

is a solution to the so called Cauchy-problem

$$\partial_t v(t,x) + \sum_{i=1}^{d} \partial_{x_i} v(t,x) x_i r(t,x) + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j} v(t,x) x_i x_j (\sigma\sigma^T)_{ij}(t,x) = r(t,x)v(t,x)$$

with final condition

$$\lim_{t \nearrow T} v(t,x) = h(x).$$

This solution can usually not explicitly be determined, but numerical procedures can be applied to compute approximately the value function.
A side-effect of this approach is that a hedge can be determined either.

$$dV(t) = dv(t,S(t)) = v(t,S(t))r(t)dt + \sum_{i=1}^{d} \partial_{x_i} v(t,S(t)) S_i(t) \sum_{k=1}^{d} \sigma_{ij}(t,S(t)) dW_j^\star(t)$$

implies with integration by parts

$$\begin{aligned} dV^\star(t) = d\frac{v(t,S(t))}{\beta(t)} &= \sum_{i=1}^{d} \partial_{x_i} v(t,S(t)) S_i^\star(t) \sum_{j=1}^{d} \sigma_{ij}(t,S(t)) dW_j^\star(t) \\ &= \sum_{i=1}^{d} \partial_{x_i} v(t,S(t)) dS_i^\star(t). \end{aligned}$$

Hence it follows

$$V^\star(t) = V^\star(0) + \sum_{i=1}^{d} \int_0^t \underbrace{\partial_{x_i} v(u,S(u))}_{H_i(u)} dS_i^\star(u)$$

with $V^\star(0) = V(0) = \mathbb{E}^\star C^\star = v(0,S(0))$.
We obtain the so called δ-Hedge by

$$\begin{aligned} H_i(t) &= \partial_{x_i} v(t,S(t)) \quad \text{for all } 1 \le i \le d, 0 \le t < T \\ K(t) &= V^\star(t) - \sum_{i=1}^{d} H_i(t) S_i^\star(t). \end{aligned}$$

4.3.3 Examples

Black-Scholes Model

Here under the equivalent martingale measure we have the dynamics

$$dS(t) = S(t)(rdt + \sigma dW^*(t))$$

with money-market account

$$d\beta(t) = \beta(t) r dt.$$

For a call option with strike K and maturity T we get the price function

$$v(t,x) = e^{-r(T-t)}\mathbb{E}^*(S(T)-K)^+|S(t)=x).$$

Due to the fact that $\frac{S(t+s)}{S(t)}$ is independent of $\mathcal{F}_t$ and also $S(t)$ the value $v(t,x)$ coincides with the initial price of a call-option in a BS-model with running time $T-t$, volatility σ and interest rate r. Hence

$$v(t,x) = x\Phi(h_1(x,T-t)) - Ke^{-r(T-t)}\Phi(h_2(x,T-t)).$$

The Δ-hedge we get by taking the partial derivative of v w.r.t. x, i.e.

$$H(t) = \partial_x v(t,S(t)) = \Phi(h_1(S(t),T-t)) \quad 0 \leq t \leq T$$

with

$$\begin{aligned} h_1(x,T) &= \frac{\log\left(\frac{x}{K}\right) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ h_2(x,T) &= \frac{\log\left(\frac{x}{K}\right) + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \end{aligned}$$

The price-function solves the Cauchy-problem

$$\partial_t v(t,x) + xr\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v(t,x) = rv(t,x)$$

for all $0 < t < T, x > 0$ with terminal condition

$$\lim_{t\to T} v(t,x) = (K-x)^+.$$

Black-Scholes Model with deterministic coefficients

Here under the equivalent martingale measure we have the dynamics

$$dS(t) = S(t)(r(t)dt + \sigma(t)dW^*(t))$$

with money-market account

$$d\beta(t) = \beta(t)r(t)dt.$$

The volatility and interest-rate are deterministic functions of time.
For a call option with strike K and maturity T we get the price function

$$v(t,x) = e^{-R(t,T)}\mathbb{E}^*(S(T)-K)^+|S(t)=x)$$

with

$$R(t,T) = \int_t^T r(s)ds.$$

Due to the fact that $\frac{S(t+s)}{S(t)}$ is independent of $\mathcal{F}_t$ the above conditional expected value can be calculated as in a Black-Scholes model with running-time $T-t$ and dynamics given by

$$dZ(u) = Z(u)(r(t+u)du + \sigma(t+u)dW^*(u))$$

with initial price x. We obtain

$$v(t,x) = x\Phi(g_1(x,t)) - Ke^{-r(T-t)}\Phi(g_2(x,t))$$

with

$$\begin{aligned} g_1(x,t) &= \frac{\log\left(\frac{x}{K}\right) + (R(t,T) + \frac{1}{2}\eta(t,T)}{\sqrt{\eta(t,T)}} \\ g_2(x,t) &= \frac{\log\left(\frac{x}{K}\right) + (R(t,T) - \frac{1}{2}\eta(t,T)}{\sqrt{\eta(t,T}}. \end{aligned}$$

and

$$\eta(t,T) = \int_t^T \sigma^2(s)ds.$$

The price-function solves the Cauchy-problem

$$\partial_t v(t,x) + xr(t)\partial_x v(t,x) + \frac{1}{2}\sigma(t)^2x^2\partial_x^2 v(t,x) = r(t)v(t,x)$$

for all $0 < t < T, x > 0$ with terminal condition

$$\lim_{t\to T} v(t,x) = (K-x)^+.$$

The Δ-hedge is obtained by

$$H(t) = \partial_x v(t, S(t)) = \Phi(g_1(x,t)).$$

The two-dimensional Black-Scholes Model

We consider a BS-model with two stocks, i.e.

$$\begin{aligned} dS_1(t) &= S_1(t)(rdt + \sigma_1 dW_1^*(t)) \\ dS_2(t) &= S_2(t)(rdt + \sigma_2 dW_2^*(t)) \end{aligned}$$

with $\langle W_1^*, W_2^*\rangle_t = \rho t$ for some $-1 < \rho < 1$. As numeraire we consider a money market account with constant interest rate r.
The exchange option gives its holder the right to exchange stock 1 with stock 2 at T. This corresponds to the T-claim

$$C = (S_2(T) - S_1(T))^+.$$

The price function depends on time and the observed stock prices x_1, x_2 and is given by

$$v(t,x_1,x_2) = \mathbb{E}^*(e^{-r(T-t)}(S_2(T) - S_1(T))^+ | S_1(t) = x_1, S_2(t) = x_2)$$

$$
\begin{aligned}
= \quad & x_2\Phi\left(\frac{\log\frac{x_2}{x_1}+\frac{1}{2}(\sigma_1^2+\sigma_2^2)T-\sigma_1\sigma_2\varrho(T-t)}{\sqrt{(T-t)(\sigma_2^2-2\varrho\sigma_1\sigma_2+\sigma_1^2)}}\right)\\
& -x_1\Phi\left(\frac{\log\frac{x_2}{x_1}-\frac{1}{2}(\sigma_1^2+\sigma_2^2)T+\sigma_1\sigma_2\varrho T}{\sqrt{T(\sigma_2^2-2\varrho\sigma_1\sigma_2+\sigma_1^2)}}\right). \qquad (4.2)
\end{aligned}
$$

It solves the PDE

$$
\begin{aligned}
& \partial_t v(t,x_1,x_2)+rx_1\partial_{x_1}v(t,x_1,x_2)+rx_2\partial_{x_2}v(t,x_1,x_2)\\
& +\frac{1}{2}\sigma_1^2x_1^2\partial_{x_1}^2v(t,x_1,x_2)+\frac{1}{2}\sigma_2^2x_2^2\partial_{x_2}^2v(t,x_1,x_2)+\rho\sigma_1\sigma_2x_1x_2\partial_{x_1,x_2}^2v(t,x_1,x_2)\\
= \quad & rv(t,x_1,x_2). \qquad (4.3)
\end{aligned}
$$

The Δ-hedge is obtained by

$$
H_i(t)=\partial_{x_i}v(t,S(t)) \quad 0\le t\le T, i=1,2.
$$

4.3.4 PDE Approach for Barrier Options

The PDE approach works in a diffusion-model for path-independent claims. Only for special cases an analogous analysis for path-dependent claims can be done. As such an example we consider a barrier option in a complete one-dimensional diffusion-model of the form

$$
\begin{aligned}
dS(t) &= S(t)\Big(r(t,S(t))dt+\sigma(t,S(t))dW^\star(t)\Big)\\
d\beta(t) &= \beta(t)r(t,S(t))dt
\end{aligned}
$$

w.r.t.the equivalent local martingale measure $\mathbb{P}^\star$.
According to

$$
h:(0,\infty)\longrightarrow\mathbb{R}
$$

with

$$
\mathbb{E}^\star\frac{|h(S(T))|}{\beta(T)}<\infty
$$

we would like to determine the price-function of a barrier-option.
Hereby a barrier-option with barrier $0\le K<L\le\infty$ becomes worthless if the price-process of the underlying falls below a barrier K or exceeds the barrier L during the running-time. This is a so called knock-out option and is defined by the T-claim

$$
C=h(S(T))\mathbb{1}_{\{\tau_0>T\}}
$$

with

$$
\tau_t:=\inf\{u\ge t:S(u)\le K \text{ oder } S(u)\ge L\}
$$

for all $0\le t<T$.
It is an example of a path dependent option.

The price of C at t fulfills

$$\begin{aligned}
p_t(C) &= \beta(t)\mathbb{E}^\star(C^\star|\mathcal{F}_t) \\
&= \mathbb{E}^\star\left(h(S(T))\exp\left(-\int_t^T r(u,S(u))du\right)\mathbb{1}_{\{\tau_0>T\}}|\mathcal{F}_t\right) \\
&= \mathbb{E}^\star\left(h(S(T))\exp\left(-\int_t^T r(u,S(u))du\right)\mathbb{1}_{\{\tau_0>t\}}\mathbb{1}_{\{\tau_t>T\}}|\mathcal{F}_t\right) \\
&= \mathbb{1}_{\{\tau_0>t\}}\mathbb{E}^\star\left(h(S(T))\exp\left(-\int_t^T r(u,S(u))du\right)\mathbb{1}_{\{\tau_t>T\}}|\mathcal{F}_t\right) \\
&= \mathbb{1}_{\{\tau_0>t\}}v(t,S(t))
\end{aligned}$$

with

$$v(t,x) = \mathbb{E}^\star\left(h(S(T))\exp\left(-\int_t^T r(u,S(u))du\right)\mathbb{1}_{\{\tau_t>T\}}|S(t)=x\right) \qquad \text{for all } K<x<L.$$

The function v fulfills a PDE with boundary- and terminal condition.
Derivation of the PDE:
Due to $\beta^{-1}(t)p_t(C) = \mathbb{E}^\star(C^\star|\mathcal{F}_t)$ is a $\mathbb{P}^\star$-martingale, the process

$$\beta^{-1}(t\wedge\tau_0)p_{t\wedge\tau_0}(C) = \beta^{-1}(t\wedge\tau_0)v(t\wedge\tau_0, S(t\wedge\tau_0)) \quad 0\le t<T$$

is a $\mathbb{P}^\star$-martingale either.
The Ito-formula applied to $v(t\wedge\tau_0, S(t\wedge\tau_0))$ leads to the PDE

$$\partial_t v(t,x) + \frac{1}{2}x^2\sigma^2(t,x)\partial_x^2 v(t,x) + r(t,x)x\partial_x v(t,x) = r(t,x)v(t,x)$$

for all $0\le t<T, K<x<L$.
Due to $\lim_{t\nearrow\tau_0} v(t,S(t)) = 0$ $\mathbb{P}^\star-$a.s. the boundary conditions follow

$$\begin{aligned}
\lim_{x\nearrow L} v(t,x) &= 0 \quad \text{for all } 0\le t<T \\
\lim_{x\searrow K} v(t,x) &= 0 \quad \text{for all } 0\le t<T
\end{aligned}$$

and the final condition

$$\lim_{t\nearrow T} v(t,x) = h(x) \quad \text{for all } K<x<L.$$

4.3.5 Sharpe Ratio

We consider a one-dimensional complete market

$$\begin{aligned} dS(t) &= S(t)\Big(\mu(t)dt + \sigma(t)dW(t)\Big) \\ d\beta(t) &= \beta(t)r(t)dt. \end{aligned}$$

The subjective probability measure $\mathbb{P}$ can be seen as the belief of an investor how the risky asset evolves. The return-rate $\mu(t)$ represents the investor's opinion on the chances whereas the volatility $\sigma(t)$ expresses the market-risk.

- $\mu(t) - r(t)$ is the so called excess return of the stock.
- $\frac{\mu(t)-r(t)}{\sigma(t)}$ is the Sharpe Ratio of the stock, i.e. the proportion of excess return to risk.

  Meaning: Valuation of the return in units of risk (volatility), also called Market Price of Risk.
- $\vartheta(t) = -\frac{\mu(t)-r(t)}{\sigma(t)} \quad 0 \leq t < T$ defines the density to the equivalent local martingale measure $\mathbb{P}^\star$:

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \vartheta(s)dW(s) - \frac{1}{2}\int_0^t \vartheta^2(s)ds\right) \qquad \text{for all } 0 \leq t < T.$$

Let $C > 0$ be a T-claim with $\mathbb{E}^\star|C^\star| < \infty$ and $\mathbb{E}^\star(C^\star|\mathcal{F}_t) > 0$ $\mathbb{P}$-almost sure. Then C can be seen as additional traded risky asset in the market with arbitrage-free price process

$$C(t) = \beta(t)\mathbb{E}^\star(C^\star|\mathcal{F}_t) \quad 0 \leq t < T.$$

Since C is a positive semi-martingale w.r.t. $\mathbb{P}$, there exists a representation of the form

$$dC(t) = C(t)\Big(\mu_C(t)dt + \sigma_C(t)dW(t)\Big)$$

with previsible processes μ_C and σ_C. Hereby

- μ_C is the return of C and
- σ_C its volatility.

According to $\mathbb{P}^\star$ we obtain

$$dC(t) = C(t)\Big(r(t)dt + \sigma_C(t)dW^\star(t)\Big),$$

since $\left(\frac{1}{\beta(t)}C(t)\right)_{0\leq t<T}$ is a $\mathbb{P}^\star$-martingale.

Due to

$$dW^\star(t) = dW(t) - \vartheta(t)dt$$

it follows

$$dC(t) = C(t)\Big((r(t) - \sigma_C(t)\vartheta(t))dt + \sigma_C(t)dW(t)\Big).$$

Hence

$$\mu_C(t) = r(t) - \sigma_C(t)\vartheta(t).$$

Therefore

$$\mu_C(t) - r(t)$$

is the excess-return of C.
It holds true:

$$\frac{\mu_C(t) - r(t)}{\sigma_C(t)} = -\vartheta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}.$$

Conclusion:

- The Sharpe-ratio is invariant under all reasonable traded assets in the market and is determined by the Girsanov-transformation.

- An investor can invest money in an arbitrary asset. The proportion of chance to risk is always the same and given by the Sharpe-ratio.

4.3.6 Construction of a Money Market Account in a Multi-Dimensional Complete Market

We consider a complete market in the form

$$dS_i(t) = S_i(t)\Big(\mu_{S,i}(t) + \sum_{j=1}^{d} \sigma_{Sij}(t)dW_j(t)\Big) \quad \text{for all } 1 \leq i \leq d$$
$$dN(t) = N(t)\Big(\mu_N(t) + \sum_{j=1}^{d} \sigma_{Nj}(t)dW_j(t)\Big)$$

with some $d-$dimensional Wiener-process W determining the uncertainty.
The completeness means

$$\sigma_{ij}(t) := \sigma_{Sij}(t) - \sigma_{Nj}(t) \quad \text{for all } 1 \leq i, j \leq d, 0 \leq t < T$$

is invertible and by

$$\frac{d\mathbb{P}^\star}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\left(\int_0^t \vartheta(s)dW(s) - \frac{1}{2}\int_0^t |\vartheta(s)|^2 ds\right) \quad 0 \leq t < T$$

a unique equivalent local martingale measure is defined, whereat

$$\vartheta(t) = \sigma^{-1}(t)\Big((\mu_N(t) - |\sigma_N(t)|^2)\mathbb{1} + \sigma_S(t)\sigma_N(t) - \mu_S(t)\Big)$$

$$W^\star(t) = W(t) - \int_0^t \vartheta(s)ds \quad d-\text{dimensional Wiener-process w.r.t. } \mathbb{P}^\star.$$

According to $\mathbb{P}^\star$ we have the dynamics

$$dS^\star(t) = S^\star(t)\sigma(t)dW^\star(t)$$

and

$$d\frac{1}{N(t)} = \frac{1}{N(t)}\Big((|\sigma_N(t)|^2 - \mu_N(t) - \sigma_N(t)\vartheta(t))dt - \sigma_N(t)dW^\star(t)\Big).$$

Put

$$r^\star(t) = -\Big(|\sigma_N(t)|^2 - \mu_N(t) - \sigma_N(t)\vartheta(t)\Big) \quad \text{for all } 0 \le t < T.$$

Then we obtain

$$d\frac{1}{N(t)} = \frac{1}{N(t)}(-r^\star(t)dt - \sigma_N(t)dW^\star(t)).$$

This means, that the arbitrage-free interest-rate process $(r(t))_{0\le t<T}$ of a money market account is uniquely determined by the above expression.
We define

$$\beta(t) = \exp\left(\int_0^t r^\star(s)ds\right) \qquad \text{for all } 0 \le t < T.$$

Then β is the price-process of a money market account with interest-rate process $r^\star$, i.e.

$$d\beta(t) = \beta(t)r^\star(t)dt, \quad \beta(0) = 1$$

and

$$\begin{aligned} d\frac{\beta(t)}{N(t)} &= \beta(t)d\frac{1}{N(t)} + \frac{1}{N(t)}d\beta(t) \\ &= -\frac{\beta(t)}{N(t)}\left(r^\star(t)dt + \sigma_N(t)dW^\star(t)\right) + \frac{\beta(t)}{N(t)}r^\star(t)dt \\ &= -\frac{\beta(t)}{N(t)}\sigma_N(t)dW^\star(t). \end{aligned}$$

Therefore $\mathbb{P}^\star$ is an equivalent local martingale measure w.r.t. the $d+1$ risky assets $S_1, ..., S_d, \beta$ and the numeraire asset N. A market with these $d+2$ assets admits no admissible arbitrage opportunities.
Question: How can the money market account be replicated?

Therefore we seek for a previsible process H, such that

$$\frac{\beta(t)}{N(t)} = \frac{1}{N(0)} + \int_0^t H(u) dS^\star(u) \quad \text{for all } 0 \le t < T.$$

Then there is a self-financing trading strategy with value-process

$$V(t) = \beta(t) \quad \text{for all } 0 \le t < T.$$

It holds

$$\begin{aligned} d\frac{\beta(t)}{N(t)} &= -\frac{\beta(t)}{N(t)} \sigma_N(t) dW^\star(t) \\ &= -\sum_{j=1}^{d} \frac{\beta(t)}{N(t)} \sigma_{Nj}(t) dW_j^\star(t) \end{aligned}$$

and

$$\begin{aligned} d\beta^\star(t) &= H(t) dS^\star(t) \\ &= \sum_{i=1}^{d} H_i(t) dS_i^\star(t) \\ &= \sum_{i=1}^{d} H_i(t) S_i^\star(t) \sum_{j=1}^{d} \sigma_{ij}(t) dW_j^\star(t) \\ &= \sum_{j=1}^{d} \left(\sum_{i=1}^{d} H_i(t) S_i^\star(t) \sigma_{ij}(t) \right) dW_j^\star(t). \end{aligned}$$

This leads to the equation

$$\sum_{i=1}^{d} H_i(t) S_i^\star(t) \sigma_{ij}(t) = -\frac{\beta(t)}{N(t)} \sigma_{Nj}(t) \quad \text{for all } 1 \le j \le d$$

resp.

$$\sigma^T(t) \begin{pmatrix} H_1(t) S_1^\star(t) \\ \vdots \\ H_d(t) S_d^\star(t) \end{pmatrix} = -\frac{\beta(t)}{N(t)} \sigma_N(t)$$

and

$$\begin{pmatrix} H_1(t) S_1^\star(t) \\ \vdots \\ H_d(t) S_d^\star(t) \end{pmatrix} = (\sigma^T(t))^{-1} \left(-\frac{\beta(t)}{N(t)} \sigma_N(t) \right) = -\frac{\beta(t)}{N(t)} (\sigma^T)^{-1} \sigma_N(t).$$

For the fraction of money invested in each stock

$$\pi_i(t) := \frac{H_i(t) S_i(t)}{V(t)} = \frac{H_i(t) S_i^\star(t)}{V^\star(t)} = \frac{H_i(t) S_i^\star(t)}{\frac{\beta(t)}{N(t)}} \quad 1 \le i \le d, 0 \le t < T$$

this means

$$\pi(t) = -(\sigma^T(t))^{-1}\sigma_N(t) \quad \text{for all } 0 \le t < T.$$

If

$$\exp\left(-\int_0^t \sigma_N(s)dW^\star(s) - \frac{1}{2}\int_0^t |\sigma_N(s)|^2 ds \right) \quad 0 \le t < T$$

is a uniformly integrable $\mathbb{P}^\star$-martingale, then the money market account is replicable by the self-financing trading strategy that belongs to H.

Corollary 4.3.13. *Are the coefficients $\sigma_S, \sigma_N, \mu_S, \mu_N$ constant, then $r^\star$ is constant. Therefore the market coincides with a multi-dimensional Black-Scholes model with interest rate*

$$r^\star = \mu_N - |\sigma_N|^2 + \sigma_N\vartheta.$$

Prices of derivatives coincide with their Black-Scholes prices.

4.3.7 Pricing in Incomplete Markets

We consider a market with d risky assets $S = (S_1, \cdots, S_d)$ and a numeraire asset N as in 4.1.3 specified. It is assumed that the NFLVR condition holds true which implies that the set of equivalent local martingale measures $\mathbb{P}$is non-empty. Contrary to the complete case we assume that $\mathcal{M}$ has more than one element which implies that $\mathcal{M}$ is a convex set with infinite elements. For a pricing purpose the following problems arise:

1. For each $\mathbb{Q} \in \mathcal{M}$ there exists some non-negative T-claim $C \in L_1(\mathbb{Q})$ that is not attainable.

2. For such a T-claim C the risk of a short position in the claim cannot be totally eliminated.

3. There is no unique initial arbitrage-free price, resp. price-process for such a T-claim C.

In general one can only construct lower- and upper bounds on arbitrage-free prices by considering super- and sub replicating strategies.
Two main theorems come into play for an analysis.

Theorem 4.3.14. *Let $C \ge 0$ be a non-negative T-claim. Then the process*

$$X(t) = \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}(C^*|\mathcal{F}_t) \quad 0 \le t \le T$$

defines a $\mathbb{Q}$-super-martingale for any $\mathbb{Q} \in \mathcal{M}$.

Proof. A proof of this non-trivial statement can be found in the book of Pham [5] p. 174. □

Theorem 4.3.15. *Let X be a super-martingale w.r.t any $\mathbb{Q} \in \mathcal{M}$. Then there exists a previsible process H and a non-decreasing adapted cadlag process A such that $A_0 = 0$ and*

$$X(t) = X_0 + \int_0^t H(u)dS^*(u) - A(t) \quad 0 \le t \le T.$$

Proof. This so called optional decomposition theorem can be found in [6] □

We first take the view of a seller of a contingent T-claim C. Its pay-off quoted in shares of the numeraire asset is given by

$$C^* = \frac{C}{N(T)}.$$

The seller can eliminate his short position risk totally if there exists an initial value x and a previsible process H such that

$$x + \int_0^T H(u)dS^*(u) \ge C^*.$$

The initial value x and the previsible process H uniquely determines a self-financing trading strategy (ϕ, H) with discounted value-process V^* such that

$$V^*(t) = x + \int_0^t H(u)dS^*(u) \quad 0 \le t \le T.$$

In particular

$$V^*(T) \ge C^*.$$

Note that ϕ is uniquely defined by

$$\phi(t) + H(t)S^*(t) = V^*(t) = x + \int_0^t H(u)dS^*(u) \quad 0 \le t \le T.$$

By taking the infimum we get the super-replication cost of the seller.

Definition 4.3.16. *Let $C \ge 0$ be a T-claim. Then the super-replication cost $p^+(C)$ is defined by*

$$p^+(C) = \inf\{x \in \mathbb{R} : \textit{there exists some admissible } H \textit{ with } x + \int_0^T H(u)dS^*(u) \ge C^*\}.$$

At time t the super-replication cost is an $\mathcal{F}_t$-measurable random variable $p_t^+(C)$ that fulfills

$$p_t^+(C) = \operatorname{ess\,inf}\{X \in L_0(\mathcal{F}_t) : \textit{there ex. an admissible } H \textit{ with } X + \int_t^T H(u)dS^*(u) \ge C^*\}.$$

This super-replication cost resp. price can be expressed by the set of equivalent local martingale measures.

Theorem 4.3.17. *Let $C \geq 0$ be a T-claim with $C \in L_1(\mathbb{Q})$ for all $\mathbb{Q} \in \mathcal{M}$. Then*

$$p^+(C) = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^*$$

and

$$p_t^+(C) = \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}(C^*|\mathcal{F}_t)$$

for all $0 \leq t \leq T$.

Proof. If we find an initial capital x and a previsible H such that

$$x + \int_0^T H(u)dS^*(u) \geq C^*,$$

then ,due to the fact that $\int_0^t H(u)dS^*(u)$ is a $\mathbb{Q}$-super-martingale for any $\mathbb{Q} \in \mathcal{M}$,

$$x \geq \mathbb{E}_{\mathbb{Q}}(x + \int_0^T H(u)dS^*(u)) \geq \mathbb{E}_{\mathbb{Q}} C^*$$

for all $\mathbb{Q} \in \mathcal{M}$. Hence

$$p^+(C) \geq \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^*.$$

More difficult is to prove the reverse inequality. Here the dominating super-martingale

$$X(t) = \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}(C^*|\mathcal{F}_t) \quad 0 \leq t \leq T$$

comes into play. Due to the optional decomposition theorem 4.3.15 there exists some previsible process H and a non-decreasing cadlag process A with $A_0 = 0$ such that

$$X(t) = X(0) + \int_0^t H(u)dS^*(u) - A(t) \quad 0 \leq t \leq T.$$

In particular

$$X(0) + \int_0^T H(u)dS^*(u) = X(T) + A(T) \geq C^*$$

and therefore

$$p^+(C) \leq \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^*$$

since $X(0) = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^*$.
The second assertion follows in the same way. $\square$

With the help of the minimal dominating super-martingale X a super-replicating trading strategy can be defined. Due to the decomposition

$$X(t) = X(0) + \int_0^t H(u)dS^*(u) - A(t) \quad 0 \le t \le T$$

a self-financing trading strategy according to the initial capital

$$X(0) = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^* \quad \text{and previsible } \ H$$

can be defined with discounted value-process V^* satisfying

$$V^*(t) = X(0) + \int_0^t H(u)dS^*(u) = X(t) + A(t) \quad 0 \le t \le T.$$

- $X(t)$ can be seen as that value in t quoted in shares of the numeraire asset such that the risk of the short position in the T-claim C can be fully eliminated.
- $A(t)$ can be seen as that amount of money that can be withdrawn until t without jeopardising the risk-elimination.
- A super-replication strategy not only finances the expenses X of eliminating risk but also the consumption w.r.t. A.
- Since only a minimal capital for super-replication will be supplied X and A are minimal. This means that A is the minimal consumption that can be allowed in order to hedge fully the risk.

Now we change our point of view and think as a buyer of a T-claim. Then one has to borrow money in order to finance the long position at the beginning. The pay-off at the end can be used to neutralise the trading's short position. A buyer could eliminate his risk from the long-position if according to an initial capital x and an admissible previsible H

$$x + \int_0^T H(u)dS^*(u) \le C^*.$$

He would accept this price x since the trading w.r.t. H does not exceed the pay-off C^*. This leads to the sub-replication cost

Definition 4.3.18. *Let $C \ge 0$ be a T-claim. Then the sub-replication cost $p^-(C)$ is defined by*

$$p^-(C) = \sup\{x \in \mathbb{R} : \textit{there exists some admissible } H \textit{ with } x + \int_0^T H(u)dS^*(u) \le C^*\}.$$

At time t the sub-replication cost is an $\mathcal{F}_t$-measurable random variable $p_t^-(C)$ that fulfills

$$p_t^-(C) = \operatorname{ess\,sup}\{X \in L_0(\mathcal{F}_t) : \textit{there ex. an admissible } H \textit{ with } X + \int_t^T H(u)dS^*(u) \le C^*\}.$$

The sub replication cost can be seen as the maximum amount of money one can borrow to buy the T-claim C without any risk. Also the sub-replication cost can be expressed by the set of equivalent local martingale measures.

Theorem 4.3.19. *Let $C \geq 0$ be a T-claim such that C^* is bounded above. Then*

$$p^-(C) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^*$$

and

$$p_t^-(C) = \operatorname{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}(C^* | \mathcal{F}_t)$$

for all $0 \leq t \leq T$.

Proof. Let x be an initial capital and H an admissible trading strategy w.r.t. x such that

$$x + \int_0^T H(u) dS^*(u) \leq C^*.$$

Then

$$-x + \int_0^T -H(u) dS^*(u) \geq -C^*$$

. Due to the boundedness of C^* the process $(\int_0^t -H(u)dS^*(u))_{0 \leq t \leq T}$ is a $\mathbb{Q}$-supermartingale for all $\mathbb{Q} \in \mathcal{M}$. But this implies

$$-x \geq \mathbb{E}_{\mathbb{Q}}(-x + \int_0^T -H(u) dS^*(u)) \geq -\mathbb{E}_{\mathbb{Q}} C^*$$

for all $\mathbb{Q} \in \mathcal{M}$. Hence

$$p^-(C) \leq \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^*.$$

More difficult is to prove the reverse inequality. Here the dominating super-martingale

$$X(t) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}(-C^* | \mathcal{F}_t) \quad 0 \leq t \leq T$$

comes into play. Due to the optional decomposition theorem 4.3.15 there exists some previsible process H and a non-decreasing cadlag process A with $A_0 = 0$ such that

$$X(t) = X(0) + \int_0^t H(u) dS^*(u) - A(t) \quad 0 \leq t \leq T.$$

In particular

$$-X(0) + \int_0^T -H(u) dS^*(u) = -X(T) - A(T) \leq -X(T) = C^*.$$

Hence

$$p^-(C) \geq -X(0) = -\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}(-C^*) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^*.$$

The second assertion follows in the same way. □

The minimal dominating super-martingale X can be used to define the sub-replicating strategy. Due to the optional-decomposition theorem 4.3.15 there exists an admissible H and a non-increasing cadlag process A such that

$$X(0) + \int_0^t H(u)dS^*(u) - A(t) = X(t).$$

According to the initial capital

$$X(0) = \sup_{\mathbb{Q}\in\mathfrak{M}} \mathbb{E}(-C^*) \quad \text{and admissible } H$$

there exists a self-financing strategy (ϕ, H) with value process V^* such that

$$V^*(t) = X(0) + \int_0^t H(u)dS^*(u) = X(t) + A(t) \quad 0 \leq t \leq T.$$

- $-X(0) = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}} C^*$ is that amount of money a buyer can borrow to cover his risk.

- $-X(t) = \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}(C^*|\mathcal{F}_t)$ can be seen as that amount of money the buyer needs at time t to cover his risk.

- $A(t)$ can be seen as that amount of money the buyer may consume up to t in order to still match his risk.

Note that the boundedness assumption for C^* is necessary for the proof.
With the help of the super-and sub-replication strategy one can determine an interval of arbitrage-free initial prices for a T-claim C. To give a precise formulation an arbitrage-free price for a T-claim C has to be defined.

Definition 4.3.20. *Let $C \geq 0$ be a T-claim. We say that an initial price x leads to an arbitrage opportunity for the seller if there exists some admissible H such that*

$$x + \int_0^T H(u)dS^*(u) \geq C^* \quad \text{and} \;\; \mathbb{P}(x + \int_0^T H(u)dS^*(u) > C^*) > 0.$$

We say that an initial price x leads to an arbitrage opportunity for the buyer if there exists some admissible H such that

$$-x + \int_0^T H(u)dS^*(u) \geq -C^* \quad \text{and} \;\; \mathbb{P}(-x + \int_0^T H(u)dS^*(u) > -C^*) > 0).$$

The set of arbitrage-free prices will be denoted by $\pi(C)$.

Theorem 4.3.21. *Let $C \geq 0$ be a T-claim such that C^* is bounded. Then*

1. *C is attainable if and only if $p^-(C) = p^+(C)$.*

2. *If C is not attainable the set of arbitrage-free prices for C coincides with the open*

interval $(p^-(C), p^+(C))$, *i.e.*

$$\pi(C) = (p^-(C), p^+(C)).$$

Proof. If C is attainable, then their exists some x, admissible H and $\mathbb{Q} \in \mathcal{M}$ such that

$$x + \int_0^T H(u) dS^*(u) = C^*$$

and $\int_0^t H(u)dS^*(u)$ is a uniformly $\mathbb{Q}$-martingale. This means that

$$x + \int_0^t H(u) dS^*(u) = \mathbb{E}_{\mathbb{Q}}(C^* | \mathcal{F}_t) \quad 0 \le t \le T.$$

Due to the boundedness of C^* for any $\mathbb{P}^* \in \mathcal{M}$

$$x + \int_0^t H(u) dS^*(u) = \mathbb{E}_{\mathbb{P}^*}(C^* | \mathcal{F}_t) \quad 0 \le t \le T$$

since the left-hand side is as bounded local $\mathbb{P}^*$-martingale a uniformly integrable martingale. Hence

$$x = \mathbb{E}^* C^* \quad \text{ for all } \mathbb{P}^* \in \mathcal{M}.$$

For the only if part let $p^-(C) = x = p^+(C)$. Then

$$\mathbb{E}^* C^* = x \quad \text{for all } \ \mathbb{P}^* \in \mathcal{M}.$$

There exists some admissible H such that

$$x + \int_0^T H(u) dS^*(u) \ge C^*.$$

If the inequality is strict we would obtain

$$\mathbb{E}^* C^* < \mathbb{E}^*(x + \int_0^T H(u) dS^*(u)) \le x$$

which would provide a contradiction. Thus

$$x + \int_0^T H(u) dS^*(u) = C^*$$

Due to the fact that

$$\mathbb{E}^* C^* = x \quad \text{for all } \ \mathbb{P}^* \in \mathcal{M}$$

we obtain that

$$x + \int_0^t H(u) dS^*(u) \quad 0 \le t \le T$$

is a $\mathbb{P}^*$-martingale for any $\mathbb{P}^* \in \mathcal{M}$. Thus C is attainable.

□

4.4 Pricing of American Derivatives

In the case of an American derivative the pay-off time is not initially determined. The holder of the derivative can choose a random time-date for expiration that is non-anticipating, hence a stopping time. Let's first recall the financial market as in 4.1.3.

$$dS_i(t) = S_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{d}\bar{\sigma}_{ij}(t)dW_j(t)\right) \quad i = 1, \cdots d$$
$$dN(t) = N(t)(r(t)dt + \sum_{j=1}^{d}\sigma_{Nj}(t)dW_j(t) \tag{4.4}$$

We assume a complete market. This means that there exists a unique equivalent local martingale measure $\mathbb{P}^*$ and a Wiener-process w.r.t. $\mathbb{P}^*$ such that

$$dS_i^*(t) \;=\; S_i^*(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_j^*(t) \quad i = 1, \cdots d \tag{4.5}$$

with

$$\sigma_{ij}(t) = \bar{\sigma}_{ij}(t) - \sigma_{Nj}(t) \quad 1 \leq i, j \leq d.$$

4.4.1 American Claim

Definition 4.4.1. *An American claim is an adapted process Y with non-negative continuous paths. We assume*

$$\mathbb{E}^* \sup_{0 \leq t \leq T} Y^*(t) < \infty$$

with $Y^*(t) = \frac{Y(t)}{N(t)}$ for all $0 \leq t \leq T$.

The buyer of an American claim has the right to choose a stopping time τ that determines his pay-off $Y(\tau)$. He is faced with optimizing

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}^* Y^*(\tau).$$

Here $\mathcal{S}$ denotes the set of all stopping times with values in $[0, T]$.
The seller of a claim would like to eliminate his risk by running a trading strategy with value-process V^* given by

$$x + \int_0^t H(u)dS^*(u) \geq Y^*(t) \quad \text{for all} \quad t \leq T.$$

Then the initial capital x quoted in number of shares of the numeraire asset would be sufficient to cover his risk from the short position. At each time t he would be able to settle the pay-off $Y^*(t)$.

How can the buyer cover his risk. He has to borrow money $x > 0$ to finance the selling-price and tries to find a trading strategy H such that at some time he can use the pay-off from the claim to settle his loan obligation. This means he seeks for an initial capital $x > 0$, a previsible, admissible process H and a stopping time τ such that

$$Y^*(\tau) + (-x + \int_0^\tau H(u) dS^*(u)) \geq 0$$

The larger x can be chosen, the larger could be the price a buyer is willing to pay.
This leads to the definition of the super- and sub-replication price

Definition 4.4.2. *Let Y be an American claim. The super-replication price $p^+(Y)$ is defined by*

$$\begin{aligned} &p^+(Y) \\ &= \inf\{x > 0 : \textit{th. ex. an admissible } H \textit{with } x + \int_0^t H(u) dS^*(u) \geq Y^*(t) \textit{for all } 0 \leq t \leq T\}. \end{aligned}$$

The sub-replication price $p^-(Y)$ is defined by

$$\begin{aligned} &p^-(Y) \\ &= \sup\{x : \textit{th ex. an admissible } H, \textit{ a stopping time } \tau \textit{with } Y^*(\tau) + (-x + \int_0^\tau H(u) dS^*(u)) \geq 0\}. \end{aligned}$$

The super-replication price can be seen as the lowest price a seller could demand in order to cover his risk and the sub-replication price as the highest price a buyer would accept in order to cover his risk.
We will see in the following that in a complete market both prices coincide and are given by the value of an optimal stopping problem.

Proposition 4.4.3. *Let*

$$v := \sup_{\tau \in \mathcal{S}} \mathbb{E}^* Y^*(\tau)$$

Then

$$0 \leq Y^*(0) \leq p^-(Y) \leq v \leq p^+(Y).$$

Proof. According to $x = Y^*(0)$ choose $\tau = 0$ and $H = 0$. Then

$$Y^*(\tau) + (-x + \int_0^\tau H(u) dS^*(u)) = Y^*(0) - x = 0.$$

Hence

$$Y^*(0) \leq p^-(Y).$$

Let $0 < x < p^-(Y)$ be arbitrary. Then there exist an admissible H, a stopping time τ such that

$$Y^*(\tau) + (-x + \int_0^\tau H(u) dS^*(u)) \geq 0,$$

hence

$$x - \int_0^\tau H(u)dS^*(u) \le Y^*(\tau).$$

Since H is admissible, the process $(x - \int_0^t H(u)dS^*(u))$ is a $\mathbb{P}^*$-submartingale. Optional-sampling implies

$$x \le \mathbb{E}^*(x - \int_0^\tau H(u)dS^*(u)) \le \mathbb{E}^* Y^*(\tau).$$

Therefore $x \le v$ and this implies

$$p^-(Y) \le v.$$

If $x > p^+(Y)$ arbitrary chosen, then there exists some admissible H such that

$$x + \int_0^t H(u)dS^*(u) \ge Y^*(t) \quad \text{for all} \quad 0 \le t \le T.$$

Thus for each stopping time τ

$$x + \int_0^\tau H(u)dS^*(u) \ge Y^*(\tau)$$

and therefore

$$x \ge \mathbb{E}^*(x + \int_0^\tau H(u)dS^*(u)) \ge \mathbb{E} Y^*(\tau)$$

since $(x + \int_0^t H(u)dS^*(u))$ is a $\mathbb{P}^*$-supermartingale. Hence we obtain

$$p^+(C) \ge \sup_{\tau \in \mathcal{S}} \mathbb{E}^* Y^*(\tau) = v.$$

□

That all these three values coincide in the case of a complete market can be shown by exploiting some techniques from optimal stopping.

4.4.2 Snell-Envelope

Definition 4.4.4. *Let Y be an American claim. The Snell-envelope Z is defined by*

$$Z(t) = \operatorname{ess\,sup}_{\tau \in \mathcal{S}_t} \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t) \quad \textit{for all } 0 \le t \le T.$$

Hereby $\mathcal{S}_t$ denotes the set of all stopping times with values larger than t.

At time t the best a buyer can expect based on the observation up to t is the Snell-envelope $Z(t)$. If this coincides with the actual pay-off $Y^*(t)$ the buyer should stop at t and accept the pay-off. One might guess that

$$\tau^* = \inf_{t: Z(t) = Y^*(t)}$$

is the optimal stopping time for the buyer.
This is indeed often true and will be clarified in the following.
But first some properties of the sup resp. ess sup of a set of random variables has to be established.

Definition 4.4.5. *A family $(X_i)_{i\in I}$ of non-negative real-valued random variables is called a lattice if for each $i, j \in I$ there exists some $k \in I$ such that*

$$X_i \vee X_j \leq X_k.$$

Proposition 4.4.6. *If $(X_i)_{i\in I}$ is a lattice then for each sub-σ-field $\mathcal{G}$*

1. $\mathbb{E}\Big(\operatorname{ess\,sup}_{i\in I} X_i\Big) = \sup_{i\in I} \mathbb{E}X_i$

2. $\mathbb{E}\Big(\operatorname{ess\,sup}_{i\in I} X_i|\mathcal{G}\Big) = \operatorname{ess\,sup}_{i\in I} \mathbb{E}(X_i|\mathcal{G})$

Proof. The assertion follows by a tricky application of the monotone convergence theorem. First there is a sequence $(j_n)_{n\in\mathbb{N}} \in I$ such that

$$\operatorname{ess\,sup}_{i\in I} X_i = \sup_{n\in\mathbb{N}} X_{j(n)}.$$

Due to the lattice property we may find a sequence $(i(n))_{n\in\mathbb{N}}$ such that

$$X_{i(1)} \leq X_{i(2)} \leq \cdots \quad , \sup_{n\in\mathbb{N}} X_{i(n)} = \operatorname{ess\,sup}_{i\in I} X_i$$

in the following way. We set $i(1) = j(1)$. Then there exists some k such that

$$X_{i(1)} \vee X_{j(2)} \leq X_k.$$

Set $i(2) = k$. Then there exists some k such that

$$X_{i(2)} \vee X_{j(3)} \leq X_k.$$

Set $i(3) = k$ and continue this procedure until infinity. Then

$$X_{i(1)} \leq X_{i(2)} \leq X_{i(3)} \cdots$$

and

$$\operatorname{ess\,sup}_{i\in I} X_i \geq \sup_{n\in\mathbb{N}} X_{i(n)} \geq \sup_{n\in\mathbb{N}} X_{j(n)} = \operatorname{ess\,sup}_{i\in I} X_i,$$

hence

$$\sup_{n\in\mathbb{N}} X_{i(n)} = \sup_{n\in\mathbb{N}} X_{j(n)} = \operatorname{ess\,sup}_{i\in I} X_i.$$

With the monotone convergence theorem we obtain

$$\mathbb{E}\Big(\operatorname{ess\,sup}_{i\in I} X_i\Big) = \mathbb{E}\Big(\sup_{n\in\mathbb{N}} X_{i(n)}\Big) = \sup_{n\in\mathbb{N}} \mathbb{E}X_{i(n)} = \sup_{i\in I} \mathbb{E}X_i$$

and also for the conditional expectation

$$\mathbb{E}\Big(\operatorname{ess\,sup}_{i\in I} X_i|\mathcal{G}\Big) = \mathbb{E}\Big(\sup_{n\in\mathbb{N}} X_{i(n)}|\mathcal{G}\Big) = \sup_{n\in\mathbb{N}} \mathbb{E}(X_{i(n)}|\mathcal{G}) = \operatorname{ess\,sup}_{i\in I} \mathbb{E}(X_i|\mathcal{G})$$

$\square$

Theorem 4.4.7. *Let Y be an American claim and*

$$Z(t) = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t) \quad \text{for all } 0 \le t \le T$$

its Snell-envelope. Then the following assertions hold true.

1. *The process Z is a $\mathbb{P}^*$ supermartingale.*
2. *For any $0 \le t \le T$*
$$\mathbb{E}^* Z(t) = \sup_{\tau\in\mathcal{S}_t} \mathbb{E}^* Y^*(\tau)$$
3. *The process Z has a right-continuous modification.*

Proof. That the Snell-envelope Z is a $\mathbb{P}^*$-supermartingale is not that obvious. Fix a time-point $t < T$ and define for any stopping time $\tau \in \mathcal{S}_t$

$$X_\tau = \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t au).$$

Then the family of random variables $(X_\tau)_{\tau\in\mathcal{S}_t}$ forms a lattice. To show this we define for stopping times $\tau_1, \tau_2 \in \mathcal{S}_t$ the stopping time

$$\tau = \tau_1 \mathbb{1}_{\{X_{\tau_1} \ge X_{\tau_2}\}} + \tau_2 \mathbb{1}_{\{X_{\tau_2} \ge X_{\tau_1}\}}.$$

Since X_{τ_1} and X_{τ_2} are measurable w.r.t. $\mathcal{F}_t$ we get

$$\begin{aligned} X_\tau &= \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t) = \mathbb{E}^*(Y^*(\tau_1)\mathbb{1}_{\{X_{\tau_1} \ge X_{\tau_2}\}}|\mathcal{F}_t) + \mathbb{E}^*(Y^*(\tau_2)\mathbb{1}_{\{X_{\tau_2} \ge X_{\tau_1}\}}|\mathcal{F}_t) \\ &= \mathbb{1}_{\{X_{\tau_1} \ge X_{\tau_2}\}} X_{\tau_1} + \mathbb{1}_{\{X_{\tau_2} \ge X_{\tau_1}\}} X_{\tau_2} \ge X_{\tau_1} \vee X_{\tau_2}, \end{aligned}$$

from which the lattice property can be deduced. This now can be used to show the supermartingale property of Z. For $s < t$

$$\begin{aligned} \mathbb{E}^*(Z(t)|\mathcal{F}_s) &= \mathbb{E}^*\Big(\operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t)|\mathcal{F}_s\Big) \\ &= \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} \mathbb{E}^*\Big(\mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t)|\mathcal{F}_s\Big) \\ &= \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_s) \\ &\ge \operatorname{ess\,sup}_{\tau\in\mathcal{S}_s} \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_s) \\ &= Z(s) \end{aligned} \tag{4.6}$$

The lattice property can also be used to show the second assertion.

$$\begin{aligned} \mathbb{E}^* Z(t) &= \mathbb{E}^*\Big(\operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} X_\tau\Big) \\ &= \sup_{\tau\in\mathcal{S}_t} \mathbb{E}^* X_\tau \\ &= \sup_{\tau\in\mathcal{S}_t} \mathbb{E}^* Y^*(\tau), \end{aligned} \tag{4.7}$$

since

$$\mathbb{E}^* X_\tau = \mathbb{E}^*\mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t) = \mathbb{E}^* Y^*(\tau).$$

The third assertion follows from the fact that supermartingales have a right continuous modification.
The process Z is called Snell-envelope of Y^* since it is the smallest right continuous supermartingale that dominates Y^*. □

Corollary 4.4.8. *If V is a right continuous supermartingale with $V(t) \geq Y^*(t)$ for all $0 \leq t \leq T$ then $Z(t) \leq V(t)$ for all $0 \leq t \leq T$.*

Proof. Let's fix some $t \leq T$. Then for any $\tau \in \mathcal{S}_t$

$$\mathbb{E}(Y^*(\tau)|\mathcal{F}_t) \leq \mathbb{E}^*(V(\tau)|\mathcal{F}_t) \leq V(t).$$

Hence

$$Z(t) = \operatorname{ess\,sup}_{\tau \in \mathcal{S}_t} \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t) \leq V(t).$$

□

Proposition 4.4.9. *Let Y be an American claim with Snell-envelope Z. Then*

$$\limsup_{t \uparrow T} Y^*(t) = \lim_{t \uparrow} Z(t).$$

Proof. The non-negative supermartingale Z converges for $t \uparrow T$. Hence

$$\limsup_{t \uparrow T} Y^*(t) \leq \limsup_{t \uparrow T} Z(t) = \lim_{t \uparrow} Z(t).$$

For any $t \geq T_1$ we have

$$\mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t) \leq \mathbb{E}^*(\sup_{s \geq T_1} Y^*(s)|\mathcal{F}_t)$$

Hence

$$Z(t) \leq \mathbb{E}^*(\sup_{s \geq T_1} Y^*(s)|\mathcal{F}_t).$$

Due to

$$\mathbb{E}^* \sup_{s \geq T_1} Y^*(s) \leq \mathbb{E}^* \sup_{0 \leq s \leq T} Y^*(s) < \infty$$

we obtain

$$\mathbb{E}^*(\sup_{s \geq T_1} Y^*(s)|\mathcal{F}_t) \underset{t \to T}{\longrightarrow} \sup_{s \geq T_1} Y^*(s).$$

Therefore

$$\limsup_{t \uparrow T} Z(t) = \lim_{t \uparrow T} Z(t) \leq \sup_{s \geq T_1} Y^*(s)$$

for all $T_1 < T$. Hence the reverse inequality

$$\lim_{t \uparrow} Z(t) \leq \limsup_{t \uparrow T} Y^*(t)$$

is true. □

4.4.3 Optimal Stopping

With the Snell-envelope the optimal stopping problem can be solved and an optimal stopping time determined.

Definition 4.4.10. *Let Y be an American claim. Then a stopping time $\sigma \in \mathcal{S}$ is called optimal, if*

$$\mathbb{E}^* Y^*(\sigma) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^* Y^*(\tau).$$

First we can give a characterisation of an optimal stopping time via the Snell-envelope.

Proposition 4.4.11. *Let Y be an American claim with Snell-envelope Z. Then a stopping time σ is optimal if and only if the following two conditions are satisfied*

(i) $Z(\sigma) = Y^*(\sigma)$

(ii) The stopped process Z^σ is a $\mathbb{P}^*$-martingale.

Proof. To show the if part we consider an optimal stopping time σ. Then

$$\mathbb{E}^* Z(\sigma) \leq \mathbb{E}^* Z(0) = \sup_{\tau \in \mathcal{S}} Y^*(\tau) = \mathbb{E}^* Y^*(\sigma) \leq \mathbb{E}^* Z(\sigma). \tag{4.8}$$

Hence

$$\mathbb{E}^* Y^*(\sigma) = \mathbb{E}^* Z(\sigma)$$

and (i) follows due to $Z(\sigma) \geq Y^*(\sigma)$. The condition (ii) follows from

$$\begin{aligned} \mathbb{E}^*(Z^\sigma)_\tau &= \mathbb{E}^*(Z^\tau)_\sigma = \mathbb{E}^* Z(\sigma \wedge \tau) \\ &\geq \mathbb{E}^* Z(\sigma) = \mathbb{E}^* Y^*(\sigma) = \mathbb{E}^* Z(0) \geq \mathbb{E}^*(Z^\sigma)_\tau \end{aligned} \tag{4.9}$$

for each stopping time τ, since

$$\mathbb{E}^*(Z^\sigma)_\tau = \mathbb{E}^* Z(0) = \mathbb{E}^*(Z^\sigma)_0.$$

For the only if part we consider a stopping time σ that fulfills the conditions (i) and (ii). Then for any stopping time we have due to (ii)

$$\mathbb{E}^* Z(0) = \mathbb{E}^*(Z^\sigma)_\tau.$$

In particular for $\tau \equiv T$ we obtain

$$\mathbb{E}^* Z(0) = \mathbb{E}^*(Z^\sigma)_T = \mathbb{E}^* Z(\sigma) = \mathbb{E} Y^*(\sigma)$$

from which the optimality of σ follows. □

A tool for computing a super-replication strategy is based on the Doob-Meyer decomposition for supermartingales.

Theorem 4.4.12. *Let $(Z(t))_{0\le t\le T}$ be a right continuous non-negative supermartingale with*

$$\mathbb{E}\sup_{0\le t\le T} Z(t) < \infty.$$

Then there exist a right continuous uniformly integrable martingale M and a non-decreasing previsible right continuous process A with $A(0) = 0$ such that

$$Z = M - A.$$

This decomposition is unique up to indistinguishability.
If the process Z is regular then A has a.s. continuous paths.

The process Z is called regular if two conditions are fulfilled.

1. For any stopping time τ the random variable $Z(\tau$ is integrable.

2. For any increasing sequence of stopping times

$$\tau_1 \le \tau_2 \le \tau_3 \cdots \quad \text{with} \quad \sup_{\tau_n} = \tau$$

$$\lim_{n\to\infty} \mathbb{E}Z(\tau_n) = \mathbb{E}Z(\tau).$$

The Doob-Meyer decomposition can be used to prove properties of ε-optimal stopping times.

Definition 4.4.13. *Let Y be an American claim with Snell-envelope Z. For any $\varepsilon > 0$ and $0 \le t < T$ we define the so called ε-optimal stopping time D_t^ε by*

$$D_t^\varepsilon = \inf\{s \ge t : Y^*(s) \ge Z(s) - \varepsilon\}.$$

The following lemma is very useful in tha analysis of the optimal stopping problem.

Lemma 4.4.14. *Let Y be an American claim with Snell-envelope Z that has a Doob-Meyer decomposition*

$$Z = M - A$$

. Then for any $0 \le t < T$ and $\varepsilon > 0$

$$A(D_t^\varepsilon) = A(t) \quad \textit{almost sure.}$$

In particular the process A and $(A(D_t^\varepsilon))_{0\le t<T}$ are indistinguishable.

Proof. Let $0 \le t < T$ be arbitrary. Due to

$$\mathbb{E}^* Z(t) = \sup_{\tau\in\mathcal{S}_t} \mathbb{E}^* Y^*(\tau)$$

we can choose a sequence of stopping times $(\tau_j)_{j\in\mathbb{N}}$ with $\tau_j \in \mathcal{S}_t$ and

$$\lim_{j\to\infty} \mathbb{E}Y^*(\tau_j) = \mathbb{E}^* Z(t).$$

It follows

$$\begin{aligned}\mathbb{E}^*Y^*(\tau_j) &\leq \mathbb{E}^*Z(\tau_j) = \mathbb{E}^*(M(\tau_j) - A(\tau_j)) \\ &= \mathbb{E}^*M(t) - \mathbb{E}^*A(\tau_j) = \mathbb{E}Z(t) - \mathbb{E}^*(A(\tau_j) - A(t)).\end{aligned} \tag{4.10}$$

Therefore

$$0 \leq \mathbb{E}^*(Z(\tau_j) - Y^*(\tau_j)) \leq \mathbb{E}^*(Z(t) - Y^*(\tau_j)) = Er^*(A(\tau_j) - A(t)) \longrightarrow 0.$$

From the L_1-convergence we may deduce the a.s. convergence of a subsequence. Hence we can assume

$$Z(\tau_j) - Y^*(\tau_j)) \longrightarrow 0 \quad \text{a.s.}.$$

Hence we have

$$D_t^\varepsilon \leq \tau_j \quad \text{for lagre} \;\; j,$$

and thus

$$A(D_t^\varepsilon) \leq A(\tau_j) \quad \text{for lagre} \;\; j.$$

Due to $\lim_{j\to\infty} A(\tau_j) = A(t)$ we obtain

$$A(D_t^\varepsilon) \leq A(t).$$

The reverse inequality follows immediately since A has non-decreasing paths and $D_t^\varepsilon \geq t$. To show that A and $(A(D_t^\varepsilon))_{0\leq t<T}$ are indistinguishable we use the right-continuity of the processes A, Z, Y^*. On all rational t the random variables $A(D_t^\varepsilon)$ and $A(t)$ coincide. For t arbitrary consider a sequence of rational $t_n \geq t$ such that $t_n \downarrow t$. Then

$$A_t \leq A(D_t^\varepsilon) \leq A(D_{t_n}^\varepsilon) = A(t_n) \longrightarrow A_t.$$

Hence

$$A_t = A(D_t^\varepsilon) \quad \text{for all } 0 \leq t < T\mathbb{P}^* - a.s.$$

□

As corollary we obtain

Corollary 4.4.15. *Let Z be the Snell-envelope of the American claim Y. Then for all $0 \leq t < T$*

$$\begin{aligned}\mathbb{E}^*Z(D_t^\varepsilon) &= \mathbb{E}^*Z(t) \\ \mathbb{E}^*Z(D_t^\varepsilon) &\geq \mathbb{E}^*Z(t) - \varepsilon\end{aligned} \tag{4.11}$$

Proof. Let $0 \leq t < T$ be arbitrary. Then

$$Z(D_t^\varepsilon) = M(D_t^\varepsilon) - A(D_t^\varepsilon) = M(D_t^\varepsilon) - A(t),$$

hence

$$\mathbb{E}^*Z(D_t^\varepsilon) = \mathbb{E}^*M(t) - \mathbb{E}^*A(t) = \mathbb{E}^*Z(t).$$

Furthermore

$$Y^*(D^\varepsilon_t) \geq Z(D^\varepsilon_t) - \varepsilon = M(D^\varepsilon_t) - A(D^\varepsilon_t) - \varepsilon,$$

hence

$$\mathbb{E}^* Y^*(D^\varepsilon_t) \geq \mathbb{E}^* M(t) - \mathbb{E}^* A(t) - \varepsilon = \mathbb{E}^* Z(t) - \varepsilon.$$

□

Together with the regularity of Z this can be used to construct an optimal stopping time.

Theorem 4.4.16. *Let Y be an American claim. Then the Snell-envelope is regular and*

$$\sigma = \inf\{t \geq 0 : Z(t) = Y^*(t)\}$$

is an optimal stopping time.

Proof. Due to

$$\mathbb{E}^* \sup_{0 \leq t \leq T} Y^*(t) < \infty$$

the regularity of Y^* follows, since Y^* has continuous paths and dominated convergence can be applied. Note, that the Snell-envelope Z need not to be a continuous process. To show the regularity of Z we consider a sequence of stopping times τ_n such that

$$\tau_1 \leq \tau_2 \leq \tau_3 \leq \cdots \longrightarrow \tau.$$

Since Z is a $\mathbb{P}^*$-supermartingale, we obtain

$$\mathbb{E}^* Z(\tau_1) \geq Er^* Z(\tau_2) \geq \cdots \geq \mathbb{E}^Z(\tau).$$

For the reverse inequality we consider

$$D^\varepsilon_{\tau_n} = \inf\{s \geq \tau_n : Y^*(s) \geq Z(s) - \varepsilon\}.$$

Then due to 4.4.14

$$A(D^\varepsilon_{\tau_n}) = A(\tau_n)$$

and due to 4.4.15

$$\mathbb{E}^* Z(\tau_n) = \mathbb{E}^* Z(D^\varepsilon_{\tau_n})$$

and therefore

$$\mathbb{E}^* Y^*(D^\varepsilon_{\tau_n}) \geq \mathbb{E}^* Z(D^\varepsilon_{\tau_n}) - \varepsilon.$$

The sequence $(D^\varepsilon_{\tau_n})$ is non-decreasing and dominated by D^ε_τ. Hence

$$\bar{\sigma} = \lim_{n \to \infty} D^\varepsilon_{\tau_n}$$

satisfies

$$\tau \leq \bar{\sigma} \leq D^\varepsilon_\tau.$$

The regularity of Y^* implies

$$\mathbb{E}^* Y^*(\bar{\sigma}) = \lim_{n\to\infty} \mathbb{E}^* Y^*(D^{\varepsilon}_{\tau_n}).$$

Thus it follows

$$\begin{aligned} \mathbb{E}^* Z(\tau) &\geq \mathbb{E}^* Z(\bar{\sigma}) \geq \mathbb{E}^* Y^*(\bar{\sigma}) = \lim_{n\to\infty} \mathbb{E}^* Y^*(D^{\varepsilon}_{\tau_n}) \\ &\geq \lim_{n\to\infty} Z(D^{\varepsilon}_{\tau_n}) - \varepsilon = \lim_{n\to\infty} \mathbb{E}^* Z(\tau_n) - \varepsilon. \end{aligned} \tag{4.12}$$

Hence

$$\mathbb{E}^* Z(\tau) \geq \lim_{n\to\infty} \mathbb{E}^* Z(\tau_n.$$

To show the optimality of σ we consider a non-increasing sequence $\varepsilon_n \downarrow 0$. It is $D_0^{\varepsilon_n}$ non-decreasing in n with

$$D_0^{\varepsilon_n} \leq \sigma \quad \text{for all } n \in \mathbb{N}.$$

Hence

$$\mathbb{E}^* Z(0) = \mathbb{E}^* Z(D_0^{\varepsilon_n}) \leq \mathbb{E}^* Y^*(D_0^{\varepsilon_n}) + \varepsilon_n.$$

For

$$D_0^+ := \lim_{n\to\infty} D_0^{\varepsilon_n}$$

we obtain due to the regularity of Z

$$\mathbb{E}^* Z(D_0^+) = \lim_{n\to\infty} \mathbb{E}^* Z(D_0^{\varepsilon_n}) = \mathbb{E}^* Z(0)$$

and

$$\mathbb{E}^* Z(D_0^+) \leq \lim_{n\to\infty} \mathbb{E}^* Y^*(D_0^{\varepsilon_n}) + \varepsilon_n = \mathbb{E}^* Y^*(D_0^+).$$

This implies that D_0^+ is an optimal stopping time. Due to 4.4.16 it follows

$$Y^*(D_0^+) = Z(D_0^+) \quad \text{and} \quad \sigma \leq D_0^+$$

. All together yields

$$\sigma = D_0^+$$

and the assertion. □

In the case of a complete financial market the super-and sub-replication price coincide.

Theorem 4.4.17. *Let Y be an American claim in a complete financial market with unique equivalent local martingale measure $\mathbb{P}^*$. Then*

$$p^-(Y) = v = p^+(Y),$$

whereat

$$v = \sup_{\tau\in\mathcal{S}} \mathbb{E}^* Y^*(\tau)$$

denotes the optimal stopping problem for the buyer.

Proof. Let Z denote the Snell-envelope of Y^* with Doob-Meyer decomposition

$$Z = M - A$$

. Then due to 4.4.14

$$v = \mathbb{E}^* Z(0) = Z(0) = M(0).$$

The martingale representation theorem **??** provides a previsible process $\alpha = (\alpha_1, \cdots, \alpha_d)$ such that

$$M(t) = M(0)0 \int_0^t \alpha(s) dW(s) \quad \text{for all } 0 \leq t \leq t.$$

As in the case of a European claim we may find a previsible H such that

$$\int_0^t \alpha(s) dW(s) = \int_0^t H8u) dS^*(u) = \int_0^t H(u) S^*(u) \sigma(u) dW(u)$$

. With v and H a super replication strategy can be established with discounted value-process

$$V^*(t) = v + \int_0^t H(u) dS^*(u) = M(0) + \int_0^t \alpha(s) dW^*(s)$$

$$= \; = M(t) \geq Z(t) \geq Y^*(t) \quad \text{for all } 0 \leq t \; (4.17$$

Hence $v \geq p^+(Y)$ and therefore $v = p^+(Y)$.
Due to v we choose the optimal stopping time

$$\sigma = \inf\{t \geq 0 : Z(t) \leq Y^*(t)\}$$

for the buyer. With the help of $-H$ he can run a self-financing trading strategy with discounted value-process

$$V^*(t) = -v + \int_0^t (-H(u)) dS^*(u) \quad \text{for all } 0 \leq t \leq T$$

and therefore

$$V^*(\sigma) + Y^*(\sigma) = -M(\sigma) + Y^*(\sigma) = -Z(\sigma) + Y^*(\sigma) = 0$$

Thus the optimal expired claim amount can cover the value of the shorted trading strategy. Therefore $p^-(Y) \geq v$ which $p^-(Y) = v$ implies.

$\square$

Note that according to the optimal stopping time σ there exists some previsible H such that

$$v + \int_0^\sigma H(u) dS^*(u) = M(\sigma) = Z(\sigma) = Y^*(\sigma).$$

The discounted value process

$$v + \int_0^t H(u) dS^*(u) = M(t) \quad \text{for all } 0 \leq t \leq T$$

is a $\mathbb{P}^*$-martingale.

4.4.4 Computing

Although a replicating strategy exists in principle it can in general not explicitly determined. As in the case of European derivatives further assumptions are necessary. For example considering

- a diffusion model
- with an American claim $Y(t) = h(S(t))$

leads to an optimal stopping problem that can be handled with a PDE-approach. In the following this will be explained. We consider a complete diffusion model with d risky assets that perform under the unique equivalent martingale measure $\mathbb{P}^*$ by

$$dS_i(t) = S_i(t)(r(t, S(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t, S(t))dW_j(t) \quad \text{for all } 1 \leq i \leq d. \tag{4.14}$$

As numeraire asset we consider a money market account β with random interest rate r that depends on time and state of the risky assets, i.e.

$$d\beta(t) = r(t, S(t))dt.$$

Furthermore we consider an American claim of the form

$$Y(t) = h(S(t)) \text{ for all } 0 \leq t \leq T.$$

that fulfills

$$\mathbb{E}^* \sup_{0 \leq t \leq T} \frac{Y(t)}{\beta(t)} < \infty.$$

The most important example is an American put with strike K and running time T which can be seen as American claim of the form

$$Y(t) = (K - S(t))^+.$$

Due to the Markov-property the Snell-envelope Z of Y^* can be written as function of time t and state $S(t)$ due to

$$\begin{aligned}
\beta(t)Z(t) &= \Big(\operatorname{ess\,sup}_{\tau \in \mathcal{S}_t} \mathbb{E}^*(Y^*(\tau)|\mathcal{F}_t)\Big)\beta(t) \\
&= \operatorname{ess\,sup}_{\tau \in \mathcal{S}_t} \mathbb{E}^*(\exp(-\int_t^\tau (r(u, S(u))du)\, h(S(\tau))|\mathcal{F}_t) \\
&= \operatorname{ess\,sup}_{\tau \in \mathcal{S}_t} \mathbb{E}^*(\exp(-\int_t^\tau (r(u, S(u))du)\, h(S(\tau))|S(t)),
\end{aligned}$$

hence

$$\beta(t)Z(t) = v(t, S(t)) \quad \text{for all } 0 \leq t \leq T$$

with

$$v(t,x) = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} \mathbb{E}^*(\exp(-\int_t^\tau (r(u,S(u))du)\, h(S(\tau))|S(t)=x)$$

for all $0 \le t \le T$ and $x \in (0,\infty)^d$. The value $v(t,x)$ can be seen as price in Euro, when in t the state x of the risky assets are seen. The state-space $(0,\infty)^d$ decomposes in a continuation and stopping region. We define the early exercise region and continuation region by

$$\begin{aligned} \mathcal{E} &= \{(t,x) : 0 \le t \le T, v(t,x) = h(x)\} \\ \mathbb{C} &= \{(t,x) : 0 \le t \le T, v(t,x) > h(x)\} \end{aligned} \tag{4.15}$$

Applied the general result 4.4.16 to this specific situation leads to the following theorem

Theorem 4.4.18. *We consider a complete diffusion model as described above and an American claim $Y = h(S(\cdot))$ that fulfills*

$$\mathbb{E}^* \sup_{0\le t\le T} \frac{Y(t)}{\beta(t)} < \infty.$$

Then an optimal stopping time σ is given by the first exit-time from the continuation region, i.e.

$$\begin{aligned} \sigma &= \inf\{0 \le t \le T : v(t,S(t)) = h(S(t))\} \\ &= \inf\{0 \le t \le T : (t,S(t)) \notin \mathbb{C}\} \end{aligned} \tag{4.16}$$

Furthermore on suitable conditions on the volatility and h the continuation region $\mathbb{C}$ is an open set and the price function v satisfies the following PDE on $\mathbb{C}$.

$$\partial_t v(t,x) + \frac{1}{2}\sum_{i,j=1}^d x_i x_j (\sigma\sigma^T)_{ij}(t,x)\partial_i\partial_j v(t,x) + \sum_{i=1}^d x_i r(t,x)\partial_i v(t,x) = r(t,x)v(t,x)$$

for all $(t,x) \in \mathbb{C}$.

Proof. We would like to give a sketch of the proof. From 4.4.16 we know that the optimal stopping time is given by

$$\begin{aligned} \sigma &= \inf\{t \ge 0 : Z(t) = Y^*(t)\} \\ &= \inf\{t \ge 0 : v(t,S(t)) = Y(t) = h(S(t))\} \\ &= \inf\{t \ge 0 : (t,S(t) \notin \mathbb{C}\} \end{aligned} \tag{4.17}$$

From the general theory on optimal stopping it is well known that the price function v is continuous as function of t and x. This implies that the continuation region is an open set and that v is an harmonic function on $\mathbb{C}$, in particular it is a $C^{1,2}$ function on $\mathbb{C}$. That v satisfies the PDE follows from an application of Ito's formula and the fact that $(\frac{v(t\wedge\sigma,S(t\wedge\sigma)}{\beta(t\wedge\sigma)})_{0\le t\le T}$ is a $\mathbb{P}^*$-martingale. □

Note, that the above PDE is a so called free-boundary value problem, since the boundary is not initially fixed. Simultaneously a function v and the boundary of $\mathbb{C}$ has ro be found to give a solution. This provides an extra difficulty in solving these PDE-problems. This can be illustrated in the Black-Scholes model when analysing the American-put.

American Put in the Black-Scholes Model

We consider a BS-model with volatility $\sigma > 0$ and interest rate r, i.e.

$$\begin{aligned} dS(t) &= S(t)(rdt + \sigma dW(t)) \\ d\beta(t) &= r\beta(t)dt \end{aligned} \tag{4.18}$$

according to the equivalent martingale measure $\mathbb{P}^*$. As American claim the put is analysed that has a pay-off process

$$Y(t) = (K - S(t))^+ \quad 0 \leq t \leq T.$$

The put price

$$v(t,x) = \sup_{\tau \in \mathcal{S}_t} \mathbb{E}^*(e^{-r(T-t)}(K - S(\tau))^+ | S(t) = x)$$

is a solution to the following free-boundary value problem due to 4.4.18.

$$\partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v(t,x) + rxv(t,x) \quad = \quad rv(t,x) \tag{4.19}$$

for all $0 < t < T$ and $x > b(t)$ with boundary conditions

$$v(t,x) = K - x \qquad \text{for all } 0 < t < T, x \leq b(t)$$

$$\partial_x v(t, b(t)) = -1 \qquad \text{for all } 0 < t < T.$$

The function v and the unknown boundary function b have to be determined. The early exercise and continuation region can be written as

$$\begin{aligned} \mathcal{E} &= \{(t,x) : 0 \leq t \leq T : x \leq b(t)\} \\ \mathbb{C} &= \{(t,x) : 0 \leq t \leq T : x > b(t)\}. \end{aligned} \tag{4.20}$$

The boundary function b is a continuous, increasing convex function with $\lim_{t \uparrow T} b(t) = K$. The function b is the so called free boundary of v. Even in this simple case an explicit representation for b and v is unknown. Only with numerical methods a valuation can be done.

4.5 Volatility Models

Objective: Specification of a practically relevant model.

4.5.1 Calibration of a Black-Scholes Model

Model-equation:

$$dS(t) = S(t)(rdt + \sigma dW^\star(t))$$
$$d\beta(t) = \beta(t)rdt$$

for all $0 \leq t < T$ with volatility σ and constant interest rate r.
The Black-Scholes model is mainly used for short running-times of $T = 3, 6$ or 9 months. The interest rate can then be taken from the over-night rate or the rate of a bond with short maturity. These parameters are endogenously fixed.
Problem: How to determine the volatility σ?
Solution: The volatility determines the price of a derivative. Initial prices of traded securities (market-prices) are further informations from the market that can be used for a calibration of the model, here a fixing of the unknown volatility. In the Black-Scholes model usually market-prices of calls or puts with different maturities and strikes are taken.
More precise:

$$C(x, T, \sigma, K) := \mathbb{E}^{\star} e^{-rT} (S(T) - K)^{+}$$

denotes the model-price of a call with maturity T, strike K and volatility σ. If we denote by $C_M(T, K)$ the corresponding observable market-price then there exists a unique volatility

$$\sigma = \sigma_{\text{impl.}}(T, K)$$

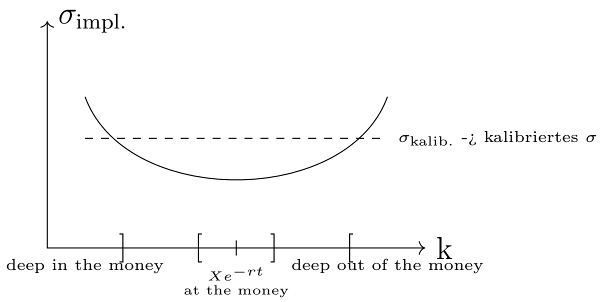
such that

$$C(x, T, \sigma, K) = C_M(T, K).$$

$\sigma_{\text{impl.}}$ is called the implied volatility die of this call. If the Black-Scholes model were a perfect fit then the implied volatility $\sigma_{\text{impl.}}(t, k)$ would remain constant. In real markets effectively an implied volatility-surface is visible.

$$(t, K) \mapsto \sigma_{\text{impl.}}(t, K) \quad 0 < t < T, K > 0.$$

The observed curvature of the volatility-surface is denoted as smile-effect. For fixed t the intersection line $K \mapsto \sigma_{\text{impl.}}(t, K)$ is of the form



To calibrate a Black-Scholes model calls with different strikes are used and that volatility taken that minimises the error between market-prices and model-prices.

4.5.2 Calibration of a Black-Scholes model with deterministic volatility

Model-equation:

$$\begin{aligned} dS(t) &= S(t)(rdt + \sigma(t)dW^\star(t)) \\ d\beta(t) &= \beta(t)rdt \end{aligned}$$

The function $\sigma : [0,T] \longrightarrow \mathbb{R}_{>0}$ is non-random with $\int_0^T \sigma^2(s)ds < \infty$. The parameters T and r are again exogenously fixed.
To determine the volatility function σ the market-prices $(C_M(t,K))_{0\le t<T, K>0}$ resp. their implied volatility surface $(\sigma_{\text{impl.}}(t,K))_{0\le t<T,K>0}$ are needed
If the model were a perfect fit then the unknown volatility-function σ would be fixed by the market-prices $(C_M(t,K))_{0\le t<T}$ according to a specific K.
Explanation:
With $C_M(t,K)$ the integral $\int\limits_0^t \sigma^2(s)ds$ is fixed by the implied volatility $\sigma_{\text{impl.}}(t,K)$ for every t. Taking the derivative determines σ^2 and therefore $\sigma(t)$ for all $0 \le t < T$.
With this procedure we obtain for every $K > 0$ a function $(\sigma_K(t))_{0\le t<T}$.
A curvature in the implied volatility surface w.r.t. K means, that a Black-Scholes model with deterministic volatility is not reasonable too.
To calibrate such a model one would use the collection of functions $(\sigma(K))_{K>0}$ in order to determine some function that explains the market-prices best.

4.5.3 Calibration of a Local Volatility Model

Model-equation:

$$\begin{aligned} dS(t) &= S(t)(rdt + \sigma(t,S(t))dW^\star(t)) \\ d\beta(t) &= \beta(t)rdt \end{aligned}$$

The function $\sigma(t,x)$ denotes for every $0 \le t < T$ and $x > 0$ the local volatility and is fixed by the market-prices

$$(C_M(t,K))_{0\le t<T,K>0}\,.$$

This is the formula of Dupire:

$$\frac{1}{2}K^2\sigma^2(t,K) = \frac{\partial_t C(t,K) + K\partial_K C(t,K)}{\partial_K^2 C(t,K)} \quad \text{for all } 0 \le t < T, K > 0.$$

Proof. Let $f(t,\cdot)$ denote the density of $S(t)$, i.e.

$$f(t,x)dx = \mathbb{P}^\star(S(t) \in dx).$$

Then

$$
\begin{aligned}
C(t,K) &= e^{-rt}\int_0^\infty (x-K)^+ f(t,x)dx \\
&= e^{-rt}\int_K^\infty (x-K) f(t,x)dx \\
&= e^{-rt}\int_K^\infty \int_K^x dy f(t,x)dx \\
&\overset{\text{Fubini}}{=} e^{-rt}\int_K^\infty \int_y^\infty f(t,x)dxdy.
\end{aligned}
$$

Hence

$$
\partial_K C(t,K) = -e^{-rt}\int_K^\infty f(t,x)dx
$$

and therefore

$$
\partial_K^2 C(t,K) = e^{-rt} f(t,K). \tag{4.21}
$$

The second partial derivative w.r.t. K of the call-price determines this density. In the diffusion model the density f satisfies a forward Kolmogorov equation

$$
\begin{aligned}
\partial_t f(t,x) &= \frac{1}{2}\partial_x^2(x^2\sigma^2(t,x)f(t,x)) - \partial_x(rxf(t,x)) \\
&= \frac{1}{2}\partial_x^2(x^2\sigma^2(t,x)f(t,x)) - rf(t,x) - rx\partial_x f(t,x).
\end{aligned}
$$

Due to Equation 4.21 it also holds true:

$$
\partial_t f(t,x) = \partial_t(e^{rt}\partial_x^2 C(t,x)) = re^{rt}\partial_x^2 C(t,x) + e^{rt}\partial_t\partial_x^2 C(t,x).
$$

Hence it follows with $f(t,x) = e^{rt}\partial_x^2 C(t,x)$:

$$
re^{rt}\partial_x^2 C(t,x) + e^{rt}\partial_x^2\partial_t C(t,x) = e^{rt}\frac{1}{2}\partial_x^2(x^2\sigma^2(t,x)\partial_x^2 C(t,x)) - re^{rt}\partial_x^2 C(t,x) - re^{rt}\partial_x\partial_x^2 C(t,x)
$$

and therefore

$$
\begin{aligned}
\partial_x^2\partial_t C(t,x) &= \frac{1}{2}\partial_x^2(x^2\sigma^2(t,x)\partial_x^2 C(t,x)) - r\partial_x^2(x\partial_x C(t,x)) \\
&= \partial_x^2\Big(\frac{1}{2}x^2\sigma^2(t,x)\partial_x^2 C(t,x) - rx\partial_x C(t,x)\Big).
\end{aligned}
$$

Twice integration provides functions $\alpha(t), \beta(t)$ such that

$$
\frac{1}{2}x^2\sigma^2(t,x)\partial_x^2 C(t,x) = rx\partial_x C(t,x) + \partial_t C(t,x) + \alpha(t)x + \beta(t)
$$

Due to the boundary condition

$$x^2\sigma^2(t,x)\partial_x^2 C(t,x) = e^{-rt}x\sigma^2(t,x)f(t,x) \overset{x\to\infty}{\longrightarrow} 0$$
$$x\partial_x C(t,x) = -e^{-rt}x\int\limits_x^\infty f(t,y)dy \overset{x\to\infty}{\longrightarrow} 0$$
$$\partial_t C(t,x) \overset{x\to\infty}{\longrightarrow} 0$$

we obtain $\alpha(t) = \beta(t) = 0$ for every $0 \le t < T$ and this implies the formula of Dupire. $\square$

4.5.4 Stochastic Volatility Models in General

The previous models all describe a complete financial market that vary in their volatility assumptions. This means that the whole uncertainty in the market can be explained by the randomness of the asset prices. In real world markets this assumption is not reasonable. One should distinguish between external and internal factors that drive the asset's prices. Examples for external uncertainty are

- Political decisions,
- Catastrophic events like Hurricanes, floods, pandemics,
- Other catastrophic events like 9/11

These risks can in principal not be hedged by trading in the market.
A class of models that are reasonable for describing these effects are so called stochastic volatility models where an external factor affects the volatility of the risky asset. The price-process of a stock in such a model is given by

$$dS(t) = S(t)(\mu dt + f(Y(t))\,dW(t))$$

and we assume that the volatility depends on a stochastic process Y via the function f with dynamics given by

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dZ(t).$$

The driving one-dimensional Wiener-processes are denoted by W and Z. The process W stands for the internal uncertainty whereas Z describes the external one. It is assumed that these Wiener-process are correlated with factor $\varrho \in (-1,1)$. This means that

$$\langle W, Z\rangle_t = \varrho t \quad \text{ for all } t \ge 0.$$

The correlation ϱ is usually assumed to be negative since stock-prices and volatilities behave in opposite directions. An increasing stock-price tends to reduce the volatility while a decreasing price leads to an increase in volatility. This is the so called Leverage-effect. Above a very general approach is formulated and the question arises.
Question: When can the above stochastic differential equation be solved?

The path to be gone is the following.
First the equation for the volatility Y has to be solved. Since the equation for the stock S is linear a solution can be expressed by an exponential semi-martingale. A first result can be found in many textbooks.

Theorem 4.5.1. *Let the functions b and σ fulfill a linear growth- and Lipschitz condition, i.e.*

$$
\begin{aligned}
|b(x)| &\leq c_1 + c_2|x| \quad \text{for all } x \in \mathbb{R} \\
|b(x) - b(y)| &\leq c|x-y| \quad \text{for all } x, y \in \mathbb{R}
\end{aligned}
$$

,analogous for σ, then there exists to every $y \in \mathbb{R}$ a unique strong solution of the SDE

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dW(t), \quad Y_0 = y.$$

However, the Lipschitz condition for σ fails in interesting cases. A weaker assumption is the Yamada-Watanabe condition.

Definition 4.5.2. *A function $g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Yamada-Watanabe condition, if there exists a strictly increasing function*

$$\rho : [0, \infty) \longrightarrow [0, \infty)$$

such that

$$\int_0^{\epsilon} \frac{1}{(\rho(x))^2} ds = +\infty$$

for some $\epsilon > 0$ and

$$|g(x) - g(y)| \leq \rho(|x-y|)$$

for all $x, y \in \mathbb{R}$.

Example 4.5.3. Consider $g(x) = \sqrt{|x|}$. Then g is not Lipschitz-continuous but satisfies the Yamada Watanabe condition with $\rho(x) = \sqrt{x}$ due to

$$\lim_{x \searrow 0} g'(x) = \lim_{x \searrow 0} \frac{1}{2} \frac{1}{\sqrt{x}} = +\infty.$$

Theorem 4.5.4. *Let the function b fulfill the linear growth- and Lipschitz condition. Let the function σ be continuous satisfying the linear growth- and Yamada-Watanabe condition.*
Then the equation

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dZ(t), \quad Y(0) = y$$

has a unique strong solution.

Proof. The continuity of b and σ provides, that the equation can be weakly solved. The Yamada-Watanabe condition implies a path-wise uniqueness of the solution.
Both together implies the existence of a unique strong solution. $\square$

Remark 4.5.5. The above solution fulfills

$$\mathbb{E}\int_0^t Y^2(s)ds = \int_0^t \mathbb{E}Y^2(s)ds < \infty \qquad \text{for all } t > 0.$$

This can be used to solve the equations of the stochastic volatility equations.

Solving the Stochastic Volatility Equation

Model equations:

$$\begin{aligned} dS(t) &= S(t)(\mu dt + f(Y(t))dW(t)) \\ dY(t) &= b(Y(t))dt + \sigma(Y(t))dZ(t) \\ d\langle W, Z\rangle_t &= \varrho dt \end{aligned}$$

Let b and σ fulfill the assumptions of 4.5.1 or 4.5.4 and let f be a continuous function. Then there exists for any starting point $y \in \mathbb{R}$ and $S(0) \in \mathbb{R}$ a unique strong solution for the above system of stochastic differential equations.

Proof. The assumptions apply that the equation for Y has a strong solution. Since the equation for S is linear we obtain

$$S(t) = S(0)\exp\left(\int_0^t f(Y(s))dW(s) - \frac{1}{2}\int_0^t f(Y(s))^2 ds\right) e^{\mu t} \qquad \text{for all } t \geq 0.$$

$\square$

Alternatively the model-equations can also be expressed by uncorrelated Wiener-processes.
Approach: Let

$$\widetilde{W} = \frac{1}{\sqrt{1-\varrho^2}}W - \frac{\varrho}{\sqrt{1-\varrho^2}}Z$$

Then $\widetilde{W}$ and Z are independent Wiener-prozesses with

$$W = \sqrt{1-\varrho^2}\widetilde{W} + \varrho Z.$$

This follows from Lévy's theorem:

$\widetilde{W}$ is a local martingale with

$$\begin{aligned}
\langle \widetilde{W} \rangle_t &= \langle \frac{1}{\sqrt{1-\varrho^2}} W - \frac{\varrho}{\sqrt{1-\varrho^2}} Z \rangle_t \\
&= \frac{1}{1-\varrho^2} t + \frac{\varrho^2}{1-\varrho^2} t - 2 \frac{\varrho}{1-\varrho^2} \underbrace{\langle W, Z \rangle_t}_{\varrho t} \\
&= \frac{1}{1-\varrho^2} t + \frac{\varrho^2}{1-\varrho^2} t - \frac{2\varrho^2}{1-\varrho^2} t \\
&= t.
\end{aligned}$$

Furthermore

$$\langle \widetilde{W}, Z \rangle_t = \frac{1}{\sqrt{1-\varrho^2}} \langle W, Z \rangle_t - \frac{\varrho}{\sqrt{1-\varrho^2}} \langle Z, Z \rangle_t = \frac{\varrho}{\sqrt{1-\varrho^2}} t - \frac{\varrho}{\sqrt{1-\varrho^2}} t = 0$$

Then the model-equations are given by:

$$\begin{aligned}
dS(t) &= S(t)\Big((\mu dt + f(Y(t))(\sqrt{1-\varrho^2} d\widetilde{W}(t) + \varrho dZ(t))\Big) \\
dY(t) &= b(Y(t))dt + \sigma(Y(t))dZ(t)
\end{aligned}$$

with $\widetilde{W}$ and Z independent Wiener-processes.

Examples

(i) Hull-White Model:

$$\begin{aligned}
dS(t) &= S(t)(\mu dt + Y(t)dW(t)) \\
dY(t) &= Y(t)(\theta dt + \xi dZ(t)) \\
\langle W, Z \rangle_t &= \varrho t
\end{aligned}$$

with $\mu, \theta \in \mathbb{R}, \xi > 0$ and $\varrho \in (-1, 1)$.
(ii) Stein-Stein Model:

$$\begin{aligned}
dS(t) &= S(t)(\mu dt + Y(t)dW(t)) \\
dY(t) &= q(m - Y(t))dt + \sigma dZ(t) \\
\langle W, Z \rangle_t &= \varrho t
\end{aligned}$$

with $\mu \in \mathbb{R}, q, \sigma > 0, m \geq 0$ and $\varrho \in (-1, 1)$.
The volatility is determined by a Vasicek-process which is mean reverting.
(iii) Heston Model:

$$\begin{aligned}
dS(t) &= S(t)(\mu dt + \sqrt{Y(t)} dW(t)) \\
dY(t) &= (a - bY(t))dt + c\sqrt{Y(t)} dZ(t) \\
\langle W, Z \rangle_t &= \varrho t
\end{aligned}$$

with $\mu \in \mathbb{R}, c > 0, a, b \geq 0$ and $\varrho \in (-1, 1)$.
The process Y is a CIR (Cox-Ingersoll-Ross) process.
It is mean reverting and remains in $[0, \infty)$ for all times. If $b > 0$, then

$$dY(t) = q_L(m - Y(t))dt + c\sqrt{Y(t)}dZ(t)$$

$q_L = b, m = \frac{a}{b}$, m return level and q_L mean rate of return.

Call Price in the Heston model

The Heston model is of significance importance and a very popular stochastic volatility model. In the following we will show how an explicit call-price can be computed. To be more precise we consider the following Heston-model under a subjective probability measure $\mathbb{P}$.

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sqrt{Y(t)}dW(t) \\ dY(t) &= q(m - Y(t))dt + \sigma\sqrt{Y(t)}dZ(t). \langle W, Z\rangle_t = \rho t \end{aligned}$$

with parameters $q, m, \sigma > 0$ and $\rho \in (-1, 1)$.
The question of existence of equivalent local martingale measures is in general difficult to answer for stochastic volatility models. In the Heston-model it is usually assumed that the so called market-price of volatility risk is proportional to the volatility. Then a unique local equivalent martingale measure can be constructed. But first we express the model by independent Wiener-processes. Let us introduce

$$\widetilde{W} = \frac{1}{\sqrt{1-\varrho^2}}W - \frac{\varrho}{\sqrt{1-\varrho^2}}Z$$

Then $\widetilde{W}$ and Z are independent Wiener-processes with

$$W = \sqrt{1-\varrho^2}\widetilde{W} + \varrho Z$$

and the Heston equations are given by

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sqrt{Y(t)}\Big(\sqrt{1-\varrho^2}d\widetilde{W}(t) + \varrho dZ(t)\Big) \\ dY(t) &= q(m - Y(t))dt + \sigma\sqrt{Y(t)}dZ(t) \end{aligned}$$

with independent Wiener-process $\widetilde{W}, Z$. We define

$$\begin{aligned} \gamma(t) &= \alpha\sqrt{Y(t)} \\ \xi(t) &= \frac{\mu - r}{\sqrt{1-\varrho^2}\sqrt{Y(t)}} - \varrho\frac{\gamma(t)}{\sqrt{1-\varrho^2}} \\ L(t) &= \exp\left(-\int_0^t \gamma(s)dZ(s) - \int_0^t \xi(s)d\widetilde{W}(s) - \frac{1}{2}\int_0^t \gamma^2(s) + \xi^2(s)ds\right) \quad (4.22) \end{aligned}$$

for every $0 \leq t \leq T$. Then there exists an equivalent probability measure $\mathbb{P}^*$ with density process L w.r.t. $\mathbb{P}$, i.e.

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L(t) \quad \text{for all } 0 \leq t \leq T.$$

Girsanov's theorem implies that $W^\star(t) = \widetilde{W}(t) + \int_0^t \xi(s)ds$ and $Z^\star(t) = Z(t) + \int_0^t \gamma(s)ds$ are independent Wiener-processes w.r.t. $\mathbb{P}^\star$. The dynamics w.r.t. $\mathbb{P}^\star$ transforms to

$$\begin{aligned}
dS(t) &= S(t)(rdt + \sqrt{Y(t)}(\sqrt{1-\varrho^2}dW^\star(t) + \varrho dZ^\star(t))) \\
dY(t) &= q(m - Y(t))dt + \sigma\sqrt{Y(t)}dZ^\star(t) - \underbrace{\sigma\sqrt{Y(t)}\gamma(t)}_{\sigma\alpha Y(t)}\, dt \\
&= (qm - (q+\sigma\alpha)Y(t))dt + \sigma\sqrt{Y(t)}dZ^\star(t) \\
&= (q+\sigma\alpha)(\frac{qm}{q+\sigma\alpha} - Y(t))dt + \sigma\sqrt{Y(t)}dZ^\star(t).
\end{aligned}$$

We recognize that the change to an equivalent local martingale measure varies the parameters of the Heston-model. But the principal structure remains unchanged. The new parameters are given by

$$\begin{aligned}
b &= q + \sigma\alpha \\
a &= \frac{qm}{q+\sigma\alpha}
\end{aligned} \tag{4.23}$$

and according to $\mathbb{P}^*$ the Heston-equation becomes

$$\begin{aligned}
dS(t) &= S(t)(rdt + \sqrt{Y(t)}(\sqrt{1-\varrho^2}dW^\star(t) + \varrho dZ^\star(t))) \\
dY(t) &= b(a - Y(t)dt) + \sigma\sqrt{Y(t)}dZ^*(t)
\end{aligned} \tag{4.24}$$

For a further analysis it is more convenient to express the Heston equation by correlated Wiener-processes. Hence we introduce the Wiener-process

$$B(t) = \sqrt{1-\varrho^2}W^\star(t) + \varrho Z^\star(t)), \quad t \geq 0.$$

Then B, Z^* are Wiener-processes with $\langle B, Z^* \rangle_t = \rho t$ and

$$\begin{aligned}
dS(t) &= S(t)(rdt + \sqrt{Y(t)}dB(t)) \\
dY(t) &= b(a - Y(t)dt) + \sigma\sqrt{Y(t)}dZ^*(t)
\end{aligned} \tag{4.25}$$

To compute the call price we note that as usual

$$\begin{aligned}
\mathbb{E}^\star e^{-rT}(S(T)-K)^+ &= \mathbb{E}^\star e^{-rT}S(T)\mathbb{1}_{\{S(T)>K\}} - e^{-rT}K\mathbb{P}^\star(S(T)>K) \\
&= S(0)\mathbb{P}_1^\star(S(T)>K) - e^{-rT}K\mathbb{P}^\star(S(T)>K)
\end{aligned}$$

with
$$\left.\frac{d\mathbb{P}_1^\star}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t} := \frac{1}{S(0)}e^{-rt}S(t) \quad \text{for all } 0 \le t < T.$$
Thus we have to determine $\mathbb{P}_1^\star(S(T) > K)$ and $\mathbb{P}^\star(S(T) > K)$. This will be done by computing the Fourier-transform of
$$X(T) := \ln S(T)$$
according to $\mathbb{P}_1^\star$ and $\mathbb{P}^\star$.
Ito's formula provides
$$\begin{aligned}
dX(t) &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}d\langle S\rangle_t \\
&= \frac{1}{S(t)}S(t)(rdt + \sqrt{Y(t)}dB(t)) - \frac{1}{2}\frac{1}{S^2(t)}S^2(t)Y(T)dt \\
&= (r - \frac{1}{2}Y(t))dt + \sqrt{Y(t)}dB(t).
\end{aligned}$$
We have to consider
$$\mathbb{E}^\star e^{i\lambda X(T)} = \mathbb{E}^\star h(X(T), Y(T))$$
with $h(x, y) = e^{i\lambda x}$. Note that the bivariate process (X, Y) is a Markov-process and therefore the expectation on the right-hand side can be evaluated with a PDE approach. We define
$$u(t, x, y) := \mathbb{E}^\star(h(X(T), Y(T)|X(t) = x, Y(t) = y).$$
Then, due to the Markov-property
$$\begin{aligned}
\mathbb{E}^\star(h(X(T), Y(T)|\mathcal{F}_t) &= \mathbb{E}^\star(h(X(T), Y(T))|X(t), Y(t)) \\
&= u(t, X(t), Y(t)).
\end{aligned}$$
Since $u(t, X(t), Y(t)), t \ge 0$, is as conditional expectation a $\mathbb{P}^\star$-martingale an application of Ito's formula leads to the desired PDE.

$$
\begin{aligned}
du(t,X(t),Y(t)) &= \partial_t u(t,X(t),Y(t))dt + \partial_x u(t,X(t),Y(t))dX(t) \\
&\quad + \partial_y u(t,X(t),Y(t))dY(t) + \frac{1}{2}\partial_x^2 u(t,X(t),Y(t))d\langle X\rangle_t \\
&\quad + \frac{1}{2}\partial_y^2 u(t,X(t),Y(t))d\langle Y\rangle_t + \partial_x\partial_y u(t,X(t),Y(t))d\langle X,Y\rangle_t \\
&= \partial_t u(t,X(t),Y(t))dt + \partial_x u(t,X(t),Y(t))(r - \frac{1}{2}Y(t))dt \\
&\quad + \partial_x u(t,X(t),Y(t))\sqrt{Y(t)}dB(t) + \partial_y u(t,X(t),Y(t))b(a - Y(t))dt \\
&\quad + \partial_y u(t,X(t),Y(t))\sigma\sqrt{Y(t)}dZ^\star(t) + \frac{1}{2}\partial_x^2 u(t,X(t),Y(t))Y(t)dt \\
&\quad + \frac{1}{2}\partial_y^2 u(t,X(t),Y(t))\sigma^2 Y(t)dt + \partial_x\partial_y u(t,X(t),Y(t))\sigma Y(t)\varrho dt
\end{aligned}
$$

since $\langle X,Y\rangle_t = \sigma\sqrt{Y(t)}\sqrt{Y(t)}d\langle B,Z^\star\rangle_t = \sigma Y(t)\varrho dt$

$$
\begin{aligned}
&= \Bigg[\partial_t u(t,X(t),Y(t)) + (r - \frac{1}{2}Y(t))\partial_x u(t,X(t),Y(t)) \\
&\quad + b(a - Y(t))\partial_y u(t,X(t),Y(t)) + \frac{1}{2}Y(t)\partial_x^2 u(t,X(t),Y(t)) \\
&\quad + \frac{1}{2}\sigma^2 Y(t)\partial_y^2 u(t,X(t),Y(t)) + \varrho\sigma y\partial_x\partial_y u(t,X(t),Y(t))\Bigg]dt \\
&\quad + \partial_x u(t,X(t),Y(t))\sqrt{Y(t)}dB(t) + \partial_y u(t,X(t),Y(t))\sigma\sqrt{Y(t)}dZ^\star(t).
\end{aligned}
$$

Hence u satisfies the partial differential equation

$$
\begin{aligned}
&\partial_t u(t,x,y) + (r - \frac{1}{2}y)\partial_x u(t,x,y) \\
&\quad + b(a - y)\partial_y u(t,x,y) + \frac{1}{2}y\partial_x^2 u(t,x,y)) \\
&\quad + \frac{1}{2}\sigma^2 y\partial_y^2 u(t,x,y) + \varrho\sigma y\partial_x\partial_y u(t,x,y) \\
&= 0
\end{aligned}
$$

on $(0,T)\times\mathbb{R}\times[0,\infty)$ with final condition

$$
\lim_{t\nearrow T} u(t,x,y) = e^{i\lambda x} \qquad \text{for all } x\in\mathbb{R}, y\in[0,\infty).
$$

As guess for a solution one can choose

$$
u(t,x,y) = \exp(C_\lambda(T-t) + D_\lambda(T-t)y + i\lambda x)
$$

with functions $C_\lambda, D_\lambda : [0,\infty) \longrightarrow \mathbb{R}$ that have to be determined.

Computation of the partial derivatives and inserting in the PDE leads to the ordinary differential equations for the real functions D, C.

$$\begin{aligned} D_\lambda'(s) &= (-b + i\lambda\varrho\sigma)D(s) + \frac{1}{2}\sigma^2 D^2(s) - \frac{1}{2}i\lambda - \frac{1}{2}\lambda^2 \\ C_\lambda'(s) &= abD(s) + ri\lambda \end{aligned}$$

with initial value $C_\lambda(0) = 0 = D_\lambda(s)$.
This ODE for D is a Ricatti-equation and a solution is given by

$$D_\lambda(t) = \frac{b - i\sigma\varrho\lambda + d}{\sigma^2}\frac{1 - e^{dt}}{1 - ge^{dt}}$$

with

$$\begin{aligned} g &= \frac{b - \varrho\sigma\lambda i + d}{b - \varrho\sigma\lambda i - d} \\ d &= \sqrt{(b - i\varrho\sigma\lambda)^2 + \sigma^2(i\lambda + \lambda^2)} \end{aligned}$$

An integration leads to a solution for C:

$$C_\lambda(t) = ri\lambda t + \frac{ab}{\sigma^2}\left((b - i\varrho\sigma\lambda + d)t - 2\ln\left(\frac{1 - ge^{dt}}{1 - g}\right)\right).$$

Hence the Fourier-transform is given by

$$\begin{aligned} u(\lambda) &= \mathbb{E}^\star\left(e^{i\lambda X(T)} | X(0) = x_0, Y(0) = y_0\right) \\ &= u(0, x_0, y_0) \\ &= \exp\left(C_\lambda(T) + D_\lambda(T)y_0 + i\lambda x_0\right) \end{aligned}$$

and

$$\mathbb{P}^\star(S(T) > K) = \mathbb{P}^\star(X(T) > \ln K)$$

can be calculated by Fourier-inversion

$$\mathbb{P}^\star(S(T) > K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\lambda \ln K}u(\lambda)}{i\lambda}\right)d\lambda.$$

This integral can be computed numerically.
Furthermore one has to determine $\mathbb{P}_1^\star(S(T) > K)$ with

$$\left.\frac{d\mathbb{P}_1^\star}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t} = \frac{1}{S(0)}e^{-rt}S(t) = \frac{1}{S(0)}S^\star(t) =: L(t).$$

Due to

$$dS^\star(t) = S^\star(t)\sqrt{Y(t)}dB(t).$$

one obtains the Doleans-exponential

$$L(t) = \exp\left(\int_0^t \sqrt{Y(s)}dW^\star(s) - \frac{1}{2}\int_0^t Y(s)ds\right).$$

Girsanov implies that

$$\begin{aligned} W^{\star\star} &= B(t) - \langle B, \int_0^\cdot \sqrt{Y(s)}dB(s)\rangle_t \\ Z^{\star\star} &= Z^\star(t) - \langle Z^\star, \int_0^\cdot \sqrt{Y(s)}dB(s)\rangle_t \end{aligned}$$

are Wiener-processes with

$$\langle W^{\star\star}, Z^{\star\star}\rangle_t = \langle B, Z^\star\rangle_t = \varrho t.$$

It holds true

$$\begin{aligned} \langle B, \int_0^\cdot \sqrt{Y(s)}dB(s)\rangle_t &= \int_0^t \sqrt{Y(s)}d\langle B\rangle_s = \int_0^t \sqrt{Y(s)}ds \\ \langle Z^\star, \int_0^\cdot \sqrt{Y(s)}dB(s)\rangle_t &= \int_0^t \sqrt{Y(s)}d\langle Z^\star, B\rangle_s = \int_0^t \sqrt{Y(s)}\varrho ds. \end{aligned}$$

Inserting provides

$$\begin{aligned} dX(t) &= (r - \frac{1}{2}Y(t))dt + \sqrt{Y(t)}dB(t) \\ &= (r - \frac{1}{2}Y(t))dt + \sqrt{Y(t)}dW^{\star\star} + \sqrt{Y(t)}\sqrt{Y(t)}dt \\ &= (r + \frac{1}{2}Y(t))dt + \sqrt{Y(t)}dW^{\star\star}(t) \\ dY(t) &= b(a - Y(t))dt + \sigma\sqrt{Y(t)}dZ^\star(t) \\ &= b(a - Y(t))dt + \sigma\sqrt{Y(t)}dZ^{\star\star}(t) + \sigma\sqrt{Y(t)}\sqrt{Y(t)}\varrho dt \\ &= (b - \varrho\sigma)\left(\frac{ab}{b - \varrho\sigma} - Y(t)\right)dt + \sigma\sqrt{Y(t)}dZ^{\star\star}(t) \\ &=: b_1(a_1 - Y(t))dt + \sigma\sqrt{Y(t)}dZ^{\star\star}(t) \end{aligned}$$

The same method as above can be applied to determine the Fourier-transform of $X(T) = \ln S(T)$.
We obtain as result

$$\begin{aligned} u_1(\lambda) &= \mathbb{E}_1^\star\left(e^{i\lambda X(T)}|X(0) = x_0, Y(0) = y_0\right) \\ &= \exp\left(C_\lambda^{(1)}(T) + D_\lambda^{(1)}(T)y_0 + i\lambda x_0\right) \end{aligned}$$

with

$$
\begin{aligned}
C_\lambda^{(1)}(t) &= ri\lambda t + \frac{a_1 b_1}{\sigma^2}\left((b_1 - i\varrho\sigma\lambda + d_1)t - 2\ln\left(\frac{1 - g_1 e^{d_1 t}}{1 - g_1}\right)\right)\\
D_\lambda^{(1)}(t) &= \frac{b_1 - i\varrho\sigma\lambda + d_1}{\sigma^2}\,\frac{1 - e^{d_1 t}}{1 - g_1 e^{d_1 t}}
\end{aligned}
$$

and

$$
\begin{aligned}
g_1 &= \frac{b_1 - \varrho\sigma i\lambda + d_1}{b_1 - \varrho\sigma i\lambda - d_1}\\
d_1 &= \sqrt{(\varrho\sigma\lambda i - b_1)^2 + \sigma^2(\lambda^2 - i\lambda)}.
\end{aligned}
$$

By Fourier-inversion one may calculate

$$
\mathbb{P}_1^\star(S(T) > K) = \mathbb{P}_1^\star(X(T) > \ln(K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\lambda \ln K} u_1(\lambda)}{i\lambda}\right) d\lambda.
$$

<u>Practical application:</u>
The Heston-model is incomplete and depends on five parameters. The observable call-prices in the market can be used to calibrate the model.
The model's parameters are

q return rate,

m return level of the volatility,

α proportional factor in the market price of volatility risk,

σ expected fluctuation of the volatility,

ϱ correlation between stock and volatility.

These parameters determine the model-price of a call option in a Heston-model. One chooses those parameters that explain the market-prices best and use this calibrated model for general pricing purposes.

5 Modelling Bond Markets

5.1 Basic Concepts

A bond-market can be seen as financial market where the risky assets are given by Zero-coupon bonds. Since there are infinite maturities there are infinite risky assets. This means that the basic concepts of finance are still true but have to be adapted to handle bond-markets too. The main ingredients are

- trading interval $[0, T^\star]$.
- the source of uncertainty is given by some n-dimensional Wiener-process $(W(t))_{0\leq t\leq T^\star}$.
- the information in the market is given by that Wiener-process filtration $(\mathcal{F}_t)_{0\leq t\leq T^\star}$ that is generated by W.
- the risky assets in this model are T-Bonds. A T-Bond is a security with pay-off 1-Euro at T. The expiration date $T < T^*$ denotes the maturity of the bond. During the running-time there is no payment of coupons. The price-process of a T-bond is denoted by
$$(B(t,T))_{0\leq t\leq T}\,.$$

The following assumptions are reasonable:

(i) $B(T,T) = 1$,

(ii) $B(t,T)$, $0 \leq t \leq T$, is a strictly positive semi-martingale with continuous paths,

(iii) the locally bounded variation part of $(B(t,T))_{0\leq t\leq T}$ has absolutely-continuous paths w.r.t. Lebesgue-measure.

(iv) $(B(t,T))_{t\leq T\leq T^\star}$, as function in T, has $\mathbb{P}$ a.s. differentiable paths, i.e. $B(t,T)$ is differentiable in T for $\mathbb{P}-$ every ω when t is fixed.

Conclusions:
From (ii) and (iii) we obtain as in 4.1.3, that $(B(t,T))_{0\leq t\leq T}$ satisfies a stochastic differential equation of the form

$$\begin{aligned} dB(t,T) &= B(t,T)(\mu(t,T)dt + \sigma(t,T)dW(t)) \\ &= B(t,T)(\mu(t,T)dt + \sum_{j=1}^{n} \sigma_j(t,T)dW_j(t)) \end{aligned}$$

with previsible processes $(\mu(t,T))_{0\le t\le T}$ and $(\sigma(t,T))_{0\le t\le T}$. This can be seen by considering $X(t) = \ln B(t,T)$ and applying Ito's formula.
From (iv) it follows, that the so called short rate process

$$r(t) := -\frac{\partial}{\partial T}\ln(B(t,T))|_{T=t}$$

is well-defined.
With the short-rate a money-market account can be defined by

$$\beta(t) := \exp\left(\int_0^t r(s)ds\right), \quad 0 \le t \le T^\star$$

resp.

$$d\beta(t) = \beta(t)r(t)dt, \quad 0 \le t \le T^\star.$$

5.1.1 Existence of an Equivalent Local Martingale Measure

The uncertainty in the model is driven by n Wiener-processes. But there exists an infinite number of risky assets. Therefore it is clear that restrictions on the drift and volatility have to be made in order to get an arbitrage-free market.

Definition 5.1.1. *A probability measure $\mathbb{P}^*$ is called an equivalent local martingale measure if*

(i) $\mathbb{P}$ *and* $\mathbb{P}^*$ *are equivalent on* $(\Omega, \mathcal{F}_T)$,

(ii) For each maturity $T < T^*$ the discounted price-process $(\frac{B(t,T)}{\beta(t)})_{t\le T}$ is a local martingale w.r.t. $\mathbb{P}^*$.

With the same methods as in 4.2.8 we can determine an equivalent local martingale measure in a bond-market.

1. We consider first the case $n = 1$.

   With the help of Girsanov we would like to construct an equivalent local martingale measure.

   Only one maturity T and one T-bond is necessary to formulate sufficient and necessary conditions.

   We choose $T = T^\star$ and consider

$$\begin{aligned} d\beta(t) &= \beta(t)r(t)dt \\ dB(t,T^\star) &= B(t,T^\star)(\mu(t,T^\star)dt + \sigma(t,T^\star)dW(t)) \end{aligned}$$

   In a market with these two assets there exists an equivalent local martingale measure if and only if a density process can be defined by

$$L(t) = \exp(\int_0^t \vartheta(t)dW(t) - \frac{1}{2}\int_0^t \vartheta(s)^2 ds) \tag{5.1}$$

with

$$\vartheta(t) = -\frac{\mu(t,T^\star) - r(t)}{\sigma(t,T^\star)} \quad \text{for all } 0 \le t \le T^\star$$

and this is the case iff

$$\mathbb{E}\exp\left(\int_0^{T^\star} \vartheta(s)dW(s) - \frac{1}{2}\int_0^{T^\star} \vartheta^2(s)ds\right) = 1.$$

Then an equivalent local martingale measure $\mathbb{P}^\star$ exists and is defined by

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \vartheta(s)dW(s) - \frac{1}{2}\int_0^t \vartheta^2(s)ds\right), \quad 0 \le t \le T^\star$$

on $\mathcal{F}_{T^\star}$. Furthermore

$$W^\star(t) = W(t) - \int_0^t \vartheta(s)ds, \quad 0 \le t \le T^\star$$

is a Wiener-process w.r.t $\mathbb{P}^\star$ and according to $\mathbb{P}^\star$:

$$dB(t,T^\star) = B(t,T^\star)(r(t)dt + \sigma(t,T^\star)dW^\star(t)).$$

For T-Bonds with shorter running-time $T < T^\star$ we obtain

$$\begin{aligned} dB(t,T) &= B(t,T)(\mu(t,T)dt + \sigma(t,T)dW(t)) \\ &= B(t,T)((\mu(t,T) + \sigma(t,T)\vartheta(t))dt + \sigma(t,T)dW^\star(t)). \end{aligned}$$

$\left(\frac{B(t,T)}{\beta(t)}\right)_{0\le t\le T}$ is a local $\mathbb{P}^\star$-martingale if and only if

$$\mu(t,T) + \sigma(t,T)\vartheta(t) = r(t).$$

Hence the drift functions $\mu(\cdot,T)$ and volatilities $\sigma(\cdot,T)$ must fulfill

$$\mu(t,T) + \sigma(t,T)\vartheta(t) = r(t) \quad \text{for all } 0 \le t \le T$$

resp.

$$\frac{r(t) - \mu(t,T)}{\sigma(t,T)} = \vartheta(t) = \frac{r(t) - \mu(t,T^\star)}{\sigma(t,T^\star)} \quad \text{for all } 0 \le t \le T$$

This condition is also natural from another point of view. The Sharpe-ratio in a market that is driven by some one-dimensional Wiener-process is an invariant.

Thus the following argumentation is also valid.

The Sharpe-ratio of the $T^\star-$bond is given by

$$\frac{\mu(t,T^\star)-r(t)}{\sigma(t,T^\star)} = -\vartheta(t)$$

Each further asset in the market has the same Sharpe-ratio. Therefore for every T-bond we obtain

$$\frac{\mu(t,T)-r(t)}{\sigma(t,T)} = -\vartheta(t) = \frac{\mu(t,T^\star)-r(t)}{\sigma(t,T^\star)} \quad \text{for all } 0 \le t \le T.$$

2. We consider secondly the case $n = d \in \mathbb{N}$ and choose d maturities with

$$T_1 < T_2 < ... < T_d.$$

The market that consists of these d bonds and the money-market account fulfills

$$\begin{aligned} d\beta(t) &= \beta(t)r(t)dt \\ dB(t,T_i) &= B(t,T_i)(\mu(t,T_i)dt + \sigma(t,T_i)dW(t)) \\ &= B(t,T_i)(\mu(t,T_i)dt + \sum_{j=1}^{d} \sigma_j(t,T_i)dW_j(t)) \end{aligned}$$

If the matrix

$$\sigma(t) := \sigma_j(t,T_i)_{\substack{1\le i\le d \\ 1\le j\le d}}$$

is invertible for all $t \le T_1$, then $\vartheta(t)$ can be defined by

$$\vartheta(t) = \sigma^{-1}(t)\left(r(t)\mathbb{1} - \begin{pmatrix} \mu(t,T_1) \\ \vdots \\ \mu(t,T_d) \end{pmatrix}\right).$$

If furthermore

$$\mathbb{E}\exp\left(\int_0^{T_1} \vartheta(s)dW(s) - \frac{1}{2}\int_0^{T_1} |\vartheta(s)|^2 ds\right) = 1$$

then by

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \vartheta(s)dW(s) - \int_0^t \frac{1}{2}|\vartheta(s)|^2 ds\right)$$

an equivalent local martingale measure $\mathbb{P}^*$ can be defined and

$$W^\star(t) = W(t) - \int_0^t \vartheta(s)ds$$

is a Wiener-process according to $\mathbb{P}^\star$.

It holds true

$$dB(t,T_i) = B(t,T_i)(r(t)dt + \sum_{j=1}^{d} \sigma_j(t,T_i)dW_j^\star(t)).$$

Hence $\left(\frac{B(t,T_i)}{\beta(t)}\right)_{0\leq t\leq T_1}$ is a local $\mathbb{P}^*$-martingale for all $1 \leq i \leq d$.

There exists an equivalent local martingale measure for the whole bond-market on $[0,T_1]$ if and only if $\left(\frac{B(t,T)}{\beta(t)}\right)_{0\leq t\leq T}$ is a local $\mathbb{P}^\star$-martingale for every $T \leq T_1$.

Then for $T \leq T_1$:

$$\begin{aligned} dB(t,T) &= B(t,T)(\mu(t,T)dt + \sum_{j=1}^{d} \sigma_j(t,T)dW_j(t)) \\ &= B(t,T)((\mu(t,T) + \sum_{j=1}^{d} \sigma_j(t,T)\vartheta_j(t))dt + \sum_{j=1}^{d} \sigma_j(t,T)dW_j^\star(t)). \end{aligned}$$

Hence we have the following condition

$$\mu(t,T) + \sum_{j=1}^{d} \sigma_j(t,T)\vartheta_j(t) = r(t) \quad \text{ for all } 0 \leq t \leq T.$$

Then all T-bonds with maturity before T_1 are local $\mathbb{P}^\star$-martingales.

5.1.2 The Short Rate Approach

Based of the evolution of the short rate r we construct an arbitrage-free bond-market model. We have to specify the following ingredients:

- a probability space $(\Omega, \mathcal{F}_{T^\star}, \mathbb{P})$, where $\mathcal{F}_{T^\star}$ is generated by some Wiener-process W.
- a family of bond-prices $(B(t,T))_{0\leq t\leq T}$ for every $T \leq T^\star$,
- an equivalent probability measure $\mathbb{P}^\star$ on $(\Omega, \mathcal{F}_{T^\star})$, such that $\left(\frac{B(t,T)}{\beta(t)}\right)_{0\leq t\leq T}$ is a local martingale w.r.t. $\mathbb{P}^\star$ for all $T \leq T^\star$.

Therefore we consider some n-dimensional Wiener-process W with Wiener-filtration $(\mathcal{F}_t)_{t\geq 0}$ according to a probability measure $\mathbb{P}$. We fix a maximal maturity $T^\star > 0$. Then the probability-space $(\Omega, \mathcal{F}_{T^\star}, \mathbb{P})$ is fixed.

1. Assumption: The short rate is a diffusion. This means that it is a strong solution to the stochastic differential equation

$$\begin{aligned} dr(t) &= m(t, r(t))dt + \delta(t, r(t))dW(t) \\ &= m(t, r(t))dt + \sum_{j=1}^{n} \delta_j(t, r(t))dW_j(t) \end{aligned}$$

with initial condition $r(0) = r_0 \in \mathbb{R}$.
The functions $m : [0, T^\star] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\delta : [0, T^\star] \times \mathbb{R} \longrightarrow \mathbb{R}$ have to be chosen, such that the above solution exists.
2. Assumption: There exists according to $\mathbb{P}$ an equivalent probability measure $\mathbb{P}^\star$ on $(\Omega, \mathcal{F}_{T^\star})$ such that

$$\left.\frac{d\mathbb{P}^\star}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L(t) = \exp\left(\int_0^t \vartheta(s)dW(s) - \frac{1}{2}\int_0^t |\vartheta(s)|^2 ds \right) \qquad \text{for all } 0 \leq t \leq T^\star$$

with $\vartheta(t) = \vartheta(t, r(t))$ for all $t \leq T^\star$ with some function $\vartheta : [0, T^\star] \times \mathbb{R} \longrightarrow \mathbb{R}^n$.
3. Assumption: Let $\mathbb{E}^\star \frac{1}{\beta(T)} < \infty$ for all $T \leq T^\star$, with

$$\beta(t) = \exp\left(\int_0^t r(s)ds \right).$$

Then

$$\begin{aligned} B(t, T) &= \beta(t)\mathbb{E}^\star\left(\frac{1}{\beta(T)} | \mathcal{F}_t \right) \\ &= \mathbb{E}^\star\left(\frac{\beta(t)}{\beta(T)} | \mathcal{F}_t \right) \\ &= \mathbb{E}^\star\left(\exp\left(-\int_t^T r(s)ds \right) | \mathcal{F}_t \right) \end{aligned}$$

for all $t \leq T$ defines an arbitrage-free bond-market with equivalent local martingale measure $\mathbb{P}^\star$, since

$$\frac{B(t, T)}{\beta(t)} = \mathbb{E}^\star\left(\frac{1}{\beta(T)} | \mathcal{F}_t \right), \quad t \leq T$$

is a $\mathbb{P}^\star$-martingale.
In the following we compute $B(t, T)$ and its volatility $\sigma(t, T)$ by a PDE-approach:
Due to the 2. assumption a Wiener-process W^* w.r.t. $\mathbb{P}^*$ is given by

$$W^\star(t) = W(t) - \int_0^t \vartheta(s, r(s))ds.$$

Furthermore it holds true:

$$\begin{aligned} dr(t) &= m(t, r(t))dt + \delta(t, r(t))dW(t) \\ &= (m(t, r(t)) + \sum_{j=1}^{n} \vartheta_j(t, r(t))\delta_j(t, r(t)))dt + \delta(t, r(t))dW^\star(t). \end{aligned}$$

Hence r is also a diffusion w.r.t. $\mathbb{P}^\star$:

$$dr(t) = b(t, r(t))dt + \delta(t, r(t))dW^\star(t)$$

with

$$b(t, r(t)) = m(t, r(t)) + \vartheta(t, r(t))\delta(t, r(t)).$$

The Markov-property of r w.r.t. $\mathbb{P}^\star$ implies

$$\begin{aligned} B(t, T) &= \mathbb{E}^\star \left(\exp\left(-\int_t^T r(s)ds \right) |\mathcal{F}_t \right) \\ &= \mathbb{E}^\star \left(\exp\left(-\int_t^T r(s)ds \right) |r(t) \right) \\ &= v_T(t, r(t)) \end{aligned}$$

with

$$v_T(t, r) = \mathbb{E}^\star \left(\exp\left(-\int_t^T r(s)ds \right) |r(t) = r \right).$$

Ito's formula applied to v_T implies:

$$\begin{aligned} dB(t, T) &= dv_T(t, r(t)) \\ &= \partial_t v_T(t, r(t)) + \partial_x v_T(t, r(t))dr(t) + \frac{1}{2}\partial_x^2 v_T(t, r(t))d\langle r\rangle_t \\ &= \left[\partial_t v_T(t, r(t)) + \partial_x v_T(t, r(t))b(t, r(t)) + \frac{1}{2}\partial_x^2 v_T(t, r(t))|\delta(t, r(t))|^2 \right] dt \\ &\quad + \partial_x v_T(t, r(t))\delta(t, r(t))dW^\star(t) \end{aligned}$$

Thus v_T satisfies the PDE

$$\partial_t v_T(t, r) + b(t, r)\partial_x v_T(t, r) + \frac{1}{2}|\delta(t, r)|^2 \partial_x^2 v_T(t, r) = r v_T(t, r)$$

on $(0, T) \times \mathbb{R}$ with final condition

$$\lim_{t \nearrow T} v_T(t, r) = 1.$$

By solving this PDE the bond-prices can be calculated explicitly. Furthermore the volatilities are determined by

$$dB(t,T) = B(t,T)(r(t)dt + \underbrace{\frac{\partial_x v_T(t,r(t))}{v_T(t,r(t))}\delta(t,r(t))}_{\sigma(t,T)}dW^\star(t).$$

Remark 5.1.2. Actually there is no change to $\mathbb{P}^\star$ necessary. This belongs to the case $\vartheta(t,r) = 0$. Then $\mathbb{P}^\star = \mathbb{P}$ and this is called martingale modelling.

Examples for short rate models

a) *Vasicek Modell*

One factor model, $n = 1$,

$$dr(t) = b(a - r(t))dt + \delta dW(t)$$

with $b, a, \delta > 0$.

- Vasicek-process
- return level a
- return-rate b

Solving the partial differential equation leads to

$$B(t,T) = \exp\left(-h(T-t) - r(t)g(T-t)\right)$$

with

$$\begin{aligned} h(s) &= \left(a - \frac{\delta^2}{2b^2}\right)s + \left(\frac{\delta^2}{b^2} - a\right)\left(1 - e^{-bs}\right)\frac{1}{b} - \frac{\sigma^2}{2b^2}\frac{1}{2b}\left(1 - e^{-2bs}\right) \\ g(s) &= \frac{1}{b}\left(1 - e^{-bs}\right) \end{aligned}$$

The Yield $Y(t,T)$ can be obtained by

$$\begin{aligned} \exp\left((T-t)Y(t,T)\right) &= \frac{1}{B(t,T)} \\ \Leftrightarrow Y(t,T) &= \frac{1}{T-t}(h(T-t) + g(T-t)r(t)) \end{aligned}$$

The function $Y(t,\cdot)$ denotes the yield as function of maturity for fixed t. The initial yield curve is given by

$$Y(0,T) = \frac{1}{T}(g(T)r(0) + h(T)).$$

The yield depends linear affine on the short rate. Therefore the Vasicek-model is an example for an affine bond market model.

Note, that the bond-price satisfies

$$dB(t,T) = B(t,T)(r(t)dt \underbrace{-g(T-t)\delta}_{\sigma(t,T)} dW(t)).$$

Therefore the volatility is given by

$$\sigma(t,T) = -g(T-t)\delta$$

The function $t \to \sigma(t,T)$ is non-random. This is a remarkable property and very useful in pricing of derivatives. Shortly one can say that pricing of derivatives in a Vasicek-model can be nearly done as in a Black-Scholes model with deterministic volatility function.

b) *Cox-Ignersoll-Ross Modell (CIR Modell)* Here the short rate is assumed to evolve as a CIR-process

$$dr(t) = b(a - r(t))dt + \delta\sqrt{r(t)}dW(t)$$

with $b, a, \delta > 0$ and $2ab \geq \delta^2$.

Solving the partial differential equation provides

$$B(t,T) = \exp\left(-h(T-t) - g(T-t)r(t)\right)$$

with

$$\begin{aligned} h(s) &= -\frac{2ab}{\delta^2}\ln\left(\frac{4\gamma e^{(\gamma+\frac{b}{2})s}}{(2\gamma+b)(e^{2\gamma s}-1)+4\gamma}\right) \\ g(s) &= \frac{2(e^{2\gamma s}-1)}{(2\gamma+b)(e^{2bs}-1)+4\gamma} \end{aligned}$$

and

$$\gamma = \frac{1}{2}\sqrt{b^2 + 2\delta^2}.$$

Therefore also the CIR model is an affine bond-market model with

$$dB(t,T) = B(t,T)(r(t)dt \underbrace{-g(T-t)\delta\sqrt{r(t)}}_{\sigma(t,T)} dW(t)).$$

5.2 Pricing of Derivatives

We consider a bond-market with an equivalent local martingale measure $\mathbb{P}^*$ and assume that

$$\mathbb{E}^*\frac{1}{\beta(T)} < \infty$$

for each maturity T. Since there are an infinite number of T-bonds but only a finite dimensional Wiener-process that represents the uncertainty it is reasonable to assume that the market is complete and therefore the local equivalent martingale measure unique. Every T-Claim C with $\mathbb{E}^\star|\frac{C}{\beta(T)}| < \infty$ has a unique arbitrage-free price given by

$$p_0(C) = \mathbb{E}^\star \frac{C}{\beta(T)}$$

in Euro.
Respectively in t by

$$p_t(C) = \beta(t)\mathbb{E}^\star \left(\frac{C}{\beta(T)} | \mathcal{F}_t \right).$$

5.2.1 Forward Martingale Measure

To compute this price it is useful to determine the so called forward-price

$$F(0, T; C) = \frac{p_0(C)}{B(0, T)}$$

resp.

$$F(t, T; C) = \frac{p_t(C)}{B(t, T)}$$

with the so called forward martingale measure $\mathbb{P}_T$ in T.

Definition 5.2.1. *The forward martingale measure $\mathbb{P}_T$ w.r.t. the time date $T > 0$ is that local equivalent martingale measure that belongs to the numeraire $(B(t, T))_{0 \le t \le T}$. More precise:*

(i) $\mathbb{P}_T \sim \mathbb{P}^\star$ on $(\Omega, \mathcal{F}_T)$.

(ii) For each risky asset S the forward-price process $\left(\frac{S(t)}{B(t,T)}\right)_{0 \le t \le T}$ is a local $\mathbb{P}_T$-martingale.

$\frac{S(t)}{B(t,T)}$ is the so called forward-price in t.

Computation of $\mathbb{P}_T$:
Due to

$$\frac{B(t, T)}{\beta(t)} = \mathbb{E}^\star \left(\frac{1}{\beta(T)} | \mathcal{F}_t \right), \quad 0 \le t \le T$$

the process $\frac{B(t,T)}{\beta(t)}$ is as conditional expectation a $\mathbb{P}^\star$-martingale for all $0 \le t \le T$. Therefore we can define an equivalent probability measure $\mathbb{P}^\star$ by

$$\left.\frac{d\mathbb{P}_T}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t} = \underbrace{\frac{B(t, T)}{\beta(t)}}_{\substack{\mathbb{P}^\star-\text{martingale} \\ \mathbb{E}^\star \frac{1}{\beta(T)} = B(0,T) > 0}} \underbrace{\frac{1}{B(0, T)}}_{\substack{\text{Norma-} \\ \text{lising-} \\ \text{factor}}} =: L(t), \quad 0 \le t \le T.$$

For every risky asset S it follows that

$$F(t,T;S) := \frac{S(t)}{B(t,T)}, \quad 0 \le t \le T$$

is a local $\mathbb{P}_T$-martingale if and only if

$$\frac{S(t)}{B(t,T)} L(t), \quad 0 \le t \le T$$

is a local $\mathbb{P}^\star$-martingale.
Due to

$$\frac{S(t)}{B(t,T)} L(t) = \frac{S(t)}{B(t,T)} \frac{B(t,T)}{\beta(t)} \frac{1}{B(0,T)} = \frac{S(t)}{\beta(t)} \frac{1}{B(0,T)}$$

$F(t,T;S)$ is a local $\mathbb{P}_T$-martingale for all $0 \le t \le T$. Therefore $\mathbb{P}_T$ is the forward martingale measure w.r.t. the time date T.
If $T_1 \neq T$ then

$$\left(\frac{B(t,T_1)}{B(t,T)} \right)_{0 \le t \le T \wedge T_1}$$

is a $\mathbb{P}_T$-martingale.

Remark 5.2.2. Due to

$$dB(t,T) = B(t,T)(r(t)dt + \sigma(t,T)dW^\star(t))$$

we have

$$\left.\frac{d\mathbb{P}_T}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \sigma(s,T)dW^\star(s) - \frac{1}{2}\int_0^t |\sigma(s,T)|^2 ds \right)$$

and

$$W^T(t) := W^\star(t) - \int_0^t \sigma(s,T)ds, \quad 0 \le t \le T$$

defines a Wiener-process according to $\mathbb{P}_T$.

Application to pricing of derivatives:
Let C be a T-claim with

$$\mathbb{E}^\star \frac{|C|}{\beta(t)} < \infty.$$

Its unique price-process is given by

$$p_t(C) = \beta(t)\mathbb{E}^\star \left(\frac{C}{\beta(T)} | \mathcal{F}_t \right)$$

for all $0 \le t \le T$.

The forward-price w.r.t. the time date T on the claim satisfies

$$\begin{aligned} F(t,T;C) &= \frac{p_t(C)}{B(t,T)} \\ &= \frac{\beta(t)}{B(t,T)}\mathbb{E}^\star\left(\frac{C}{\beta(T)}|\mathcal{F}_t\right) \\ \text{Bayes-Formula} \rightarrow \quad &= \frac{\beta(t)}{B(t,T)}\mathbb{E}_T\left(\frac{C}{\beta(T)}\frac{1}{L(T)}|\mathcal{F}_t\right)L(t) \\ &= \mathbb{E}_T(C|\mathcal{F}_t). \end{aligned}$$

This corresponds to the standpoint that it is not necessary to take $\mathbb{P}^\star$ as pricing measure. Since $\mathbb{P}_T$ is the local equivalent martingale measure corresponding to the numeraire $B(\cdot,T)$,

$$\mathbb{E}_T\left(\frac{C}{B(T,T)}|\mathcal{F}_t\right) = \mathbb{E}_T\left(C|\mathcal{F}_t\right)$$

is then the arbitrage-free price of C, quoted in shares of the numeraire asset. The price in Euro we get by multiplying with the price of the numeraire asset, i.e.

$$p_t(C) = \mathbb{E}_T(C|\mathcal{F}_t)B(t,T)$$

resp.

$$F(t,T;C) = \frac{p_t(C)}{B(t,T)} = \mathbb{E}_T(C|\mathcal{F}_t).$$

We apply this general framework to several examples.

5.2.2 Pricing of a Call-Option

We consider the following setup:

- Bond market model
- equivalent local martingale measure $\mathbb{P}^\star$
- T_1-Bond as risky asset
- call on the T_1-bond with running-time $T < T_1$, i.e.

$$C = \left(B(T,T_1) - K\right)^+.$$

For pricing purposes we consider the forward-martingale measure w.r.t. the time date T which is denoted by $\mathbb{P}_T$ and given by the density-process

$$\left.\frac{d\mathbb{P}_T}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t} = \frac{B(t,T)}{\beta(t)}\frac{1}{B(0,T)}.$$

We have to calculate

$$\mathbb{E}_T\left((B(T,T_1)-K)^+|\mathcal{F}_t\right), \quad \text{for all } 0\le t\le T.$$

It follows:

$$\mathbb{E}_T(C|\mathcal{F}_t)=\mathbb{E}_T\left(B(T,T_1)\mathbb{1}_{\{B(T,T_1)>K\}}|\mathcal{F}_t\right)-\mathbb{E}_T(K\mathbb{1}_{\{B(T,T_1)>K\}}|\mathcal{F}_t).$$

Furthermore

$$\left.\frac{d\mathbb{P}_T}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t}=\frac{B(t,T)}{\beta(t)}\frac{1}{B(0,T)}$$

and

$$\left.\frac{d\mathbb{P}_{T_1}}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t}=\frac{B(t,T_1)}{\beta(t)}\frac{1}{B(0,T_1)}.$$

Due to Bayes $\mathbb{E}_T(YL(T)|\mathcal{F}_t)=\mathbb{E}_{T_1}(Y|\mathcal{F}_t)L(t)$, hence

$$\begin{aligned}\mathbb{E}_T\left(B(T,T_1)\mathbb{1}_{\{B(T,T_1)>K\}}|\mathcal{F}_t\right)&=\mathbb{E}_{T_1}\left(\mathbb{1}_{\{B(T,T_1)>K\}}|\mathcal{F}_t\right)F(t,T;T_1)\\&=\mathbb{P}_{T_1}\left(B(T,T_1)>K|\mathcal{F}_t\right)F(t,T;T_1).\end{aligned}$$

Therefore it follows for the claim's pricing

$$\mathbb{E}_T(C|\mathcal{F}_t)=F(t,T;T_1)\mathbb{P}_{T_1}(B(T,T_1)>K|\mathcal{F}_t)-K\mathbb{P}_T(B(T,T_1)>K|\mathcal{F}_t)$$

resp. for the arbitrage-free price in Euro

$$\begin{aligned}p_t(C)&=B(t,T)\mathbb{E}_T(C|\mathcal{F}_t)\\&=B(t,T_1)\mathbb{P}_{T_1}(B(T,T_1)>K|\mathcal{F}_t)-KB(t,T)\mathbb{P}_T(B(T,T_1)>K|\mathcal{F}_t).\end{aligned}$$

So far the considerations are valid in each bond market model. For an explicit computation

$$\mathbb{P}_{T_1}(B(T,T_1)>K|\mathcal{F}_t) \quad \text{and} \quad \mathbb{P}_T(B(T,T_1)>K|\mathcal{F}_t)$$

have to be determined and these values depend on the specified model.

Calaculation in the Vasicek-Model:

In short-rate models an explicit computation is in principle possible.
Very easy is the case of a Vasicek-model since the volatility of the T-bonds are deterministic functions.

$$\begin{aligned}dB(t,T)&=B(t,T)(r(t)dt+\sigma(t,T)dW^\star(t))\\dB(t,T_1)&=B(t,T_1)(r(t)dt+\sigma(t,T_1)dW^\star(t))\end{aligned}$$

Then with Ito

$$dF(t,T;T_1) = F(t,T;T_1)(\underbrace{\sigma(t,T_1) - \sigma(t,T)}_{\eta(t)})dW^T(t)$$

whereby

$$W^T(t) = W^\star(t) - \int_0^t \sigma(t,T)dt$$

is a Wiener-process w.r.t. $\mathbb{P}_T$.
Due to

$$\left.\frac{d\mathbb{P}_{T_1}}{d\mathbb{P}_T}\right|_{\mathcal{F}_t} = \frac{F(t,T;T_1)}{F(0,T;T_1)} = \exp\left(\int_0^t \eta(s)dW^T(s) - \frac{1}{2}\int_0^t \eta^2(s)ds\right)$$

the process

$$W^{T_1}(t) = W^T(t) - \int_0^t \eta(s)ds$$

is a Wiener-process w.r.t. $\mathbb{P}_{T_1}$.
Hence

$$dF(t,T;T_1) = F(t,T;T_1)\eta(t)dW^{T_1}(t) + F(t,T;T_1)\eta(t)dt.$$

In the Vasicek-model η is a deterministic function and therefore the conditional probabilities are determined by the normal distribution.
More precise

$$\begin{aligned}
\mathbb{P}_T(B(T,T_1) > K|\mathcal{F}_t) &= \mathbb{P}_T(F(T,T;T_1) > K|\mathcal{F}_t) \\
&= \mathbb{P}_T\Bigg(F(t,T;T_1)\underbrace{\frac{F(T,T;T_1)}{F(t,T;T_1)}}_{\exp-\text{martingale}} > K|\mathcal{F}_t\Bigg) \\
&= \mathbb{P}_T\Bigg(\underbrace{(F(t,T;T_1)}_{\mathcal{F}_t-\text{mb}}\underbrace{\exp\left(\int_t^T \eta(s)dW^T(s) - \frac{1}{2}\int_t^T \eta^2(s)ds\right) > K}_{\text{independent of }\mathcal{F}_t}|\mathcal{F}_t\Bigg) \\
&= \mathbb{P}_T\Bigg(\exp\left(\int_t^T \eta(s)dW^T(s) - \frac{1}{2}\int_t^T \eta^2(s)ds\right) > \frac{K}{F(t,T;T_1)}\Bigg) \\
&= \Phi\left(\frac{\ln\frac{F(t,T;T_1)}{K} - \frac{1}{2}\int_t^T \eta^2(s)ds}{\sqrt{\int_t^T \eta^2(s)ds}}\right)
\end{aligned}$$

and

$$
\begin{aligned}
\mathbb{P}_{T_1}(B(T,T_1) > K|\mathcal{F}_t) &= \mathbb{P}_{T_1}(F(T,T;T_1) > K|\mathcal{F}_t) \\
&= \mathbb{P}_{T_1}\left(\exp\left(\int_t^T \eta(s)dW^{T_1}(s) + \frac{1}{2}\int_t^T \eta^2(s)ds \right) > \frac{K}{F(t,T;T_1)} \right) \\
&= \Phi\left(\frac{\ln \frac{F(t,T;T_1)}{K} + \frac{1}{2}\int_t^T \eta^2(s)ds}{\sqrt{\int_t^T \eta^2(s)ds}} \right).
\end{aligned}
$$

Note:

$$
\begin{aligned}
\sigma(t,T) &= -g(T-t)\delta = -\frac{\delta}{b}\left(1 - e^{-b(T-t)}\right), \\
\sigma(t,T_1) &= -g(T_1-t)\delta = -\frac{\delta}{b}\left(1 - e^{-b(T_1-t)}\right), \\
\eta(t) &= \frac{\delta}{b}\left(e^{-b(T_1-t)} - e^{-b(T-t)}\right) \\
&= \frac{\delta}{b}e^{bt}\left(e^{-bT_1} - e^{-bT}\right)
\end{aligned}
$$

5.2.3 Pricing of Caplets

As before we consider a general setup:

- bond market model
- local equivalent martingale measure $\mathbb{P}^\star$
- A caplet is an interest-rate derivative, that ensures to make safe a floating interest rate.

For this we consider a time-period $[T, T_1]$. The discrete floating interest-rate of a risk-less return on capital between T and T_1 is

$$
\frac{1}{T_1 - T}\left(\frac{1}{B(T,T_1)} - 1\right),
$$

for 1 Euro, that is invested in T in T_1-Bonds, one receives $\frac{1}{B(T,T_1)}$ T_1-Bonds. These are in T_1 $\frac{1}{B(T,T_1)}$ Euro worth. Hence

$$
\frac{1}{B(T,T_1)} - 1
$$

is the capital gain, which corresponds to an annual interest-rate of

$$\underbrace{\frac{1}{T_1 - T}}_{\text{period}} \underbrace{\left(\frac{1}{B(T, T_1)} - 1 \right)}_{\text{Gain}} =: \underbrace{R_d(T, T_1)}_{\text{interest-rate}}$$

A caplet according to the period $[T, T_1]$ with strike K gives the right to exchange the floating coupon

$$(T_1 - T) R_d(T, T_1) = \frac{1}{B(T, T_1)} - 1$$

with the fixed coupon

$$(T_1 - T)K.$$

A caplet provides the following pay-off in T_1:

$$\begin{aligned} ((T_1 - T) R_d(T, T_1) - (T_1 - T)K)^+ &= \left(\frac{1}{B(T, T_1)} - 1 - (T_1 - T)K \right)^+ \\ &= \left(\frac{1}{B(T, T_1)} - (1 + K(T_1 - T)) \right)^+ . \end{aligned}$$

As before the calculation of the caplet-price is based on the calculation of the caplet's forward-price in T_1. It follows

$$\begin{aligned} \mathbb{E}_{T_1} \left(\frac{1}{B(T, T_1)} - (1 + (T_1 - T)K \right)^+ &= \mathbb{E}_{T_1} \left(\frac{B(T, T)}{B(T, T_1)} - (1 + (T_1 - T)K) \right)^+ \\ &= \mathbb{E}_{T_1} \left(F(T, T_1; T) - (1 + (T_1 - T)K) \right)^+ \end{aligned}$$

The forward-price process $(F(t, T_1; T))_{0 \le t \le T}$ of a T-bond is a $\mathbb{P}_{T_1}$-martingale. Therefore

$$\begin{aligned} dF(t, T_1; T) &= F(t, T_1; T) \eta(t) dW^{T_1}(t), \\ dB(t, T) &= B(t, T)(r(t)dt + \sigma(t, T) dW^\star(t)), \\ dB(t, T_1) &= B(t, T_1)(r(t)dt + \sigma(t, T_1) dW^\star(t)) \end{aligned}$$

and

$$\eta(t) = \sigma(t, T) - \sigma(t, T_1).$$

Hence we obtain

$$\mathbb{E}_{T_1} F(T, T_1; T) \mathbb{1}_{\{F(T, T_1; T) > 1 + (T_1 - T)K\}} = F(0, T_1; T) \mathbb{P}_T \Big(F(T, T_1; T) > 1 + (T_1 - T)K \Big)$$

due to

$$\left. \frac{d\mathbb{P}_T}{d\mathbb{P}_{T_1}} \right|_{\mathcal{F}_t} = \frac{F(t, T_1; T)}{F(0, T_1; T)}.$$

Therefore it follows

$$\begin{aligned}
&\mathbb{E}_{T_1}\left(\frac{1}{B(T,T_1)}-(1+(T_1-T)K\right)^+\\
=&F(0,T_1;T)\mathbb{P}_T\Big(F(T,T_1;T)>1+(T_1-T)K\Big)\\
&\quad-(1+(T_1-T)K)\mathbb{P}_{T_1}\Big(F(T,T_1;T)>1+(T_1-T)K\Big).
\end{aligned}$$

As arbitrage-free price we obtain

$$\begin{aligned}
Cl(0) &:= B(0,T_1)\mathbb{E}_{T_1}\left(\frac{1}{B(T,T_1)}-(1+(T_1-T)K\right)^+\\
&= B(0,T)\mathbb{P}_T\Big(F(T,T_1;T)>1+(T_1-T)K\Big)\\
&\quad-\frac{(1+(T_1-T)K)}{B(0,T_1)}\mathbb{P}_{T_1}\Big(F(T,T_1;T)>1+(T_1-T)K\Big).
\end{aligned}$$

With the same way we receive the price at t:

$$\begin{aligned}
&\mathbb{E}_{T_1}\left(\frac{1}{B(T,T_1)}-(1+(T_1-T)K|\mathcal{F}_t\right)^+\\
=&F(t,T_1;T)\mathbb{P}_T\Big(F(T,T_1;T)>1+(T_1-T)K|\mathcal{F}_t\Big)\\
&\quad-(1+(T_1-T)K)\mathbb{P}_{T_1}\Big(F(T,T_1;T)>1+(T_1-T)K|\mathcal{F}_t\Big)
\end{aligned}$$

resp. as initial arbitrage-free price

$$\begin{aligned}
Cl(0) &= B(t,T_1)\mathbb{E}_{T_1}\left(\frac{1}{B(T,T_1)}-(1+(T_1-T)K|\mathcal{F}_t\right)^+\\
&= B(t,T)\mathbb{P}_T\Big(F(T,T_1;T)>1+(T_1-T)K|\mathcal{F}_t\Big)\\
&\quad-(1+(T_1-T)K)B(0,T_1)\mathbb{P}_{T_1}\Big(F(T,T_1;T)>1+(T_1-T)K|\mathcal{F}_t\Big).
\end{aligned}$$

The explicit computation of $\mathbb{P}_T$ and $\mathbb{P}_{T_1}$ depends on the chosen model. In the Vasicek model we obtain

$$dF(t,T_1;T)=F(t,T_1;T)\eta(t)dW^{T_1}(t)$$

with

$$\eta(t)=\sigma(t,T)-\sigma(t,T_1)$$

which is deterministic in t.
Due to

$$\left.\frac{d\mathbb{P}_T}{d\mathbb{P}_{T_1}}\right|_{\mathcal{F}_t}=\frac{F(t,T_1;T)}{F(0,T_1;T)}=\exp\left(\int_0^t\eta(s)dW^{T_1}(s)-\frac{1}{2}\int_0^t\eta^2(s)ds\right)$$

the process

$$W^T(t) := W^{T_1}(t) - \int_0^t \eta(s)ds$$

is a Wiener-process according to $\mathbb{P}_T$.
We obtain

$$\mathbb{P}_{T_1}\Big(F(T,T_1;T) > 1 + (T_1 - T)K|\mathcal{F}_t\Big) = \Phi\left(h_1(F(t,T_1;T),t)\right)$$

and

$$\mathbb{P}_T\Big(F(T,T_1;T) > 1 + (T_1 - T)K|\mathcal{F}_t\Big) = \Phi\left(h_2(F(t,T_1;T),t)\right).$$

Hereby

$$h_1(x,t) = \frac{\ln \frac{x}{1+(T_1-T)K} - \frac{1}{2}\int_t^T \eta^2(s)ds}{\sqrt{\int_t^T \eta^2(s)ds}}$$

and

$$h_2(x,t) = \frac{\ln \frac{x}{1+(T_1-T)K} + \frac{1}{2}\int_t^T \eta^2(s)ds}{\sqrt{\int_t^T \eta^2(s)ds}}.$$

5.2.4 Caplets, Caps, Floorlets und Floors

A Cap is a finite sequence of caplets. For a given tenor structure

$$T_0 < T_1 < ... < T_n$$

we consider the time-periods $[T_{i-1}, T_i]$ and their corresponding caplets. The i-th caplet gives the right to exchange the floating coupon, fixed at T_{i-1}, with an initially fixed coupon $(T_i - T_{i-1})K$. The exchange will take place at T_i.
Thus a Cap induces a pay-off stream:
At each T_i there is a pay-off

$$(R_d(T_{i-1}, T_i) - K)^+(T_i - T_{i-1}).$$

If we denote according to $t \leq T_0$ with $Cl_i(t)$ the price of the i-th caplet, then

$$Cap(t) = \sum_{i=1}^n Cl_i(t)$$

is the price of the Cap in t.

In practice a holder of a cap can use it to make safe a floating loan contract w.r.t. to an creasing interest-rate. In a floating loan contract one has to pay for the i-th time-period $[t_{i-1}, t_i]$ the floating coupon

$$R_d(T_{i-1}, T_i)(T_i - T_{i-1})$$

which is fixed at t_{i-1} and paid at t_i.
One would like to ensure that an initially fixed interest-rate K will not be exceeded. Therefore one buys a cap with strike K according to the suitable tenor-structure of the loan. Then a difference between floating-rate and fixed rate

$$R_d(T_{i-1}, T_i) - K$$

would be provided by the cap. The cost to avoid this risk at t for $t \leq T_0$ are given by the Cap's price $Cap(t)$.
Instead of caplet und cap one can also define floorlet und floor. These are derivatives with pay-off

$$(K - R_d(T_{i-1}, T_i))^+$$

at T_i for each time-period $[t_{i-1}, t_i]$.
Application:
One has to pay the coupons of a fixed rate loan according to a tenor-structure

$$T_0 < T_1 < ... < T_n$$

and one would like to exchange the fixed coupon with a floating coupon, if it is favourable.
Solution: Buy a floor corresponding to the tenor-structure. If in the $i-$ten period

$$K > R_d(T_{i-1}, T_i)$$

then $R_d(T_{i-1}, T_i)$ has to be effectively to be paid, since the difference

$$K - R_d(T_{i-1}, T_i)$$

is financed by the floorlet.

5.2.5 Swaps

A long position in a cap combined with a short position in a floor and vice versa yields a swap. There are

$$\text{Payer-Swap} = \text{Cap} - \text{Floor}$$
$$\text{Receiver-Swap} = \text{Floor} - \text{Cap}$$

More precise:

- tenor-structure

$$T_0 < T_1 < ... < T_n$$

- fixed-rate K,
- nominal N.

A Swap is an exchange-contract, that exchanges in each period the floating coupons with the fixed-rate coupons. There is no optionality. At the end of each period an exchange occurs. For the payer-swap we obtain in every period $[t_{i-1}, t_i]$ the pay-off

$$N(T_i - T_{i-1})(R_d(T_{i-1}, T_i) - K), \quad \text{for all } 1 \leq i \leq n$$

and for the Receiver-Swap

$$N(T_i - T_{i-1})(K - R_d(T_{i-1}, T_i)), \quad \text{for all } 1 \leq i \leq n.$$

Pricing of Swaps

Reminder: At t we can realise the gain

$$\frac{B(t, T_{i-1})}{B(t, T_i)} - 1$$

for a capital investment in the period $[T_{i-1}, T_i]$ by settling forward contracts on the T_i-bond with maturity T_{i-1}. This corresponds to an annual discrete interest-rate

$$\Phi_d(t; T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{B(t, T_{i-1})}{B(t, T_i)} - 1 \right)$$

the so called forward-rate.
Note:

$$\Phi_d(T_{i-1}; T_{i-1}, T_i) = R_d(T_{i-1}, T_i).$$

Remark 5.2.3. $\Phi_d(t; T_{i-1}, T_i)$ can be seen as forward-price with maturity T_i fixed at t on the short-rate $R_d(T_{i-1}, T_i)$.

Proof.

$$R_d(T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{1}{B(T_{i-1}, T_i)} - 1 \right)$$

and

$$\begin{aligned}
\mathbb{E}_{T_i}(R_d(T_{i-1}, T_i)|\mathcal{F}_t) &= \mathbb{E}_{T_i}\left(\frac{1}{T_i - T_{i-1}} \left(\frac{1}{B(T_{i-1}, T_i)} - 1 \right) |\mathcal{F}_t \right) \\
&= \mathbb{E}_{T_i}\left(\frac{1}{T_i - T_{i-1}} \left(\frac{B(T_{i-1}, T_{i-1})}{B(T_{i-1}, T_i)} - 1 \right) |\mathcal{F}_t \right) \\
&= \frac{1}{T_i - T_{i-1}} \left(Er_{T_i}(F(T_{i-1}, T_i; T_{i-1})|\mathcal{F}_t) - 1 \right) \\
&= \frac{1}{T_i - T_{i-1}} (F(t, T_i; T_{i-1}) - 1) \\
&= \frac{1}{T_i - T_{i-1}} \left(\frac{B(t, T_{i-1})}{B(t, T_i)} - 1 \right) \\
&= \Phi_d(t; T_{i-1}, T_i)
\end{aligned}$$

$\square$

Therefore we have calculated the arbitrage-free price at t of the T_i's-pay-off $R_d(T_{i-1}, T_i)$.

$$B(t,T_i)\Phi_d(t;T_{i-1},T_i) = \frac{1}{T_i - T_{i-1}}(B(t,T_{i-1}) - B(t,T_i)).$$

This considerations provides a valuation of the payer-swap in t:

$$\begin{aligned} swap(t) &= N\sum_{i=1}^{n}(T_i - T_{i-1})(\Phi_d(t;T_{i-1},T_i) - K)B(t,T_i) \\ &= N\sum_{i=1}^{n}(B(t,T_{i-1}) - B(t,T_i)) - N\sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i) \\ &= N(B(t,T_0) - B(t,T_n)) - N\sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i). \end{aligned}$$

What is the meaning of

> $Swap(t) > 0 \Rightarrow$ The pay-off stream of the floating coupons is more worth than that of the fixed coupons.
>
> $Swap(t) < 0 \Rightarrow$ The pay-off stream of the floating coupons is less worth than that of the fixed coupons.
>
> $Swap(t) = 0 \Rightarrow$ The pay-off stream of the floating coupons is as worth as that of the fixed coupons. A floating rate loan contract and a fixed rate loan contract can be exchanged without causing any costs.

Application:
A floating rate loan contract has no risk according to change in interest-rates. By additional taking a swap position a fixed interest loan can be exchanged to a floating rate loan contract. This gives the opportunity to eliminate the risk of a change in interest rate for fixed interest rate loans.
That fixed-rate K, at which $swap(t) = 0$, is called swaprate $R_{swap}(t)$. There are two ways how to determine the swap-rate:
1. way:

$$\begin{aligned} K = R_{swap}(t) &\Leftrightarrow \sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i)(\Phi_d(t;T_{i-1},T_i) - K) = 0 \\ &\Leftrightarrow \sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i)\Phi_d(t;T_{i-1},T_i) = K\sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i) \\ &\Leftrightarrow \sum_{i=1}^{n}\underbrace{\frac{(T_i - T_{i-1})B(t,T_i)}{\sum_{k=1}^{n}(T_k - T_{k-1})B(t,T_k)}}_{\omega_i(t)}\Phi_d(t;T_{i-1},T_i) = K. \end{aligned}$$

Hence,

$$R_{swap}(t) = \sum_{i=1}^{n} \omega_i(t)\Phi_d(t;T_{i-1},T_i).$$

2. way:

$$\begin{aligned} swap(t) = 0 \Leftrightarrow B(t,T_0) - B(t,T_n) &= K\sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i) \\ = \Leftrightarrow \frac{B(t,T_0) - B(t,T_n)}{\sum\limits_{i=1}^{n}(T_i - T_{i-1})B(t,T_i)} &= K. \end{aligned}$$

Hence,

$$R_{swap}(t) = \frac{B(t,T_0) - B(t,T_n)}{\sum\limits_{i=1}^{n}(T_i - T_{i-1})B(t,T_i)}.$$

The price of a swap can also be expressed with the swap-rate

$$\begin{aligned} swap(t) &= B(t,T_0) - B(t,T_n) - K\sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i) \\ &= R_{swap}(t)\sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i) - K\sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i) \\ &= (R_{swap}(t) - K)\underbrace{\sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i)}_{N(t)} \\ &= (R_{swap}(t) - K)N(t) \end{aligned}$$

Swaption

A Payer-Swaption gives the holder the right, to settle at T_0 in a Payer-Swap with tenor structure

$$T_0 < T_1 < ... < T_n$$

and fixed-rate K.
A Swaption will be executed in T_0, if

$$Swap(T_0) \geq 0.$$

This results in the pay-off stream of a swap in $T_1, ...T_n$ which is valued at T_0 by $Swap(T_0)$. Therefore a swaption can be seen as T_0-Claim with pay-off

$$C = Swap(T_0)^+.$$

It follows

$$Swap(T_0)^+ = (R_{wap}(T_0) - K)^+ N(T_0).$$

Pricing of a Swaption

The idea is that the swaption can nearly be seen as call on the swap-rate process $(R_{Swap}(t))_{0\le t\le T_0}$ and a suitable choose of a numeraire asset leads to a pricing-formula.

Definition 5.2.4. *Let*

$$N(t) := \sum_{i=1}^{n}(T_i - T_{i-1})B(t,T_i) \quad \text{for all } 0 \le t \le T_0.$$

A probability measure $\mathbb{P}_{swap}$ is called swap-martingale measure, if it is a local equivalent martingale measure w.r.t. the numeraire N, i.e.

(i) $\mathbb{P}_{swap} \sim \mathbb{P}_{T_0}$ *on* $(\Omega, \mathcal{F}_{T_0})$,

(ii) $\left(\frac{S(t)}{N(t)}\right)_{0\le t\le T_0}$ *is a local $\mathbb{P}_{Swap}$-martingale for all risky assets S.*

Computation of $\mathbb{P}_{swap}$:

$$\left.\frac{d\mathbb{P}_{swap}}{d\mathbb{P}_{T_0}}\right|_{\mathcal{F}_t} =: L(t).$$

It follows: $\frac{S(t)}{N(t)}$ is a local $\mathbb{P}_{Swap}$-martingale if and only if $\frac{S(t)}{N(t)}L(t)$ is a local $\mathbb{P}_{T_0}$-martingale.
Furthermore:
$\frac{S(t)}{B(t,T_0}$ is a local $\mathbb{P}_{T_0}$-martingale with

$$\frac{S(t)}{B(t,T_0)} = \frac{S(t)}{N(t)}L(t)c$$

with a constant c. Hence

$$L(t) = \frac{N(t)}{B(t,T_0)}\frac{1}{c}.$$

Due to normalisation

$$L(t) = \frac{N(t)}{B(t,T_0)}\frac{B(0,T_0)}{N(0)}.$$

Hence

$$\left.\frac{d\mathbb{P}_{Swap}}{d\mathbb{P}_{T_0}}\right|_{\mathcal{F}_t} = \frac{N(t)}{B(t,T_0)}\frac{B(0,T_0)}{N(0)}, \qquad \text{for all } 0 \le t \le T_0$$

defines the density of the swap-martingale measure with respect to the forward-martingale measure.
Thus we can obtain an arbitrage-free Euro price by

$$\begin{aligned} p_t(C) &= N(t)\mathbb{E}_{Swap}\left(\left.\frac{(R_{Swap}(T_0)-K)^+N(T_0)}{N(T_0)}\right|\mathcal{F}_t\right) \\ \Leftrightarrow p_t(C) &= N(t)\mathbb{E}_{Swap}\left((R_{Swap}(T_0)-K)^+|\mathcal{F}_t\right). \end{aligned}$$

For an explicit computation the swap-rate process under $\mathbb{P}_{Swap}$ has to be determined. The process $(R_{Swap}(t))_{0\le t\le T_0}$ is a positive $\mathbb{P}_{swap}$-martingale.
Hence, there is a previsible process σ_{Swap}, such that

$$dR_{swap}(t) = R_{swap}(t)\sigma_{swap}(t)dW_{swap}(t)$$

with a Wiener-process W_{swap} according to $\mathbb{P}_{swap}$.
In principal the swap-rate's volatility σ_{Swap} can be computed, but the formula even in short rate models is complex and not suitable for effective computations. Therefore simplifications are used in practise.
The most simple method is to assume, that $(\sigma_{Swap}(t))_{0\le t\le T_0}$ is some deterministic function in time t. This results in the so called formula of Black for the swaption price, which can be calculated in the same way as the call price in a Black-Scholes model.

5.3 Libor Market Model

The Libor market model is at this time a very popular and advance model used in banks. The reason relies in the fact that contrary to short-rate models and to the HJM-framework the Libor-market approach uses observable market-prices to fix a bond model. The main idea is to describe the evolution of the discrete forward-rates $\Phi_d(t, T_{i-1}, T_i)$ which are also often called Libor-rates.

5.3.1 Model Specification

We consider a given tenor-structure

$$T_0 < T_1 < ... < T_N$$

with interval lengths

$$\delta_i = T_i - T_{i-1}.$$

The T_i-Bonds $i = 0, ..., N-1$ are the risky assets with price-processes

$$(B(t, T_i))_{0\le t\le T_i}, i = 0, ..., N-1.$$

The T_N-bond serves as numeraire asset with price-process

$$(B(t, T_N))_{0\le t\le T_N}.$$

Hence we have defined a financial market with N risky assets and the T_N-bond as numeraire asset.
Note: The $i-$th risky asset can only be traded until T_i. Thereafter it vanishes from the market and can't be used for hedging purposes.
Assumptions:

- The source of uncertainty is given by some d-dimensional Wiener-process $(W(t))_{t\geq 0}$. This means that we consider a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $(\mathcal{F}_t)_{t\geq 0}$ is a Wiener-filtration of some $d-$dimensional Wiener-process W.

  The price-processes of all $N+1$ Bonds are continuous, strictly positive semi-martingales with $B(T_i, T_i) = 1$ for all $i = 0, ..., N$.

- The model is free of arbitrage. This means that there exists some probability measure $\mathbb{P}_{T_N}$ on $(\Omega, \mathcal{F}_{T_N})$, such that
  - (i) $\mathbb{P}_{T_N} \sim \mathbb{P}$ on $(\Omega, \mathcal{F}_{T_N})$,
  - (ii) $\left(\frac{B(t,T_i)}{B(t,T_N)}\right)_{0\leq t\leq T_i}$ is a local $\mathbb{P}_{T_N}$-martingale for every $0 \leq i \leq N-1$.

  We request here the slight stronger condition, that $\left(\frac{B(t,T_i)}{B(t,T_N)}\right)_{0\leq t\leq T_i}$ is a $\mathbb{P}_{T_N}$-martingale for every $0 \leq i \leq N-1$.

Remark 5.3.1. $\mathbb{P}_{T_N}$ is the forward-martingale measure w.r.t. the time-date T_N, due to the fact that it is the equivalent martingale measure according to the numeraire $(B(t, T_N))_{0\leq t\leq T_N}$.

Remark 5.3.2. The specification of the model avoids to use the money-market account, since the short-rate is not directly observable. The model uses only market observable quantities.

For the period from T_{i-1} to T_i the discrete forward interest-rate is defined at $t \leq T_i$ by

$$\begin{aligned} L_i(t) &:= \Phi_d(t; T_{i-1}, T_i) \\ &= \frac{1}{\delta_i}\left(\frac{B(t,T_{i-1})}{B(t,T_i)} - 1\right) \end{aligned}$$

This $L_i(t)$ is also called i-th Libor-rate.

Remark 5.3.3. For any $1 \leq i \leq N$ the Libor-rate process L_i is a semi-martingale, since

$$\frac{B(t,T_{i-1})}{B(t,T_i)} = \frac{B(t,T_{i-1})/B(t,T_N)}{B(t,T_i)/B(t,T_N)}$$

is a fraction of semi-martingales.

It is further assumed that the Libor-rates are strictly positive.

5.3.2 Terminal Measure

The probability measure $\mathbb{P}_{T_N}$ is also called terminal measure, since T_N is the last time point on the tenor structure. According to $\mathbb{P}_{T_N}$ the evolution of the Libor-rates can in principal be clarified.

We start with $(L_N(t))_{0\leq t\leq T_{N-1}}$. Due to the assumptions

$$L_N(t) = \frac{1}{\delta_N}\left(\frac{B(t,T_{N-1})}{B(t,T_N)} - 1\right)$$

is a $\mathbb{P}_{T_N}$-martingale . Therefore there exists some previsible $\mathbb{R}^d$-valued process $\left(\sigma^{(N)}(t)\right)_{0\le t\le T_{N-1}}$, such that

$$dL_N(t) = L_N(t)\sigma^{(N)}(t)dW(t) = L_N(t)\sum_{j=1}^{d}\sigma_j^{(N)}(t)dW_j(t).$$

$\left(\sigma^{(N)}(t)\right)$ determines the volatilities of the N-th Libor-rate. With the positive martingale L_N a change to the forward-martingale measure $\mathbb{P}_{T_{N-1}}$ can be done.

$$\left.\frac{d\mathbb{P}_{T_{N-1}}}{d\mathbb{P}_{T_N}}\right|_{\mathcal{F}_t} = \frac{B(t,T_{N-1})}{B(t,T_N)}\frac{B(0,T_N)}{B(0,T_{N-1})} = \frac{\delta_N L_N(t)+1}{\delta_N L_N(0)+1} =: R_N(t)$$

Hence it follows

$$\begin{aligned} dR_N(t) &= \frac{\delta_N}{\delta_N L_N(0)+1}dL_N(t) \\ &= \frac{\delta_N}{\delta_N L_N(0)+1}L_N(t)\sigma^{(N)}(t)dW(t) \\ &= R_N(t)\frac{\delta_N L_N(t)}{\delta_N L_N(t)+1}\sigma^{(N)}(t)dW(t). \end{aligned}$$

We obtain the exponential martingale expression for the density-process:

$$R_N(t) = \exp\left(\int_0^t \frac{\delta_N L_N(s)}{\delta_N L_N(t)+1}\sigma^{(N)}(s)dW(s) - \frac{1}{2}\int_0^t \left|\frac{\delta_N L_N(s)}{\delta_N L_N(s)+1}\sigma^{(N)}(s)\right|^2 ds\right).$$

Girsanov's theorem yields a Wiener-process $W^{(N-1)}$, defined by

$$W^{(N-1)}(t) = W(t) - \int_0^t \frac{\delta_N L_N(s)}{\delta_N L_N(s)+1}\sigma^{(N)}(s)ds$$

according to $\mathbb{P}_{T_{N-1}}$.
In the next step the dynamics of the $N-1$-th Libor-rate

$$L_{N-1}(t) = \frac{1}{\delta_{N-1}}\left(\frac{B(t,T_{N-2})}{B(t,T_{N-1})} - 1\right)$$

will be determined.
L_{N-1} is a positive semi-martingale w.r.t. $\mathbb{P}_{T_N}$. Therefore there exist some previsible processes $\left(\sigma^{(N-1)}(t)\right)_{0\le t\le T_{N-2}}$ and $\left(\mu^{(N-1)}(t)\right)_{0\le t\le T_{N-2}}$, such that

$$dL_{N-1} = L_{N-1}(t)(\mu^{(N-1)}(t)dt + \sigma^{(N-1)}(t)dW(t)).$$

Since L_{N-1} is a $\mathbb{P}_{T_{N-1}}-$ martingale, $\left(\mu^{(N-1)}(t)\right)$ is uniquely determined. Due to

$$dW(t) = dW^{(N-1)}(t) + \frac{\delta_N L_N(t)}{\delta_N L_N(t)+1}\sigma^{(N)}(t)dt$$

we obtain by inserting

$$dL_{N-1}(t) = L_{N-1}(t)\left((\mu^{(N-1)}(t)+\frac{\delta_N L_N(t)}{\delta_N L_N(t)+1}\sigma^{(N)}(t)\sigma^{(N-1)}(t))dt+\sigma^{(N-1)}(t)dW^{(N-1)}(t)\right)$$

Since L_{N-1} is a $\mathbb{P}_{T_{N-1}}-$martingale, it follows

$$\begin{aligned}\mu^{(N-1)}(t) &= -\frac{\delta_N L_N(t)}{\delta_N L_N(t)+1}\sigma^{(N)}(t)\sigma^{(N-1)}(t)\\ &= -\sum_{j=1}^{d}\frac{\delta_N L_N(t)}{\delta_N L_N(t)+1}\sigma_j^{(N)}(t)\sigma_j^{(N-1)}(t).\end{aligned}$$

The $\mathbb{P}_{T_{N-1}}$-martingale L_{N-1} determines the change of measure to the forward-martingale measure $\mathbb{P}_{T_{N-2}}$:

$$\left.\frac{d\mathbb{P}_{T_{N-2}}}{d\mathbb{P}_{T_{N-1}}}\right|_{\mathcal{F}_t} = \frac{B(t,T_{N-2})}{B(t,T_{N-1})}\frac{B(0,T_{N-1})}{B(0,T_{N-2})} = \frac{\delta_{N-1}L_{N-1}(t)+1}{\delta_{N-1}L_{N-1}(0)+1} =: R_{N-1}(t).$$

Therefore it follows:

$$dR_{N-1}(t) = R_{N-1}(t)\frac{\delta_{N-1}L_{N-1}(t)}{\delta_{N-1}L_{N-1}(t)+1}\sigma^{(N-1)}(t)dW^{N-1}(t).$$

Again Girsanov provides a Wiener-process $W^{(N-2)}$

$$W^{(N-2)}(t) = W^{(N-1)}(t) - \int_0^t \frac{\delta_{N-1}L_{N-1}(s)}{\delta_{N-1}L_{N-1}(s)+1}\sigma^{(N-1)}(s)ds$$

according to $\mathbb{P}_{T_{N-2}}$ for every $0 \le t \le T_{N-2}$.

By continuing this procedure we obtain.
The Libor-rates $(L_i(t))_{0\le t\le T_{i-1}}, 1 \le i \le N$ have according to $\mathbb{P}_{T_N}$ the dynamics

$$dL_i(t) = L_i(t)\left(-\sum_{k=i+1}^{N}\eta_{ik}(t)dt + \sigma^{(i)}(t)dW(t)\right)$$

with

$$\eta_{ik}(t) := \frac{\delta_k L_k(t)}{\delta_k L_k(t)+1}\sigma^{(i)}(t)\sigma^{(k)}(t).$$

The forward-martingale measures $\mathbb{P}_{T_1}, ..., \mathbb{P}_{T_N}$ fulfill

$$\left.\frac{d\mathbb{P}_{T_{i-1}}}{d\mathbb{P}_{T_i}}\right|_{\mathcal{F}_t} = \frac{\delta_i L_i(t) + 1}{\delta_i L_i(0) + 1} = R_i(t)$$

and

$$dL_i(t) = L_i(t)\sigma^{(i)}(t)dW^{(i)}(t)$$

with Wiener-process $W^{(i)}$ w.r.t. $\mathbb{P}_{T_i}$.
By specifying the volatilities $\sigma^{(i)}$, $i = 1, \cdots, n$ the model is uniquely determined. In practise a model will be specified by fitting the volatilities to observable market-prices of caps and swaptions. This means that those volatilities are determined such that the model-prices explain the market-prices of observable caps and swaptions best.

Lognormal Libor-market Model

Model-prices can be calculated when suitable assumptions on the volatilities are fulfilled. This would make the calibration by market-data easier. One often used approach is to consider log-normal Libor-rates. This is the following model

- Tenor-structure $T_0 < T_1 < ... < T_N$,
- $L_i(t) = \frac{1}{\delta_i}\left(\frac{B(t,T_{i-1})}{B(t,T_i)} - 1\right), \quad 0 \le t \le T_{i-1}$,
- $\mathbb{P}_{T_i}$ forward-martingale measure,
- $Z = (Z_1, ..., Z_N)$ some $N-$dimensional correlated Wiener-process with

  $$\langle Z_k, Z_l \rangle_t = \varrho_{kl} t$$

  for $k \neq l : -1 < \varrho_{kl} < 1$.

  Every Wiener-process is driving one Libor-rate. Due to the correlation of the Wiener-process we obtain a dependency on the d Libor-rates.

  Note, that with Girsanov as in subsection 5.3.2 N-dimensional Wiener-processes $Z^{(i)} = (Z_1^{(i)}, ..., Z_N^{(i)})$ w.r.t. $\mathbb{P}_{T_i}$ can be determined with correlation $(\varrho_{kl})_{1 \le k,l \le N, k \neq l}$.

In log-normal Libor-market-models one assumes the following dynamics:

$$dL_i(t) = L_i(t)\lambda_i(t)dZ_i^{(i)}(t)$$

for all $1 \le i \le N$ with deterministic functions $\lambda_1, ..., \lambda_N$.
According to $\mathbb{P}_{T_i}$ the i-th Libor-rate $L_i(t)$ is a log-normal distributed random variable. The benefit is, that prices of caplets can be easily calculated in log-normal Libor-market-models.

Pricing of Caplets in the Log-Normal Libor-Market Model

In the time period $[T_{i-1}, T_i]$ a Caplet provides the pay-off

$$\delta_i(L_i(T_{i-1}) - K)^+ = \delta_i \left(\frac{1}{\delta_i} \left(\frac{B(T_{i-1}, T_{i-1})}{B(T_{i-1}, T_i)} - 1 \right) - K \right)^+$$

at T_i.
<u>Valuation:</u>
Computation of the forward-price at date T_i:

$$\begin{aligned} \mathbb{E}_{T_i}\left((L_i(T_{i-1}) - K)^+ \delta_i | \mathcal{F}_t\right) &= \delta_i L_i(t) \mathbb{E}_{T_i}\left(\underbrace{\frac{L_i(T_{i-1})}{L_i(t)}}_{\text{pos. MG}} \mathbb{1}_{\{L_i(T_{i-1}) > K\}} \right) - \delta_i K \mathbb{P}_{T_i}(\underbrace{L_i(T_{i-1})}_{\sim \log \mathcal{N}} > K | \mathcal{F}_t) \\ &= \delta_i L_i(t) Q_{T_i}(L_i(T_{i-1}) > K | \mathcal{F}_t) - \delta_i K \mathbb{P}_{T_i}(L_i(T_{i-1}) > K | \mathcal{F}_t). \end{aligned}$$

Due to

$$dL_i(t) = L_i(t) \lambda_i(t) dZ_i^{(i)}(t)$$

it follows

$$\mathbb{P}_{T_i}(L_i(T_{i-1}) > K | \mathcal{F}_t) = \Phi(h_2(L_i(t), t))$$

and since

$$dL_i(t) = L_i(t) \lambda_i(t) dB^{(i)}(t) - \frac{1}{2} \lambda_i(t) L_i(t) dt$$

w.r.t. Q_{T_i} we obtain

$$Q_{T_i}(L_i(T_{i-1}) > K | \mathcal{F}_t) = \Phi(h_1(L_i(t), t)).$$

Hereby

$$h_1(x, t) = \frac{\ln \frac{x}{K} + \frac{1}{2} \int_t^{T_{i-1}} \lambda_i^2(s) ds}{\sqrt{\int_t^{T_{i-1}} \lambda_i^2(s) ds}}$$

and

$$h_2(x, t) = \frac{\ln \frac{x}{K} - \frac{1}{2} \int_t^{T_{i-1}} \lambda_i^2(s) ds}{\sqrt{\int_t^{T_{i-1}} \lambda_i^2(s) ds}}.$$

The arbitrage-free Euro price at t is therefore

$$Cl_i(t) = B(t, T_i) \delta_i \Big(L_i(t) \Phi(h_1(L_i(t), t)) - K \Phi(h_2(L_i(t), t)) \Big).$$

This is the formula of Black for caplets and hence prices of caps can be computed as well.
<u>Remarks concerning calibration:</u>
Model-parameters are

- deterministic volatility functions $\lambda_1, ..., \lambda_N$ from a finite dimensional vector space.

- correlation ϱ_{kl} of the driving Wiener-processes.

Observable data are

- cap-prices
- swaption-prices.

With respect to the tenor-structure $T_0 < T_\alpha < ... < T_\beta < T_N$ there are several caps and swaptions in the market. These market-prices are taken to calibrate the parameters. Swaptions are needed to fit the correlation.

5.3.3 Further Libor-Market-Models

a) Diffusion Models

$dL_i(t) = L_i(t)\lambda_i(t)\sigma(L_i(t))dZ^{(i)}(t)$

- λ_i is a deterministic function of time
- σ is a function of state.

This corresponds to the diffusion approach in the general framework.

b) Libor-rate-models with stochastic volatility

- The analogon to the Heston Modell

The volatility of the i-th Libor-rate is exogenously determined by

$$\sigma_i(t) = \sqrt{V(t)}\lambda_i(t), \quad 1 \leq i \leq N$$

with V a CIR-process according to $\mathbb{P}_{T_N}$ of the form

$$dV(t) = a(b - V(t))dt + c\sqrt{V(t)}dB(t)$$

and λ_i a deterministic function of time.

According to $\mathbb{P}_{T_N}$ the processes $Z_1, ..., Z_N, B$ are correlated Wiener-processes. By the change of measure to $\mathbb{P}_{T_i}$ we obtain correlated Wiener-processes $Z_1^{(i)}, ..., Z_N^{(i)}, B^{(i)}$ and there is a change in the drift of V.

We obtain the following structure

$$\begin{aligned} dL_i(t) &= L_i(t)\lambda_i(t)\sqrt{V(t)}dZ_i^{(i)}(t) \\ dV(t) &= a(b - \zeta(t)V(t))dt + c\sqrt{V(t)}dB^{(i)}(t). \end{aligned}$$

How a calibration of such a model can be established in practise one can read in the master theses of Hasow, Santen, Hülsbusch und Cresnik. (compare to Homepage of Dr. Paulsen, `http://wwwmath.uni-muenster.de/statistik/paulsen/Abschlussarbeiten/Masterarbeiten/`(21.07.2016)).

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